Total Domination Number in Graphs and Graph Modification

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Abstract *— The major purpose of this paper is to study the effect that various graph modifications have on the total domination number of a graph. In addition, we study the effect that the operation of edge lifting has on the domination number of a graph. We finish this work with a study of the total domination number of diameter two graphs and the relationship between the total domination number of a graph and its total destruction number.*

Keywords — *Dominating Graph, Stable Graphs, Edge Critical Graphs, Domination Number.*

I. INTRODUCTION

Graph theory in mathematics means the study of graphs. The basic idea of graphs was first introduced in the 18th century by Swiss mathematician Leonard Euler. A graph in this context is made up of vertices (also called nodes or points) which are connected by edges (also called links or lines). Graphs are one of the prime objects of study in discrete mathematics. In general, a graph is represented as a set of vertices connected by edges.

Graph theory is ultimately the study of relationships. Given a set of nodes and connections, which can abstract anything from city layouts to computer data, graph theory provides a helpful tool to quantity and simplicity the many moving parts of dynamic systems.

The dominating graph $D(G)$ of a graph G is the graph with the vertex set $V \cup S$ where s is the set of all minimal dominating sets of G in which two vertices u and v are adjacent if $u \in V$ and V is a minimal dominating set in G containing u.

II. TOTAL DOMINATING GRAPHS

The definition of dominating graph of a graph inspired us to define the following graph valued function in domination theory.

A. Definitions

Let G = (V, E) be a graph. Let S be the set of all minimal total dominating sets of G. The total dominating graph Dt(G) of G is the graph with the vertex set V ∪ S in which two vertices u and v are adjacent if u ∈ V and v is a minimal total dominating set of G containing u.

1) Definition

A graph G consists of a finite non-empty set $v = v(G)$ of points together with a prescribed set $E = E(G)$ of unordered pairs of distinct elements of v. v is called the **vertex set** and E is known as the **edge set** of G.The elements of v are called **vertices** and the elements of E are called **edges**.

2) Definition

A graph without loops (an edge with identical ends) and multiple edges (more than one edge joins the same pair of vertices) is called a **simple graph**.

3) Definition

A **caterpillar** is a tree with the property that the removal of its and points leaves a path.

4) Definition

In graph theory a **critical graph** is a graph G in which every vertex or edge is a critical element that is, if its deletion decreases the chromatic number of G, such a decrease Cannot be by more than 1.

5) Definition

An edge of a graph is said to be pendant if one of its vertices is a **pendant vertex**.

6) Definition

A vertex of a graph is said to be **pendant** if its neighbourhood contains exactly one vertex.

7) Definition

Double star is the graph obtained by joining the center of two stars $k_{1,n}$ and $k_{1,m}$ with an edge.

8) Definition

In a **connected graph**, it's possible to get from every vertex in the graph to every other vertex in the graph through a series of edges called a path.

9) Definition

An **isolated vertex** is a vertex with degree zero, ie) a vertex is not an endpoint of any edge.

10) Definition

A set S of vertices in a graph $G(V, E)$ is called a **dominating set** if every vertex $v \in V$ is either an element of S or is adjacent to an element of S. A set S of vertices in a graph $G(V, E)$ is called a **total dominating set** if every vertex $v \in V$is adjacent to an element of S.

B. Dominations and Total Dominations

A set D ⊆ V is called a dominating set if every vertex in V − D is adjacent to some vertex of D. Notice that D is a dominating set if and only if N[D] = V . The domination number of G, denoted γ = γ(G), is the cardinality of a smallest dominating set of V . We call a smallest dominating set a γ-set. A set S ⊆ V is a total dominating set if every vertex in V is adjacent to some vertex of S. Alternatively, we may define a dominating set D to be a total dominating set if G[D] has no isolated vertices. The total domination number of G, denoted γt = γt(G), is the cardinality of a smallest total dominating set, and we refer to such a set as a γt-set. An immediate consequence of the definitions of domination number and total domination number is that, for any graph G, γt ≥ γ.

C. Characterizations of Total Domination Number

Total domination number is easily computed for several families of graphs. First, we make the following observation. Observation 1.7.1. Let G be a graph of order n with no isolated vertices. If Δ ≥ n − 2, then γt = 2. Graphs for which Observation in applies include stars and complete graphs. Other graphs with γt = 2 include binary stars and complete bipartite graphs. In general, however, the problem of calculating the total domination number in bipartite graphs remains difficult [20]. Total domination number is easily calculated for cycles and paths.

