Some Ramanujan Integrals Associated with Generalized Riemann Zeta Function and Multivariable Gimel-Function

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ABSTRACT
In this paper, some Ramanujan integrals associated with generalized Riemann Zeta function and multivariable Gimel-function are evaluated. Importance of the results established in this paper lies in the fact that they involve gimel-function, which is sufficiently general in nature and capable of yielding a large number of results merely by specializing the parameters therein.

KEYWORDS: Multivariable Gimel-function, multiple integral contours, Ramanujan integrals, generalized Riemann Zeta function.

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1. Introduction and preliminaries.

Throughout this paper, let \( \mathbb{C}, \mathbb{R} \) and \( \mathbb{N} \) be set of complex numbers, real numbers and positive integers respectively. Also \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). We define a generalized transcendental function of several complex variables.

\[
\mathcal{I}(z_1, \ldots, z_r) = \int_{\gamma_1} \cdots \int_{\gamma_r} \psi(s_1, \ldots, s_r) \prod_{k=1}^{r} \theta_k(s_k) z_k^{s_k} ds_1 \cdots ds_r
\]

with \( \omega = \sqrt{-1} \)
\[
\psi(s_1, \ldots, s_r) = \frac{\prod_{j=1}^{R^2} \Gamma^{A_{2j}}(1 - a_{2j} + \sum_{k=1}^{2} \alpha_{2jk}^{(k)} s_k)}{\sum_{r=1}^{R^2} \prod_{j=1}^{R^2} \Gamma^{A_{2j}}(a_{2j} - \sum_{k=1}^{2} \alpha_{2jk}^{(k)} s_k) \prod_{j=1}^{R^2} \Gamma^{B_{2j}}(1 - b_{2j} + \sum_{k=1}^{2} \beta_{2jk}^{(k)} s_k)}
\]

\[
\prod_{j=1}^{R^2} \Gamma^{A_{2j}}(1 - a_{2j} + \sum_{k=1}^{2} \alpha_{2jk}^{(k)} s_k)
\]

\[
\sum_{r=1}^{R^2} \prod_{j=1}^{R^2} \Gamma^{A_{2j}}(a_{2j} - \sum_{k=1}^{2} \alpha_{2jk}^{(k)} s_k) \prod_{j=1}^{R^2} \Gamma^{B_{2j}}(1 - b_{2j} + \sum_{k=1}^{2} \beta_{2jk}^{(k)} s_k)
\]

and

\[
\theta_k(s_k) = \frac{\prod_{j=1}^{R^2} \Gamma^{D_j^{(k)}}(d_j^{(k)} - \delta_j^{(k)} s_k) \prod_{j=1}^{R^2} \Gamma^{C_j^{(k)}}(1 - c_j^{(k)} + \gamma_j^{(k)} s_k)}{\sum_{r=1}^{R^2} \prod_{j=1}^{R^2} \Gamma^{D_j^{(k)}}(d_j^{(k)} - \delta_j^{(k)} s_k) \prod_{j=1}^{R^2} \Gamma^{C_j^{(k)}}(1 - c_j^{(k)} + \gamma_j^{(k)} s_k)}
\]

1) \[(c_j^{(k)}; \gamma_j^{(k)})_{1,n_1}, \ldots, (c_j^{(k)}; \gamma_j^{(k)})_{1,n_1}.

2) n_2, \ldots, n_r, m^{(1)}, n^{(1)}, \ldots, m^{(r)}, n^{(r)}, p_{i_1}, q_{i_1}, R_2, \tau_2, \ldots, p_{i_r}, q_{i_r}, R_r, \tau_r, p_{i_1}, q_{i_1}, R_1, \tau_1, p_{i_r}, q_{i_r}, (1, \ldots, r) \in N and verify :

\[
0 \leq m_2, \ldots, 0 \leq m_r, 0 \leq n_2 \leq p_{i_2}, \ldots, 0 \leq n_r \leq p_{i_r}, 0 \leq m^{(1)} \leq q_{i_1}, \ldots, 0 \leq m^{(r)} \leq q_{i_r},
\]

\[
0 \leq n^{(1)} \leq p_{i_1}, \ldots, 0 \leq n^{(r)} \leq p_{i_1}.
\]

3) \(\tau_2 = (i_2 = 1, \ldots, R_2) \in \mathbb{R}^1; \tau_r \in \mathbb{R}^1 (i_r = 1, \ldots, R_r); \tau^{(k)}_{i_1} \in \mathbb{R}^1 (i_1 = 1, \ldots, R^{(k)}_1), (k = 1, \ldots, r).

