Acyclic Coloring on Extended Duplicate Graph of Star Graph Families

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Abstract—In this paper we present acyclic coloring and acyclic chromatic number of middle graph, central graph and total graph of the extended duplicate graph of the star graphs denoted by M(EDG(K₁,n)), C(EDG(K₁,n)) and T(EDG(K₁,n)) respectively.

Keywords—Middle graph, central graph, total graph, acyclic coloring, extended duplicate graph, star graph.

I. INTRODUCTION

The proper coloring of a graph is the coloring of the vertices of G such that no two adjacent vertices of G are assigned the same color. Throughout this paper, by a graph we mean a finite, undirected, simple graph and the term coloring is used to denote vertex coloring of graphs.

A proper vertex coloring in which the subgraph induced by any two colors does not contain a cycle is called an acyclic coloring of G. The notion of acyclic chromatic number was introduced by B.Grunbaum [5] in 1973. The least number of colors required to color the graph acyclically is called acyclic chromatic number of a graph G and is denoted by a(G).

The acyclic and star coloring of middle, central and total graphs of some class of graphs have been studied in the literature [1,2,6,7,8].

A complete bipartite graph K₁,n is a star graph with n+1 vertices and n edges.

The concept of extended duplicate graph was introduced by Thirusangu, et al. [10].

A duplicate graph of G is DG = (V₁, E₁) where the vertex set V₁ = V ⊔ V′ and V ∩ V′ = ∅ and f : V → V′ is bijective (for v ∈ V, we write f(v) = v′) and the edge set E₁ of DG is defined as follows. The edge uv is in E₁ if and only if both uv′ and u′v are edges in E₁. The extended duplicate graph of DG, denoted by EDG, is defined as, adding an edge between any vertex from V to any other vertex in V′, except the terminal vertices of V and V′. For convenience, we take v₂ ∈ V and v₂′ ∈ V and thus the edge v₂v₂′ is formed.

The extended duplicate graph of star graph, denoted by EDG(K₁,n) is obtained from the duplicate graph of star by joining the vertices v₁ and v₁′.

The graph obtained from G by inserting a new vertex into every edge of G and by joining those pairs of these new vertices with edges which lie on adjacent edges of G is called the Middle graph of G, denoted by M(G)[3].

The graph obtained from G by subdividing each edge exactly once and joining all the non-adjacent vertices of G is called the Central graph of G and is denoted by C(G)[9].

The Total graph of G, is a new graph whose vertex set is the union of vertex and edge set of G and two vertices of T(G) are adjacent if they are either two adjacent vertices or two adjacent edges or an incident vertex with an edge of G and is denoted by T(G)[3,4].
II. ACYCLIC COLORING OF M(EDG(K_1,n))

**Theorem 1.** For any extended duplicate graph of the star graph EDG(K_{1,n}), the acyclic chromatic number, a[M(EDG(K_{1,n})] = n+1, n ≥ 3.

**Proof.**

Consider the extended duplicate graph of the star graph EDG(K_{1,n}) with V(EDG(K_{1,n})) = \{v, v', v_1, v_1', v_2, v_2', ..., v_n, v_n'\}. By the definition, in middle graph M(EDG(K_{1,n})), edge v'v_k is subdivided by the vertex x_k, vv'_k is subdivided by x_{nk+1} for 1 ≤ k ≤ n and v_1v'_1 is subdivided by x_{n+1} in M(EDG(K_{1,n})). That is, V(M(EDG(K_{1,n}))) = \{v, v'_1, v_1, v_1', v_2, v_2', ..., v_n, v_n'\}.

The vertices are properly colored as follows. The color C_1 is assigned to v, v'_1, v_1 and v_1' for 1 ≤ i ≤ n. The color C_{i+1} is assigned to x_i and x_{ni+1} for 1 ≤ i ≤ n. Assign color C_3 to the vertex x_{n+1}.

As M(EDG(K_{1,n})) contains a clique of order n+1, we need at least (n+1) colors for the proper coloring. We show that the above coloring is acyclic.

![Diagram](image_url)

**Case 1.** Consider the color class <C_1, C_i> for 2 ≤ k ≤ n+1, the subgraph induced by these color classes is the union of path P_1 and (2n-1) pendant vertices.

**Case 2.** Consider the color class <C_i, C_j> for 2 ≤ i ≤ j ≤ n+1, the subgraph induced by these color classes is path P_2 and the union of paths P_1.

In both the cases, the induced subgraphs have no cycles hence the coloring is acyclic and therefore, a[M(EDG(K_{1,n})]] = n+1, n ≥ 3.

III. ACYCLIC COLORING OF C(EDG(K_{1,n}))

**Theorem 2.** For any extended duplicate graph of the star graph EDG(K_{1,n}), the acyclic chromatic number, a[C(EDG(K_{1,n}))] = 2n+1, n ≥ 3.