III. CRITICAL AND STABLE GRAPHS UPON EDGE REMOVAL

In our investigation of graphs that are stable or critical after the deletion of an edge, we say that $\gamma_t(G) = \infty$ if the graph *G*, has an isolated vertex. For instance if we delete a pendant edge $e \in E(G)$, then $\gamma_t(G - e)$.We say that a graph G is total domination edge stable or γ_t-stable for short, if the removal of any edge of G does not change the total domination number, that is, $\gamma_t(G - e) = \gamma_t(G)$ for every edge $e \in E(G)$. If $\gamma_t(G) = K$ and G is γ_t -stable, we say that G is k_t −stable. We say that G is total domination edge critical or γ_t –critical for short, if the removal of any edge in the graph changes the total domination number, that is, $\gamma_t(G - e) \neq \gamma_t(G)$ for every edge $e \in E(G)$.

We note that removing an edge from a graph cannot decrease the total domination number. Hence if G is γ_t-critical, then $\gamma_t(G - e) > \gamma_t(G)$ for every edge $e \in E(G)$. An edge e∈E(G) is a stable edge of G if $\gamma_t(G - e) = \gamma_t(G)$, while e is a critical edge of G is $\gamma_t(G - e) > \gamma_t(G)$. Thus every edge in a γ_t − stable graph is a stable graph is a stable edge, while every edge in a γ_t − critical graph is a critical edge.A vertex v in *G* is called a γ_t*-good vertex* if v is in some γ_t- (G)-set, and we define $T_t(G)$ to be the set of all γ_t- good vertices of *G*.

A. Total Domination Edge Critical Graphs

1) Definition:

A tree $T \in \mathcal{T}$ if T is a non trivial star, or a double star, or if T can be obtained from a subdivided star $K_{1,k}^*$ where $k \geq 2$, by adding zero or more pendant edges to the non-leaf vertices of $K_{1,k}^*$.

Lemma

If G is a γ_t –critical graph, then for every $\gamma_t(G)$-set $S, G[S]$ is a galaxy of nontrivial stars.

Proof

Let S be any $\gamma_t(G)$-set in the γ_t – critical graph G, and let $Gs = G[S]$.
Let e be an arbitrary edge in Gs.If both ends of e have degree at least 2 in Gs, then S is a TDS in G–e,

and so, $\gamma_t(G - e) \leq |s| = \gamma_t(G)$

Which is a contradicting the fact that G is $\gamma_t -$ critical.
Hence at least one end of the edge e is a leaf in Gs,
$\therefore$ Gs is a galaxy of nontrivial stars.

B. Total domination Edge Stable Graphs

If G is a graph with an isolated vertex, then $\gamma_t(G) = \infty$.Thus if e is an edge of a graph G incident with a leaf, then $\gamma_t(G - e) = \infty$.

Proposition:

A graph G is γ_t -stable if and only if $\delta(G) \geq 2$ and for each $e = uv \in E(G)$, there exists $\gamma_t(G) -$ set S such that one of the following conditions are satisfied.

(a) $u, v \notin S$.

(b) $u, v \in S, |N(u) \cap S| \geq 2$ and $|N(v) \cap S| \geq 2$.

(c) Without loss of generality, if $u \in S$ and $v \in S$, then $|N(v) \cap S| \geq 2$.

Proof:

Assume that G is γ_t –stable.
We know that, If G is γ_t -stable then $\delta(G) \geq 2$.
Let $e = uv$ be an arbitrary edge of G. Let $G' = G - uv$ and let S be any $\gamma_t(G') -$set. We know that, the set S is a $\gamma_t(G) -$set. If $u, v \in s$, then the set $S\gamma_t(G) -$set.
Therefore u,v∈ S
(a) holds.
Now we assume that, $u \in s$, If $v \in S$, then since S is a TDS for G',

$|NG'(u) \cap s| \geq 1$ and $|NG'(v) \cap s| \geq 1$

$|NG(u) \cap s| \geq 2 \; |NG(v) \cap s| \geq 2$

(b) holds.
If $v \notin S$, then since S is a TDS for G',

$|NG'(v) \cap s| \geq 1$ and $|NG(v) \cap s| \geq 2$

(c) holds.
Now we assume that $\delta(G) \geq 2$ and for each $e = uv \in E(G)$, there exists a $\gamma_t(G) -$set s.
In all the three conditions (a), (b),(c) are the set s is also a TDS for $G - e$.