4) \(c_j^{(k)}; \gamma_j^{(k)} \in \mathbb{R}^1; (j = 1, \ldots, n^{(k)}); (k = 1, \ldots, r); \delta_j^{(k)}; D_j^{(k)} \in \mathbb{R}^1; (j = 1, \ldots, m^{(k)}); (k = 1, \ldots, r);

C_{j^{(k)}} \in \mathbb{R}^+, (j = m^{(k)} + 1, \ldots, p^{(k)}); (k = 1, \ldots, r);

D_{j^{(k)}} \in \mathbb{R}^-, (j = n^{(k)} + 1, \ldots, q^{(k)}); (k = 1, \ldots, r);

\alpha_j^{(k)} \in \mathbb{R}^+; (j = 1, \ldots, n_k); (k = 2, \ldots, r); (l = 1, \ldots, k);

\delta_{j^{(k)}}^{(l)} \in \mathbb{R}^+; (j = m_k + 1, \ldots, q_k); (k = 2, \ldots, r); (l = 1, \ldots, k);

\delta_{j^{(k)}}^{(l)} \in \mathbb{R}^+; (i = 1, \ldots, R^{(k)}); (j = m^{(k)} + 1, \ldots, q^{(k)}); (k = 1, \ldots, r);

\gamma_j^{(k)} \in \mathbb{R}^+; (i = 1, \ldots, R^{(k)}); (j = n^{(k)} + 1, \ldots, p^{(k)}); (k = 1, \ldots, r).

5) \(c_j^{(k)} \in \mathbb{C}; (j = 1, \ldots, n^{(k)}); (k = 1, \ldots, r); d_j^{(k)} \in \mathbb{C}; (j = 1, \ldots, m^{(k)}); (k = 1, \ldots, r).
\( \alpha_{k, j, k} \in \mathbb{C}; (j = n_k + 1, \ldots, p_{i,k}); \) \((k = 2, \ldots, r)\).

\( b_{k, j, k} \in \mathbb{C}; (j = 1, \ldots, q_{i,k}); \) \((k = 2, \ldots, r)\).

\( d_{j, k}^{(k)} \in \mathbb{C}; (i = 1, \ldots, R^{(k)}); (j = m^{(k)} + 1, \ldots, q_{i,k}); \) \((k = 1, \ldots, r)\).

\( \gamma_{j, k}^{(k)} \in \mathbb{C}; (i = 1, \ldots, R^{(k)}); (j = n^{(k)} + 1, \ldots, p_{i,k}); \) \((k = 1, \ldots, r)\).

The contour \( L_k \) is in the \( s_k(k = 1, \ldots, r) \)-plane and run from \( \sigma - i \infty \) to \( \sigma + i \infty \) where \( \sigma \) is a real number with loop, if necessary to ensure that the poles of \( \Gamma^{(n)} \left( 1 - a_{j} + \sum_{k=1}^{n} \alpha_{j,k}^{(k)} s_k \right) \) \((j = 1, \ldots, n_2)\), \( \Gamma^{(n)} \left( 1 - c_{j}^{(k)} + \gamma_{j}^{(k)} s_k \right) \) \((j = 1, \ldots, n^{(k)})\) \((k = 1, \ldots, r)\) lie to the right of the contour \( L_k \) and the poles of \( \Gamma^{(n)} \left( d_{j}^{(k)} - \delta_{j}^{(k)} s_k \right) \) \((j = 1, \ldots, n^{(k)})\) \((k = 1, \ldots, r)\) lie to the left of the contour \( L_k \). The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as:

\[ |\arg(s_k)| < \frac{1}{2} A^{(k)}_i \pi \]

\[ A^{(k)}_i = \sum_{j=1}^{m^{(k)}} D^{(k)}_j d^{(k)}_j = \frac{n^{(k)}}{1} \sum_{j=1}^{n^{(k)}} C^{(k)}_j \gamma^{(k)}_j - \tau^{(k)}_i \left( i = n^{(k)} + 1 \right) + \sum_{j=1}^{n^{(k)} + 1} C^{(k)}_j \gamma^{(k)}_j \left( i = n^{(k)} + 1 \right) \]

\[ \tau_i \left( \sum_{j=1}^{p_{i,r}} A_{j; i, k} \alpha^{(k)}_{j, k} + \sum_{j=1}^{q_{i,r}} B_{j; i, k} \beta^{(k)}_{j, k} \right) - \cdots - \tau_r \left( \sum_{j=1}^{p_r} A_{r; j, k} \alpha^{(k)}_{r, k} + \sum_{j=1}^{q_r} B_{r; j, k} \beta^{(k)}_{r, k} \right) \]

