Open Support of a Graph under Multiplication

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Abstract
In this paper, the open support of a vertex \( v \) under multiplication and open support of a graph \( G \) under multiplication is defined and studied. Also, we find the value of open support of some namely graphs like Dutch windmill graph, Butterfly graph and Ladder graph. Moreover, we generalized the value of open support under multiplication for any given graph \( G \).

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I. INTRODUCTION

In this work we consider finite, undirected, simple graphs \( G = (V, E) \) with \( n \) vertices and \( m \) edges. The neighbourhood of a vertex \( v \in V(G) \) is the set \( N_G(v) \) of all the vertices adjacent to \( v \) in \( G \). For a set \( X \subseteq V(G) \), the open neighbourhood \( N_G(X) \) is defined to be \( \bigcup_{v \in X} N_G(v) \) and the closed neighbourhood \( N_G[X] = N_G(X) \cup X \). The degree of a vertex \( v \in V(G) \) is the number of edges of \( G \) incident with \( v \) and is denoted by \( d_G(v) \) or \( deg(v) \). The maximum and the minimum degrees of the vertices of \( G \) are respectively denoted by \( \Delta(G) \) and \( \delta(G) \). A vertex of a degree 0 in \( G \) is called an isolated vertex and a vertex of degree 1 is called a pendant vertex or an end vertex of \( G \). A vertex of a graph \( G \) is said to be a vertex of full degree if it is adjacent to all other vertices in \( G \). A graph \( G \) is said to be regular of degree \( r \) if every vertex of \( G \) has degree \( r \). Such graphs are called \( r \)-regular graphs.

The Dutch windmill graph \( D_{n}^{(m)} \), is the graph obtained by taking \( m \) copies of the cycle graph \( C_n \) with a vertex in common. The Butterfly graph (also called the bowtie graph and the hourglass graph) is a planar undirected graph with 5 vertices and 6 edges. It can be constructed by joining 2 copies of the cycle graph \( C_3 \) with a common vertex. It is denoted by \( B_n \). The ladder graph \( L_n \) is a planar undirected graph with 2n vertices and \( n+2(n-1) \) edges. The Ladder graph obtained as the cartesian product of two graphs one of which has only one edge: \( L_{n,1} = P_n \times P_1 \).

An open support of a vertex \( v \) under multiplication is defined by \( \prod_{u \in N_G(v)} deg(u) \) and is denoted by \( mult(v) \). An open support of a graph \( G \) under multiplication is defined by \( \prod_{u \in V(G)} mult(u) \) and it is denoted by \( mult(G) \).

II. DEFINITIONS

Definition 2.1. Let \( G = (V, E) \) be a graph. An open support of a vertex \( v \) under multiplication is defined by \( \prod_{u \in N_G(v)} deg(u) \) and is denoted by \( mult(v) \).

Definition 2.2. Let \( G = (V, E) \) be a graph. An open support of a graph \( G \) under multiplication is defined by \( \prod_{u \in V(G)} mult(u) \) and it is denoted by \( mult(G) \).

III. RESULTS

Proposition 3.1. For a Path \( P_n \) (\( n \geq 2 \)), \( mult(P_n) = 4^{n-2} \).
Proof: Let \( G \) be a path on \( n \) vertices and let \( V(G) = \{u_1, u_2, ..., u_n\} \) where \( deg(u_1) = deg(u_n) = 1 \) and \( deg(u_i) = 2 \) for \( i = 2, 3, ..., n-1 \).
If \( n = 2, 3, \) or \( 4, \) then clearly \( \mult(G) = 1, 4 \) and \( 16 \) respectively.
Let \( n \geq 5. \) Then
\[
\mult(u_1) = \mult(u_{n}) = 2,
\]
and
\[
\mult(u_2) = \mult(u_{n-1}) = 2
\]
Therefore
\[
\mult(G) = \prod_{u \in V(G)} \deg(v) = 4.
\]

**Theorem 3.2** For any \( r \)-regular connected graph \( G \) of order \( n \geq 2, \) \( \mult(G) = r^{rn}. \)

**Proof:** Let \( G \) be a \( r \)-regular connected graph on \( n \) vertices and let \( V(G) = \{u_1, u_2, \ldots, u_n\} \) where \( \deg(u_i) = r \) for all \( i. \) Let \( n \geq 2. \) Then
\[
\mult(u_i) = \prod_{v \in N(u_i)} \deg(v) = r \times r \times \cdots \times r \quad (r \text{-times}) = r^r.
\]
Thus, \( \mult(u_i) = r^r \) and hence
\[
\mult(G) = r^r \times r^r \times \cdots \times r^r = r^{rn}.
\]

