Some Labelings on Square Graph of Comb

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Abstract: In this paper, we prove the existence of square sum, cube sum and cube difference labelings on square graph of comb.

Keywords: Square sum labeling, cube sum labeling, cube difference labeling, square graph, comb graph.

I. INTRODUCTION

Rosa introduced the notion of Graph labeling in 1967 [5]. A graph labeling is a mapping that carries a set of graph elements onto a set of numbers called labels (usually the set of integers). A dynamic survey on graph labeling is regularly updated by Gallian[3]. Germina introduced and proved some results of square sum labeling. Reena Sebastian etc., all discussed the concepts of square sum labeling in 2014[4].

The concept of cube difference labeling was introduced by J. Shaima and it was proved that many graphs like Pₙ, Cₙ, complete graphs, ladder, lattice grids, wheels, comb, star graphs, crown, dragon, coconut trees and shell graphs admit cube difference labeling [8]. Cube sum labelling have also been studied in the literature.

Motivated by these study, in this paper we prove square graph of comb admits square sum labeling, cube sum labelling and cube

II. PRELIMINARIES

Definition 2.1 : (Comb graph)

Let Pₙ be a path graph with n vertices. The comb graph is defined as Pₙ⊙K₁. It has 2n vertices and 2n - 1 edges.

Definition 2.2 (Square graph)

For a given graph G, the square graph G² is a graph on the same vertex set but in which two vertices are adjacent if and only if they are at distance at most 2 in G.

Definition 2.3 (Square sum labeling)

A graph G is said to admits a square sum labeling [4] if there exist a bijection g:V(G)→{0,1,2,…,p-1} such that the induced function g*:E(G)→N given by g*(uv)=[g(u)]²+[g(v)]² is injective, for every uv∈ E(G).

Definition 2.4 (Cube sum labeling)

A graph G is said to admits a cube sum labeling if there exist a bijection g:V(G)→{0,1,2,…,p-1} such that the induced function g*:E(G)→N given by g*(uv)=[g(u)]³+[g(v)]³ is injective, for every uv∈ E(G).

Definition 2.5 (Cube difference labeling)

A graph G is said to admits a cube difference labeling [8] if there exist a bijection g:V(G)→{0,1,2,…,p-1} such that the induced function g*: E(G)→N given by g*(uv)=[|g(u)]³-[|g(v)]³| is injective, for every uv∈ E(G).

III. MAIN RESULTS

In this section we discuss the structure of square graph of comb and prove the existence of square sum labeling, cube sum labelling and cube difference labeling by presenting an algorithm.

Structure of square graph of comb:

The square graph of comb graph is denoted by (Pₙ⊙K₁)² and has vertex set V={vᵢuᵢ/1≤i≤n} and the edge set is E={vᵢuᵢ/1≤i≤n}∪{uᵢuᵢ₂/1≤i≤n-2}∪{uᵢuᵢ₁,uᵢuᵢ₃}∪{vᵢuᵢ₁}/1≤i≤n-1. This set has 2n vertices and 5(n-1) edges.
Algorithm 3.1:
Procedure: Vertex labeling of square graph of comb.
Input: \((P_n \odot K_1)^2\) graph
\[ V \leftarrow \{u_i \} \cup \{v_i\}, 1 \leq i \leq n. \]
for \(i = 1\) to \(n\) do
\{ \(u_i \leftarrow 2i - 2; v_i \leftarrow 2i - 1;\) \}
end for
end procedure
Output: The vertex labeled square graph of comb.

Theorem 3.1: The square graph of comb admits square sum labeling.

Proof: By the structure of square graph of comb, it is clear that \((P_n \odot K_1)^3\) has 2n vertices and 5(n-1) edges. Label the vertices of \((P_n \odot K_1)^3\) by defining a function \(g: V \rightarrow \{0, 1, ..., 2n-1\}\), as given in algorithm 3.1
To get the edge labels, define the induced function \(g^*: E \rightarrow \mathbb{N}\) such that \(g^*(uv) = |g(u)^2 + g(v)^2|\) which is injective.
Thus the edge labels are as follows.

(i) For \(1 \leq i \leq n-1,\)
\[ g^*(u_i u_{i+1}) = (2i-2)^2 + (2(i+1)-2)^2 \]
\[ = 4(2i^2 - 2i + 1) \]
\[ g^*(u_i v_{i+1}) = (2i-2)^2 + (2(i+1)-1)^2 \]
\[ = 8i^2 - 4i + 5 \]
\[ g^*(v_i u_{i+1}) = (2i-1)^2 + (2(i+1)-2)^2 \]
\[ = 8i^2 - 4i + 1 \]

(ii) For \(1 \leq i \leq n,\)
\[ g^*(u_i v_i) = (2i-2)^2 + (2i-1)^2 \]
\[ = 8i^2 - 12i + 5 \]

(iii) For \(1 \leq i \leq n-2,\)
\[ g^*(u_i v_{i+2}) = (2i-2)^2 + (2(i+2)-2)^2 \]
\[ = 8(i^2 + 1) \]
Thus \(g^*(E) = \{1, 4, 5, ..., 8n^2 - 12n + 5\}.\)

Clearly all the edge labels are distinct.
Hence the square graph of comb admits square sum labeling.