Following the lines of Braaksma ([2] p. 278), we may establish the asymptotic expansion in the following convenient form:

\[ N(z_1, \ldots, z_r) = 0 \left( |z_1|^{(1)}, \ldots, |z_r|^{(r)} \right), \max( |z_1|, \ldots, |z_r| ) \rightarrow 0 \]

\[ N(z_1, \ldots, z_r) = 0 \left( |z_1|^{(1)}, \ldots, |z_r|^{(r)} \right), \min( |z_1|, \ldots, |z_r| ) \rightarrow \infty \] where \( i = 1, \ldots, r \):

\[ \alpha_i = \min_{1 \leq j \leq m^{(i)}} \Re \left( D_j^{(i)} \right), \beta_i = \max_{1 \leq j \leq n^{(i)}} \Re \left( C_j^{(i)} \right) \]

**Remark 1.**
If \( n_2 = \cdots = n_r = p_{i,r} = q_{i,r} = \cdots = p_{i,r-1} = q_{i,r-1} = 0 \) and \( A_{2j; i, k} = B_{2j; i, k} = \cdots = A_{r; j, k} = B_{r; j, k} = 1 \) \( A_{r; j, k} = B_{r; j, k} = 1 \), then the multivariable Gimel-function reduces in the multivariable Aleph-function defined by Ayant [2].

**Remark 2.**
If \( n_2 = \cdots = n_r = p_{i,r} = q_{i,r} = \cdots = p_{i,r-1} = q_{i,r-1} = 0 \) and \( \tau_2 = \cdots = \tau_r = \tau^{(i)}_i = \cdots = \tau^{(r)}_i = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1 \), then the multivariable Gimel-function reduces in a multivariable I-function defined by Prathima et al. [8].

**Remark 3.**
If \( A_{2j; i, k} = B_{2j; i, k} = \cdots = A_{r; j, k} = B_{r; j, k} = 1 \) and \( \tau_2 = \cdots = \tau_r = \tau^{(i)}_i = \cdots = \tau^{(r)}_i = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1 \), then the generalized multivariable Gimel-function reduces in a multivariable I-function defined by Prasad [7].

**Remark 4.**
If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the multivariable H-function defined by Srivastava and Panda [9,10].
In your investigation, we shall use the following notations.

\[ A = [(a_{rj}; \alpha_r^{(r)}, \cdots, \alpha_r^{(r)}; A_{rj})_1, n_r,] \]

\[ \tau_r = (a_{rj}; \alpha_r^{(r)}, \cdots, \alpha_r^{(r)}; A_{rj})_1, n_r,] \]

\[ \beta_r^{(r)} = (\beta_r^{(r)}; B_{rj})_1, n_r,] \]

\[ B = [(d_{rj}^{(r)}; D_{rj}^{(r)})_1, m_{rj},] \]

\[ U = [0, n_2; 0, n_3; \cdots; 0, n_r,] \]

\[ X = p_{r}; q_{r}; \tau_{r}; R_{r}; \cdots; p_{r-1}; q_{r-1}; \tau_{r-1} = R_{r-1}; Y = p_{r}; q_{r}; \tau_{r}; R_{r}; \cdots; p_{r}; q_{r}; \tau_{r}; R_{r} \]

2. Required results.

We have the following integrals Garg et al. [4]

Lemma 1.

\[ \int_0^\infty x^{p-1} \left[ \frac{2}{1 + \sqrt{1 + 4x}} \right]^n \, dx = \frac{n\Gamma(p)\Gamma(n - 2p)}{\Gamma(n - p + 1)} \]

provided \( n > 0, 0 < p < \frac{n}{2} \)

Lemma 2.

\[ \int_0^\infty x^{p-1} \left[ \frac{1}{1 + \sqrt{1 + x^2}} \right]^n \, dx = \frac{n\Gamma(p)\Gamma\left(\frac{n-p}{2}\right)}{2^{p+1}\Gamma\left(\frac{n+p+2}{2}\right)} \]

provided \( 0 < p < n \)
Goyal and laddha ([5], p. 99-108, Vol. 11(2)) introduced an extension to the generalized Riemann Zeta function defined in the following slightly modified form.