**Proof.**

Consider the extended duplicate graph of the star graph EDG(K_{1,n}) with V(EDG(K_{1,n})) = \{v, v', v_1, v_1', v_2, v_2', ..., v_n, v_n'\}. In central graph C(EDG(K_{1,n})), by definition edge v'v_k is subdivided by x_k, vv'_k is subdivided by x_{nk+1} for 1 ≤ k ≤ n, and v_1v'_1 is subdivided by x_{n+1} in C(EDG(K_{1,n})). That is, V(C(EDG(K_{1,n}))) = \{v, v'_i / 1 ≤ i ≤ n\} ∪ \{x_i, x_{ni+1} / 1 ≤ i ≤ n\} ∪ \{v, v'_1\} ∪ \{x_{n+1}\}.

The vertices are properly colored as follows. Assign the color C_1 to v and v'. Assign the color C_i to v_i for 2 ≤ i ≤ n. Assign C_{ni} to v'_i for 1 ≤ i ≤ n and the color C_{2ni+1} to v. The color C_2 is assigned to the newly introduced vertices x_i for 1 ≤ i ≤ 2n+1, i ≠ 2 and the color C_3 is assigned to x_2.
As $\text{C}(\text{EDG}(K_{1,n}))$ contains a clique of order $2n+1$, minimum $(2n+1)$ colors are required for its proper coloring. To prove that the above said coloring is acyclic. In the above said coloring, the color classes $C_k$, $4 \leq k \leq 2n+1$ never induce a cycle (it occur only once in the coloring procedure). The subgraphs induced by $<C_i, C_j>$ for $i=1, 2$ and $j=2, 3$ with $i < j$, is as follows.

Case 1. Consider the color class $<C_1, C_2>$, the subgraph induced by these color classes is the path $P_5$ and some pendant vertices.

Case 2. Consider the color class $<C_1, C_3>$, the subgraph induced by these color class is the union of path $P_2$ and $P_3$.

Case 3. Consider the color class $<C_2, C_3>$, the subgraph induced by these color classes is a path $P_4$ and some pendant vertices.

In all the cases, the induced subgraphs has no cycles hence the coloring is acyclic and therefore, $a[C(\text{EDG}(K_{1,n}))] = 2n+1$, $n \geq 3$.

IV. ACYCLIC COLORING OF T(EDG(K_{1,n}))

**Theorem 3.** For any extended duplicate graph of the star graph EDG(K_{1,n}), the acyclic chromatic number, $a[T(\text{EDG}(K_{1,n}))] = n+1$, $n \geq 3$.

**Proof.** Consider the extended duplicate graph of the star graph EDG(K_{1,n}) with $\text{V}(\text{EDG}(K_{1,n})) = \{v, v', v_1, v'_1, v_2, v'_2, \ldots, v_n, v'_n\}$. In total graph $T(\text{EDG}(K_{1,n}))$, by definition each edge $v'v_k$ is subdivided by the vertex $x_k$, $vv'_k$ is subdivided by $x_{nk+1}$ for $1 \leq k \leq n$ and $v_i v'_i$ is subdivided by $x_{n+1}$ in $T(\text{EDG}(K_{1,n}))$. The vertices $\{x_1, x_2, \ldots, x_{n+2}, x_{n+3}, \ldots, x_{2n+1}, v\}$ induce a clique of order $n+1$ in $T(\text{EDG}(K_{1,n}))$. That is, $\text{V}(T(\text{EDG}(K_{1,n}))) = \{v, v', v_1, v'_1, v_2, v'_2, \ldots, v_n, v'_n\}$.

Now assign a proper coloring to these vertices as follows. The color $C_i$ is assigned to the vertices $x_i$ and $x_{n+i+1}$ for $i = 1, 2, \ldots, n$. Assign the color $C_{n+1}$ to the vertices $v$ and $v'$. Assign the color $C_i$ to $v_i$ and $v'_i$ for $2 \leq i \leq n$. The Color $C_2$ is assigned to $v_1$ and color $C_4$ is assigned to $v'_1$. The Color $C_3$ is assigned to the remaining vertex $x_{n+1}$.

The coloring is minimum, as $T(\text{EDG}(K_{1,n}))$ contains a clique of order $n+1$, at least $(n+1)$ colors are required for its proper coloring. To prove that the above said coloring is acyclic.

**Case 1.** Consider the color class $<C_1, C_2>$ for $k=2$ and $4$, the subgraph induced by these color class is the union of path $P_3$ and $P_4$ with $2(n-2)$ pendant vertices.

**Case 2.** Consider the color class $<C_1, C_3>$, the subgraph induced by these color class is a path $P_7$ and $2(n-2)$ pendant vertices.

**Case 3.** Consider the color class $<C_1, C_2>$ for $5 \leq k \leq n$, the subgraph induced by these color classes is the union of path $P_3$ and $2(n-2)$ pendant vertices.
Case 4. Consider the color class \(<C_1, C_{n+1}>\), the subgraph induced by these color class is the star \(K_{1,n}\).

Case 5. Consider the color class \(<C_i, C_j>\) for \(2 \leq i < j \leq n+1\), the subgraph induced by these color classes is the path \(P_2\).

In all the cases, the induced subgraphs has no cycles hence the coloring is acyclic and therefore, 
\[ a(T(EDG(K_{1,n})) = n+1, \quad n \geq 3. \]

V. CONCLUSION

In this paper, we obtained the acyclic chromatic number of middle graph, central graph and total graph of the extended duplicate graph of the star graphs.

REFERENCES