**Corollary 3.3** For a Cycle \( C_n, \) \( (n \geq 3), \) \( \mult(C_n) = 4^n. \)

**Corollary 3.4** For a complete graph \( K_n, \) \( (n \geq 2), \) \( \mult(K_n) = (n - 1)^n(n-1). \)

**Corollary 3.5** For a Petersen graph \( P, \) \( \mult(P) = 3^{30}. \)

**Proposition 3.6.** For a complete bipartite graph \( K_{m,n}, \) \( (m, n \geq 1), \) \( \mult(K_{m,n}) = (mn)^{mn}. \)

**Proof:** Let \( G = K_{m,n} \) be a complete bipartite graph with bipartition \( (V_1, V_2) \) where \( V_1 = \{u_1, u_2, \ldots, u_m\} \) and \( V_2 = \{v_1, v_2, \ldots, v_n\}. \) Then \( \deg(u_i) = m \) and \( \deg(v_j) = n \) for all \( i, j. \) Thus
\[
\mult(u_i) = \prod_{v \in N(u_i)} \deg(v) = \prod_{v \in V} m = m^n \quad \text{for} \ i = 1, 2, \ldots, m.
\]
Similarly,
\[
\mult(v_j) = n^m \quad \text{for} \ j = 1, 2, \ldots, n.
\]
Therefore
\[
\mult(G) = m^m \times n^m = (mn)^{mn}.
\]

**Proposition 3.7.** Let \( G = L_{2n}, \) \( (n \geq 4), \) be a Ladder graph. Then \( \mult(G) = 2^8 \times 3^6(n-2). \)

**Proof:** Let \( G = L_{2n}, \) \( n \geq 4. \) Let \( V(G) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\} \) where \( \deg(u_1) = \deg(u_n) = \deg(v_1) = \deg(v_n) = 2 \) and \( \deg(u_i) = \deg(v_i) = 3 \) for \( i = 2, 3, \ldots, n - 1. \) Then
\[
\mult(u_i) = \mult(v_i) = \mult(u_{n-1}) = \mult(v_{n-1}) = 6,
\]
and
\[
\mult(u_2) = \mult(u_3) = \mult(v_2) = \mult(v_3) = 18
\]
and
\[
\mult(u_i) = \prod_{v \in N(u_i)} \deg(v) = 3 \times 3 \times 3 = 27.
\]
Similarly,
\[
\mult(v_i) = 27.
\]
Therefore,
\[
\mult(G) = \prod_{u \in V(G)} \mult(u)
= 6^4 \times (18)^4 \times (27)^{2n-8}
= 2^8 \times 3^6 \times 3^8 \times 3^6(n-4)
= 2^8 \times 3^6(n-2).
\]

**Proposition 3.8.** For a Fan \( F_n, \) \( (n \geq 4), \) \( \mult(F_n) = 2^4 \times 3^3(n-3) \times (n-1)^{n-1}. \)

**Proof:** Let \( G = F_n \) \( (n \geq 4). \) Let \( V(G) = \{u, u_1, u_2, \ldots, u_{n-1}\} \) where \( \deg(u) = n - 1, \deg(u_i) = \deg(u_{n-1}) = 2 \) and \( \deg(u_i) = 3 \) for \( i = 2, 3, \ldots, n - 2. \)

Then
\[ mult(u) = 2^2 \times 3^{n-3}, \]
\[ mult(u_1) = mult(u_{n-1}) = 3 \times (n - 1), \]
\[ mult(u_2) = mult(u_{n-2}) = 6 \times (n - 1) \]

and
\[ mult(u_i) = \Pi_{v \in N(u_i)} deg(v) = 9 \times (n - 1) \]
for \( i = 2, 3, \ldots, n - 3. \)

Therefore
\[ mult(G) = \Pi_{v \in V(G)} mult(v) \]
\[ = 2^2 \times 3^{n-3} \times 3^2 \times (n - 1)^3 \times 6^2 \times (n - 1)^2 \times 9^{n-5} \times (n - 1)^{n-5} \]
\[ = 2^4 \times 3^{n+1} \times (n - 1)^4 \times 3^{2n-10} \times (n - 1)^{n-5} \]
\[ = mult(G) = 2^4 \times 3^{3(n-3)} \times (n - 1)^{n-1}. \]

**Theorem 3.9.** Let \( G = L_{2n} \). Then \( mult(G) = 6^4 \times 18^4 \times 27^{2(n-4)}. \)

**Proof.** Let \( G = L_{2n} \) be a Ladder graph with \( 2n \) vertices. Let \( V(G) = \{ v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n \} \) and \( deg(v_i) = deg(v_i) = 2 \) for all \( i = 1, n; deg(u_i) = deg(u_i) = 3 \) for all \( i = 2, 3, \ldots, n - 1. \)

Then
\[ mult(v_1) = \Pi_{v \in N(v_1)} deg(v) \]
\[ = deg(u_1) \times deg(v_2) \]
\[ mult(v_1) = 6 \]