**Definition.**

\[
\phi_h(z, s, a, g) = \sum_{g=0}^{\infty} \left( h \right)_g \frac{(a + g)^{-s} z^g}{g!} \quad (2.3)
\]

provided that \( h \geq 1, |z| < 1, Re(a) > 0 \)

If \( h = 1 \), the above function reduces and we have

\[
\phi(z, s, a) = \sum_{s=0}^{\infty} (a + g)^{-s} z^g, |z| < 1, Re(a) > 0 \quad (2.4)
\]

If \( h = s = 1 \), we have the following result, see [3]

\[
\phi(1, 1, a, g) = \frac{1}{a} \cF_{11}(1, a; 1 + a; z) \quad (2.5)
\]

3. Main integrals.

In this section, we evaluate two general integrals.

**Theorem 1.**

\[
\int_0^\infty \frac{z^n}{x^{p+1}} \left[ 1 + \frac{1}{\sqrt{1 + 4z}} \right] \phi_h(z, s, a, g) \mathcal{D} \left( z_1 x^{\sigma_1} \{ x + \sqrt{1 + 4z} \}^{-\zeta_1}, \cdots, z_r x^{\sigma_r} \{ x + \sqrt{1 + 4z} \}^{-\zeta_r} \right) \, dx =
\]

\[
\sum_{s=0}^{\infty} \left( h \right)_g \frac{(a + g)^{-s + g}}{g!} \sum_{k_0, k_1, \cdots, k_r \geq 0} \frac{1}{k_0! k_1! \cdots k_r!} A_{k_0} A_{k_1} \cdots A_{k_r} \mathbf{B}_{1-n-\zeta g, \zeta_1, \cdots, \zeta_r, 1}
\]

\[
(1-n-\zeta g + 2p + 2\sigma g; \zeta_1 - 2\sigma_1, \cdots, \zeta_r - 2\sigma_r, 1), (-n - \zeta g; \zeta_1, \cdots, \zeta_r, 1), A : A_1 \cdots (3.1)
\]

\[
\zeta, \sigma_i > 0, \zeta_i - 2\sigma_i > 0 (i = 1, \cdots, r), Re (n + \zeta g) + \min_{1 \leq j \leq m^{(t)}} Re \left[ D_j^{(i)} \left( \frac{d_j^{(i)}}{\delta_j^{(i)}} \right) \right] > 0
\]

Provided

\[
Re (n - 2p + (\zeta - 2\sigma) g) + \sum_{i=1}^{r} (\zeta_i - 2\sigma_i) \min_{1 \leq j \leq m^{(t)}} Re \left[ D_j^{(i)} \left( \frac{d_j^{(i)}}{\delta_j^{(i)}} \right) \right] > 0,
\]

\[
|\arg(z, x^{\sigma_i} y^{\zeta_i} \{ x + \sqrt{1 + 4x} \}^{-\zeta_i})| < \frac{1}{2} A_i^{(k)} \pi \text{ where } A_i^{(k)} \text{ is defined by (1.4)}.
\]

**Proof.**

To establish the theorem 1, expressing the multivariable Gimel-function in the Mellin-Barnes multiple integrals contour with the help of (1.1) and expressing the generalized Riemann Zeta function in series, interchanging the order of
integration ans summations which is justified under the conditions mentioned above, Now interchanging the order of integrations which is justified under the conditions mentioned above, evaluating the inner $x$-integral with the help of the lemma 1 and interpreting the resulting multiple integrals contour with the help of (1.1) about the gimel-function of r-variables, we obtain the desired theorem 2.

**Theorem 2.**

\[
\int_0^\infty x^{p-1} \left[ \frac{1}{1 + \sqrt{1 + x^2}} \right]^n \phi_h(z, s, a, g) \mathcal{J} \left( z_1 x^{\sigma_1} \left\{ x + \sqrt{1 + x^2} \right\}^{-\zeta_1}, \cdots, z_r x^{\sigma_r} \left\{ x + \sqrt{1 + x^2} \right\}^{-\zeta_r} \right) \, dx =
\]

\[
\sum_{s=0}^\infty \left( \frac{a + g}{2^{p+s+1} g!} \right) \mathcal{J} X, p_r + \frac{1}{2} q_r + 2, \tau_r, R_r, Y \left( z_1 \mathcal{A}_1, (1 - p - \sigma g; \sigma_1, \cdots, \sigma_r; 1), \cdots, z_r \mathcal{A}_r, (1 - n - \zeta g; \zeta_1, \cdots, \zeta_r; 1) \right) \mathcal{B} \mathcal{B}, (1 - n - \zeta g; \zeta_1, \cdots, \zeta_r; 1),
\]