Example 3.1:
Square graph of comb \((P_n \odot K_1)^2\) and its square sum labeling is given in figure 3.2
Theorem 3.2: The square graph of comb admits cube sum labeling.

Proof: Let \((P_n \circ K_1)^2\) be a square graph of comb with 2n vertices and 5(n-1) edges. Define a function \(g: V \rightarrow \{0,1,...,2n-1\}\) to label the vertices of \((P_n \circ K_1)^2\) using algorithm 3.1. To obtain the edge labels, define the induced function \(g^*: E \rightarrow \mathbb{N}\) such that, \(g^*(uv) = |g(u)^3 + g(v)^3|\) which is injective. Thus the edge labels are as follows,

(i) For \(1 \leq i \leq n-1\),
\[
g^*(u_{i+1}, u_i) = (2i-2)^3 + (2(i+1)-2)^3
\]
\[
= 8(2i^3 - 3i^2 + 3i - 1)
\]
\[
g^*(u_{i+1}, v_i) = (2i-2)^3 + (2(i+1)-1)^3
\]
\[
= 16i^3 + 12i^2 + 30i - 7
\]
\[
g^*(v_{i+1}, u_i) = (2i-1)^3 + (2(i+1)-2)^3
\]
\[
= 16i^3 + 12i^2 + 6i - 1
\]

(ii) For \(1 \leq i \leq n\),
\[
g^*(u_{i+1}, v_i) = (2i-2)^3 + (2i-1)^3
\]
\[
= 16i^3 - 36i^2 + 30i - 9
\]

(iii) For \(1 \leq i \leq n-2\),
\[
g^*(u_{i+1}, u_i) = (2i-2)^3 + (2(i+2)-2)^3
\]
\[
= 16(i^3 + 3i)
\]
Thus \(g^*(E) = \{1,8,9,16n^3 - 36n^2 + 30n - 9\}\).
Clearly all the edge labels are distinct.
Hence the square graph of comb admits cube sum labeling.

Example 3.3:
Square graph of comb \((P_n \circ K_1)^2\) and its cube sum labeling is given in figure 3.3

![Figure 3.3](image-url)

Theorem 3.3: The square graph of comb graph admits cube difference labeling.

Proof: Let \((P_n \circ K_1)^2\) be a square graph of comb with 2n vertices and 5(n-1) edges. Define a function \(g: V \rightarrow \{0,1,...,2n-1\}\), to label the vertices of \((P_n \circ K_1)^2\) using algorithm 3.1. To obtain the edge labels, define the induced function \(g^*: E \rightarrow \mathbb{N}\) such that, \(g^*(uv) = |g(u)^3 - g(v)^3|\) which is injective. Thus the edge labels are as follows,

(i) For \(1 \leq i \leq n-1\),
\[
g^*(u_{i+1}, u_i) = |(2i-2)^3 - (2(i+1)-2)^3|
\]
\[
= 8|3i^3 - 3i^2 + 3i - 1|
\]
\[
g^*(u_{i+1}, v_i) = |(2i-2)^3 - (2(i+1)-1)^3|
\]
\[
= 9|4i^3 - 2i^2 + 1|
\]
\[
g^*(v_{i+1}, u_i) = |(2i-1)^3 - (2(i+1)-2)^3|
\]
\[
= |12i^3 - 6i + 1|
\]

(ii) For \(1 \leq i \leq n\),
\[
g^*(u_{i+1}, v_i) = |(2i-2)^3 - (2i-1)^3|
\]
\[
= |12i^2 - 18i + 7|
\]

(iii) For \(1 \leq i \leq n-2\),
\[
g^*(u_{i+1}, u_i) = |(2i-2)^3 - (2(i+2)-2)^3|
\]
\[
= 16|3i^3 + 1|
\]
Thus \( g^*(E) = \{1,7,8,\ldots,9|4n^2-10n+7\} \).
Clearly all the edge labels are distinct.
Hence the square graph of comb admits cube difference labeling.

**Example 3.4:**
Square graph of comb \((P_2 \odot K_1)^2\) and its cube difference labeling is given in figure 3.4

![Figure 3.4](image)

**IV. CONCLUSION**

In this paper, we have proved the existence of square sum labeling, cube sum labelling and cube difference labelling for square graph of comb by presenting an algorithm and examples.

**REFERENCES**