Similarly,

\[ mult(v_n) = mult(u_1) = mult(u_n) = 6. \]

\[ mult(v_2) = \Pi_{v \in N(v_2)} deg(v) \]
\[ = deg(u_2) \times deg(v_1) \times deg(v_3) \]
\[ mult(v_2) = 18 \]

Similarly,

\[ mult(u_2) = mult(u_{n-1}) = mult(v_{n-1}) = 18. \]

For each \( i = 3, 4, \ldots, n - 2, \)

\[ mult(v_i) = \Pi_{v \in N(v_i)} deg(v) \]
\[ = deg(u_i) \times deg(v_{i-1}) \times deg(v_{i+1}) \]
\[ mult(v_i) = 27, \text{ for all } i = 3, 4, \ldots, n - 2. \]

Similarly,

\[ mult(u_i) = 27, \text{ for all } i = 3, 4, \ldots, n - 2. \]

Now,

\[ mult(G) = \Pi_{v \in V(G)} mult(v) \]
\[ = \Pi_{i=1}^{n} mult(v_i) \times \Pi_{i=1}^{n} mult(u_i) \times \Pi_{i=2}^{n-1} mult(v_i) \times \Pi_{i=2}^{n-1} mult(u_i) \times \Pi_{i=3}^{n-2} mult(u_i) \]
\[ = \sum_{i=1}^{n} 6 \times \Pi_{i=1}^{n} 6 \times \Pi_{i=2}^{n-1} 18 \times \Pi_{i=2}^{n-1} 27 \times \Pi_{i=3}^{n-2} 27 \]
\[ mult(G) = 6^4 \times 18^4 \times 27^{2(n-4)}. \]

**Theorem 3.10.** Let \( G=(m,m,\ldots,m) \) be a caterpillar graph. Then

\[ mult(G) = (m + 1)^2(m + 1)(m + 2)^{m+2(m+n-2)} \]

**Proof.** Let \( G=(m,m,\ldots,m) \) be a caterpillar graph.

Let \( V(G) = \{ v_1, v_2, \ldots, v_n, u_{i1}, \ldots, u_{im}, \ldots, u_{n1}, \ldots, u_{nm} \} \) such that \( deg(v_i) = deg(v_n) = m + 1; \)
\( deg(v_i) = m + 2 \) for all \( i = 2, 3, \ldots, n - 1 \) and \( deg(u_{ij}) = 1 \) for all \( i = 1, 2, \ldots, n \) and \( j = 1, 2, \ldots, m \)

\[ mult(v_1) = \Pi_{i=1}^{n} deg(u_{ij}) \times deg(v_1) \]
\[ = \Pi_{i=1}^{n} 1(m + 2) \]
\[ = m + 2 \]

Similarly,

\[ mult(v_n) = m + 2; \]

\[ mult(v_2) = \Pi_{i=1}^{n} deg(u_{ij}) \times deg(v_1) \times deg(v_3) \]
\[ = \Pi_{i=1}^{n} 1(m + 1)(m + 2) \]
\[ = (m + 1)(m + 2) \]

Similarly,

\[ mult(v_{n-1}) = (m + 1)(m + 2); \]
For $i = 3, 4, \ldots, n - 2$
\[
mult(v_i) = \Pi_{v \in \mathcal{N}(v_i)} \deg(v) \\
= \deg(v_{i-1}) \times \Pi_{i-1}^n \deg(u_{i}) \times \deg(v_{i+1}) \\
= (m + 2)\Pi_{i=1}^n 1 \times (m + 2) \\
= (m + 2)^2
\]
For $j = 1, 2, \ldots, m$
\[
mult(u_{ij}) = \deg(v_i) \\
= m + 1
\]
Similarly,
\[
mult(u_{nj}) = m + 1;
\]
For $i = 2, 3, \ldots, n - 1$ and $j = 1, 2, \ldots, m$
\[
supp(u_{ij}) = \deg(v_i) \\
= m + 2
\]
Now,
\[
mult(G) = \Pi_{v \in \mathcal{V}(G)} \mult(v) \\
= \Pi_{v \in \mathcal{V}(G)}^{m} \mult(v) \\
= \mult(v_1) \times \mult(v_2) \times \mult(v_3) \times \mult(v_{n-1}) \times \mult(v_n) \\
\times \Pi_{v \in \mathcal{V}(G)}^{m} \mult(u_{ij}) \times \mult(u_{nj}) \times \Pi_{v \in \mathcal{V}(G)}^{m} \mult(u_{ij}) \\
= (m + 2)^2 (m + 1)^2 (m + 2)^2 \Pi_{i=1}^n (m + 2) \\
= (m + 1)^3 (m + 2)^4 (m + 1)^2 (m + 2)^m (m + 1)^2 (m + 2)^m (m + 1)(m + 2)^m + 2(m + n - 2).
\]