(3.2)

where $z = cx^\sigma \left[ \frac{2}{1 + \sqrt{1 + x^2}} \right]^\zeta$

Provided

\[
\zeta_i, \sigma_i > 0, \zeta_i - \sigma_i > 0 (i = 1, \cdots, r), \quad \text{Re} (p + \sigma g) + \sum_{i=1}^r \zeta_i \min_{1 \leq j \leq m(i)} \text{Re} \left( D_j^{(i)} \left( \frac{d_j^{(i)}}{\delta_j^{(i)}} \right) \right) > 0
\]

\[
\text{Re} (n - p + (\zeta - \sigma) g) + \sum_{i=1}^r (\zeta_i - \sigma_i) \min_{1 \leq j \leq m(i)} \text{Re} \left( D_j^{(i)} \left( \frac{d_j^{(i)}}{\delta_j^{(i)}} \right) \right) > 0,
\]

\[
\left| \arg(z_i x^{\sigma_i} y^{n} \left\{ x + \sqrt{1 + x^2} \right\}^{-\zeta_i}) \right| < \frac{1}{2} A_i^{(k)} \pi \text{ where } A_i^{(k)} \text{ is defined by } (1.4).
\]

To prove the theorem 2, we use the similar methods but we use the lemma 2 instead to lemma 1.

3. Particular cases.

Taking $h = s = 1$, we obtain the following formulæ

**Corollary 1.**

\[
\int_0^\infty x^{p-1} \left[ \frac{1}{1 + \sqrt{1 + 4x}} \right]^n \phi_1(1, a; 1 + a; z) \mathcal{J} \left( z_1 x^{\sigma_1} \left\{ x + \sqrt{1 + 4x} \right\}^{-\zeta_1}, \cdots, z_r x^{\sigma_r} \left\{ x + \sqrt{1 + 4x} \right\}^{-\zeta_r} \right) \, dx =
\]

\[
\sum_{a=0}^\infty \left( \frac{c^a}{a + g} \right) \mathcal{J} X, p_r + \frac{1}{2} q_r + 2, \tau_r, R_r, Y \left( z_1 \mathcal{A}_1, (1 - p - \sigma g; \sigma_1, \cdots, \sigma_r; 1), \cdots, z_r \mathcal{A}_r, (1 - n - \zeta g; \zeta_1, \cdots, \zeta_r; 1) \right) \mathcal{B} \mathcal{B}, (1 - n - \zeta g; \zeta_1, \cdots, \zeta_r; 1),
\]
where \( z = cx^\nu \left[ \frac{2}{1 + \sqrt{1 + 4x}} \right]^\zeta \), under the same existence conditions that theorem 1.

Taking \( g = 1 \) and \( s = -m \), we have the following relation, see [3]

\[
\phi(z, -m, a, g) = \frac{m!}{z^a} \left[ \log \left( \frac{1}{m} \right) \right]^{m-1} - \frac{1}{z^a} \sum_{s=0}^{\infty} \frac{B_{m+g+1}(c)(logz)^s}{g!(m + g + 1)}
\]

We obtain

**Corollary 2.**

\[
\int_0^\infty x^{p-1} \left[ \frac{1}{1 + \sqrt{1 + 4x}} \right]^n \left[ \frac{m!}{z^a} \left[ \log \left( \frac{1}{m} \right) \right]^{m-1} - \frac{1}{z^a} \sum_{s=0}^{\infty} \frac{B_{m+g+1}(c)(logz)^s}{g!(m + g + 1)} \right]
\]

\[
\mathcal{I} \left( z_1 x^{\sigma_1} \{ x + \sqrt{1 + 4x} \}^{-\zeta_1}, \ldots, z_r x^{\sigma_r} \{ x + \sqrt{1 + 4x} \}^{-\zeta_r} \right) dx =
\]

\[
\sum_{s=0}^{\infty} c^s(a + g)^m \sum_{\nu=0,n_r+3,V}^{U;0,n_r+3,V} \sum_{\nu=0,\nu_r+2,\nu_r+R_r;Y} \left( \begin{array}{c}
\nu_1 \\
\nu_r \\
\nu_r
\end{array} \right)
\]

\[
2^{\sigma_1} z_1 \mathcal{A}; (1 - p - \sigma g; \sigma_1, \ldots, \sigma_r; 1), \ldots, 2^{\sigma_r} z_r \mathcal{B}; (1 - m - \sigma g; \zeta_1, \ldots, \zeta_r; 1),
\]