Hence $\gamma_t(G) \leq \gamma_t(G - e) \leq |s| = \gamma_t(G)$.Consequently $\gamma_t(G) = \gamma_t(G)$.

$\therefore$ The graph G is $\gamma_t -$stable.

C. Disjoint Minimum Total Dominating Sets

1) Corollary:

If a bipartite graph G has two disjoint $\gamma_t(G)$-sets, then G is γ_t –stable.

Proof:

We have let G be a bipartite graph, then G is γ_t –stable if and only if for every vertex v in G, $|N(v) \cap T_t(G)| \geq 2$.

This shows that a bipartite graph G has two disjoint $\gamma_t(G)$-sets, then G is γ_t –stable is not true for general graphs.
However if G is a graph with three pairwise disjoint $\gamma_t(G)$-sets, and if $e \in E(G)$.
Then atleast one of these three $\gamma_t(G)$-sets does not contain an end of e and is therefore also a TDS in $G - e$.Consequently, $\gamma_t(G - e) = \gamma_t(G)$ for every $e \in E(G)$,

∴ G is a γ_t –stable graph.

Hence a bipartite graph G has two disjoint $\gamma_t(G)$-sets, then G is γ_t –stable.

IV. TOTAL DOMINATION STABLE GRAPHS UPON EDGE ADDITION

In such graphs the total domination number remains unchanged upon the addition of any edge. We say that a graph *G* is total domination edge addition stable, or γ_t^+-*stable* for short, if the addition of any edge to $E(G)$ does not change the total domination number. In other words, $\gamma_t(G + e) = \gamma_t(G)$ for every edge $e \in E(G)$. We note that adding an edge to a graph cannot increase the total domination number. Hence $\gamma_t(G + e) \leq \gamma_t(G)$ for every edge $e \in E(G)$.

Since $\gamma_t(G + e) \leq \gamma_t(G)$ for every $e \in E(G)$. and for every isolate free graph G, $\gamma_t(G) \geq 2$, we note that If $\gamma_t(G) = 2$, then G is γ_t^+-stable. Accordingly, the graphs G for which $\gamma_t(G) \geq 3$. We construct γ_t^+-stable graphs having a specified total domination number and induced subgraph.

Lemma:

If G is a γ_t^+-stable graph and $\gamma_t(G) \geq 3$,then for every $\gamma_t(G)$ $-$set s and for every $v \in s$,one of the following properties hold,

(a) |pn(v,s)| ≥ 2 and |epn(v,s)| ≥ 1.

(b) If epn(v,s)=∅,then $|\mathrm{ipn}(v,s)| \geq 3$.

Proof:

Let G be a γ_t^+-stable graph with $\gamma_t(G) \geq 3$.

Suppose that S is a $\gamma_t(G)$ $-$set,and that $v \in s$.We have, If S is a minimal TDS of a connected graph G, then for each vertex $v \in s$, $|\mathrm{epn}(v,s)| \geq 1$ or $|\mathrm{ipn}(v,s)| \geq 1$.

If pn(v,s)=$\{u\}$,then the set s/$\{v\}$ is a TDS G+uw,where $w \in s, \{u, v\}$ and so G is not γ_t^+-stable.

Which is a contradiction. $|\mathrm{pn}(v,s)| \geq 2$, if $|\mathrm{epn}(v,s)| \geq 1$ Now we may assume that, $\mathrm{epn}(v,s) = \emptyset$. If $\mathrm{ipn}(v,s) =, \{x,y\}$,then the set s/$\{v\}$ is a TDS for G+xy of cardinality $\gamma_t(G) - 1$.

G is not γ_t^+-stable.Which is also a contradiction.

Hence $|\mathrm{ipn}(v,s)| \geq 3$.

Hence proved.