**Theorem 3.11** Let $G$ be a butterfly graph. Then $\mult(G) = 2^{16}$

**Proof.** Let $G$ be a butterfly graph. Let $\mathcal{V}(G) = \{x, v_1, v_2, v_3, v_4\}$ such that $\deg(x) = 4; \deg(v_i) = 2$, for all $i = 1, 2, 3, 4$;
\[
mult(x) = \Pi_{v \in \mathcal{V}(x)} \deg(v) \\
= \deg(v_1) \times \deg(v_2) \times \deg(v_3) \times \deg(v_4) \\
= 16
\]
For $i = 1, 2, 3, 4$
\[
mult(v_i) = \Pi_{v \in \mathcal{N}(v_i)} \deg(v) \\
= 2 \times 4 \\
= 8
\]
Now,
\[
mult(G) = \Pi_{v \in \mathcal{V}(G)} \mult(v) \\
= \mult(v_1) \times \mult(v_2) \times \mult(v_3) \times \mult(v_4) \times \mult(x) \\
= 2^{16}
\]

**Theorem 3.12.** Let $G = D_{(n)}^{(m)}$ be a Dutch windmill graph. Then $\mult(G)=[m2^{(2n-1)}]^{m}$

**Proof.** Let $G = D_{(n)}^{(m)}$ be a Dutch windmill graph. Let $\mathcal{V}(G) = \{x, v_1^i, v_2^i, v_3^i, \ldots, v_{n-1}^i\}$ for $i = 1, 2, \ldots, m$ such that $\deg(x) = 2m; \deg(v_i^j) = 2$, for all $i, j = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n - 1$;
\[
mult(x) = \Pi_{v \in \mathcal{V}(x)} \deg(v) \\
= \Pi_{i=1}^m(\deg(v_1^i) \times \deg(v_1^{i-1})) \\
= \Pi_{i=1}^m(2) \\
= 4^m
\]
For $i = 1, 2, \ldots, m$,
\[
mult(v_i^j) = \Pi_{v \in \mathcal{N}(v_i^j)} \deg(v) \\
= 2m \times 2
\]
Similarly,
\[
mult(v_i^{j-1}) = 4m
\]
For $j = 2, 3, \ldots, n - 2$,
\[
mult(v_j^i) = \Pi_{v \in \mathcal{N}(v_j^i)} \deg(v) \\
= 2 \times 2 = 4
\]
Now,
\[
mult (G) = \Pi_{v \in V(G)} \mult (v).
\]
\[
= \mult(x) \times \Pi_{i=1}^{m} \Pi_{j=2}^{m} \mult(v_{j}) \times \mult(v_{1}) \times \mult(v_{n-1})
\]
\[
= 4^m \times \Pi_{i=1}^{m} \Pi_{j=2}^{m} 4 \times 8m
\]
\[
= 4^m \times \Pi_{i=1}^{m} (4^{n-3} \times 8m)
\]
\[
= 4^m \times [4^{n-3} \times 8m]^m
\]
\[
= [8m \times 4^{n-2}]^m
\]
\[
= [m \times 2^{3} \times 2^{2n-4}]^m
\]
\[
= [m \times 2^{2n-1}]^m
\]

**Theorem 3.13.** Let \( G = K_m(a_1,a_2,\ldots,a_m) \). Then \( \mult(G) = n^2 \).

**Proof.** Let \( G = K_m(a_1,a_2,\ldots,a_m) \) be a multistar graph of order \( m + a_1 + a_2 + \cdots + a_m \). Let \( V(G) = \{v_1,v_2,\ldots,v_n,u_1,u_2,\ldots,u_n\} \) such that \( \deg(v_i) = n \) and \( \deg(u_i) = 1 \) for all \( i = 1,2,\ldots,n \).

\[
\mult(u_i) = \Pi_{v \in N(u_i)} \deg(v)
\]
\[
\mult(u_i) = n
\]
\[
\mult(v_i) = \Pi_{v \in N(v_i)} \deg(v)
\]
\[
\mult(v_i) = n \times n \times \cdots \times n \ (n-1 \text{times})
\]
\[
\mult(v_i) = n^{n-1}
\]

Now,
\[
\mult(G) = \Pi_{v \in V(G)} \mult(v)
\]
\[
= \Pi_{i=1}^{n} \mult(v_i) \times \Pi_{i=1}^{n} \mult(u_i)
\]
\[
= \Pi_{i=1}^{n} n^{n-1} \times \Pi_{i=1}^{n} 1
\]
\[
= n^{n(n-1)} \times n^n
\]
\[
\mult(G) = n^{n^2}.
\]

**REFERENCES**