*A. **Total Domination Changing and Stable Graphs Upon*** vertex

1) Definition

For a graph G, we define a weak partition $V(G) = V^0(G) \cup V^+(G) \cup V^-(G)$ of its vertex set, where

$$V^0(G) = v \in \frac{V(G)}{\gamma_t(G - V)} = \gamma_t(G)$$

$$V^+(G) = v \in \frac{V(G)}{\gamma_t(G - V)} > \gamma_t(G)$$

$$V^-(G) = v \in \frac{V(G)}{\gamma_t(G - V)} < \gamma_t(G)$$

In a graph G is defined to be total domination vertex removal critical or γ_t −critical for short, If $\gamma_t(G - V) < \gamma_t(G)$ for every vertex $v \in \frac{V(G)}{S(G)}$.

2) Definition

A graph G is γ_t −changing if $\gamma_t(G - V) \neq \gamma_t(G)$ for every vertex $v \in V(G)$, while a graph G is $\gamma_t(G - V) = \gamma_t(G)$ for every vertex $v \in V(G)$.

*B. **properties of vertices***

Proposition:

Let G be a graph without isolated vertices. A vertex v is in $V^-(G)$ iff there exists some $\gamma_t(G)$-set s and a vertex $v \in s$ such that $v \notin s$ and pn(u,s)=$\{v\}$

Proof:

Let G be a graph without isolated vertices, And let $v \in V^-(G)$

Let S^* be an arbitrary $\gamma_t(G - V)$-set and let $u \in N(V)$
By previous knowledge,
$|S^*| = \gamma_t(G) - 1$ and $S^* \cap N(V) = \emptyset$
Thus $S = S^* U\{u\}$ is a $\gamma_t(G)$-set, such that $v \notin s$ and pn(u,s)=$\{v\}$.
Conversely, assume that there exists a $\gamma_t(G)$-set s and such that $v \notin s$ and pn(u,s)=$\{v\}$ for some $v \in s$.
The set $s/\{u\}$ is a TDS for G-V of cardinality $\gamma_t(G) - 1$.
Hence $v \in V^-(G)$.

*C. **γ_t- Changing Graphs***

In γ_t-changing graphs G with $V(G) = V^-(G) \cup V^+(G)$.The only connected γ_t-changing graph G with $V(G) = V^+(G)$ is G=K_2. Further the γ_t-changing graphs G with $V(G) = V^-(G)$ are precisely the γ_t-critical graphs with minimum degree at least two.

1) Definition

Let F be the family of all graphs G that can be obtained from a connected graph H, where every support vertex of H is a strong vertex, by adding a new vertex v' and an edge vv' to every vertex v in H that is not a support vertex.

2) Definition

Let H be the family of all graphs G that can be obtained from a connected graph F , by adding to every vertex v in F two disjoint copies of K_2 and adding an edge from v to one vertex in each copy of K_2.

Theorem: 5.3.4

If G is a connected γ_t-changing graph of order n with $V^+(G) = \emptyset$ and $V^-(G) = \emptyset$,then $n/2 \leq \gamma_t(G) \leq 3n/5$.
Furthermore the following holds,
$\gamma_t(G) = \frac{n}{2}$ iff G=$H^\circ K$,for some connected graph H with $\delta(G) \geq 2$.
$\gamma_t(G) = \frac{3n}{5}$iff $G \in H$.

Proof:

Let G be a graph as defined in the hypothesis.
Let G be a connected γ_t-changing graph. Then $V^+(G) = \emptyset$ iff $G \in F$.

Thus G is constructed from a connected graph H, where every support vertex of H is a strong support vertex by adding to every vertex in H that is not a support vertex a new vertex v' and an edge vv'. If H=K then $G = K_2 = K_1 \circ K_1$ and $V^+(G) = V(G)$.

Hence we may assume that $|V(H)| \geq 2$ with this assumption, the graph G satisfies the five properties:

(a) $V^-(G) = L(G)$ and $V^+(G) = V(H)$.

(b) $V(H)$ is the unique $\gamma_t(G)$-set.

(c) Every support vertex of G is adjacent to exactly one leaf.

(d) No support vertex of G is a support vertex of H.

(e) Every vertex of H that is not a support vertex of G is a strong support vertex of H.

V. CONCLUSION

In this paper I have explained the concept of fundamentals of graphs and as well as the critical and stable graphs. And also explained the concept of total domination stable graphs upon edge and vertex addition. Finally we conclude the total domination changing and stable graphs upon vertex. None of the concept is difficult, but there is an accumulation of new concepts which may sometimes seems heavy.

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