\[
\begin{array}{c}
(1 - m - \sigma g; \zeta_1, \ldots, \zeta_r; 1), (1 - n - \zeta_1, \ldots, \zeta_r; 1),
\end{array}
\]

\[
\begin{array}{c}
A : A, \ldots, B : B, (1 - n - \zeta_1, \ldots, \zeta_r; 1),
\end{array}
\]

\[
\begin{array}{c}
\mathcal{F_1}(1, a; 1 + a; \nu_1; \nu_2; \nu_r; \alpha_1; \ldots; \alpha_r; \nu_1; \nu_2; \nu_r; \alpha_1; \ldots; \alpha_r; \nu_1; \nu_2; \nu_r; \alpha_1; \ldots; \alpha_r),
\end{array}
\]

\[
\begin{array}{c}
(1 - m - \sigma g; \zeta_1, \ldots, \zeta_r; 1),
\end{array}
\]

\[
\begin{array}{c}
(1 - n - \zeta_1, \ldots, \zeta_r; 1),
\end{array}
\]

\[
\begin{array}{c}
\mathcal{F_1}(1, a; 1 + a; \nu_1; \nu_2; \nu_r; \alpha_1; \ldots; \alpha_r; \nu_1; \nu_2; \nu_r; \alpha_1; \ldots; \alpha_r; \nu_1; \nu_2; \nu_r; \alpha_1; \ldots; \alpha_r),
\end{array}
\]

where \( z = cx^\nu \left[ \frac{2}{1 + \sqrt{1 + 4x}} \right]^\zeta \), under the same existence conditions that theorem 1.

**Corollary 3.**

\[
\int_0^\infty x^{p-1} \left[ \frac{1}{1 + \sqrt{1 + x^2}} \right]^n \mathcal{I} \left( z_1 x^{\sigma_1} \{ x + \sqrt{1 + x^2} \}^{-\zeta_1}, \ldots, z_r x^{\sigma_r} \{ x + \sqrt{1 + x^2} \}^{-\zeta_r} \right) dx =
\]

\[
\sum_{s=0}^{\infty} c^s(a + g)^m \sum_{\nu=0,n_r+3,V}^{U;0,n_r+3,V} \sum_{\nu=0,\nu_r+2,\nu_r+R_r;Y} \left( \begin{array}{c}
\nu_1 \\
\nu_r \\
2^{\sigma_1} z_1, \ldots, 2^{\sigma_r} z_r
\end{array} \right)
\]

\[
\begin{array}{c}
\mathcal{A}; (1 - p - \sigma g; \sigma_1, \ldots, \sigma_r; 1), \ldots, \mathcal{B}; (1 - m - \sigma g; \zeta_1, \ldots, \zeta_r; 1),
\end{array}
\]

\[
\begin{array}{c}
(1 - m - \sigma g; \zeta_1, \ldots, \zeta_r; 1), (1 - n - \zeta_1, \ldots, \zeta_r; 1),
\end{array}
\]

\[
\begin{array}{c}
A : A, \ldots, B : B, (1 - n - \zeta_1, \ldots, \zeta_r; 1),
\end{array}
\]
\[
\left(\frac{2-n-\zeta g+\pi g}{2}, \frac{\zeta g_1}{2}, \cdots, \frac{\zeta g_n}{2}; 1\right), (-n - \zeta g; \zeta_1, \cdots, \zeta_r; 1), A : A, \left(\frac{-n-\zeta g+\pi g}{2}, \frac{\zeta g_1+\pi g_1}{2}, \cdots, \frac{\zeta g_n+\pi g_n}{2}; 1\right) : B
\] 

(4.4)

where \( z = e^{x^2} \left[ \frac{2}{1 + \sqrt{1 + x^2}} \right] \) under the same existence conditions that theorem 1.

**Remark:**
We obtain the same double finite integrals with the functions defined in section I. Kumar and Nagar [6] have obtained the same relations about the multivariable H-function and generalized Riemann Zeta function.

5. Conclusion.

The importance of our all the results lies in their manifold generality. Firstly, in view of general arguments utilized in these single infinite integrals, we can obtain a large simpler single infinite integrals. Secondly by specialising the various parameters as well as variables in the generalized multivariable Gimel-function, we get a several formulae involving remarkably wide variety of useful functions (or product of such functions) which are expressible in terms of E, F, G, H, I, Aleph-function of one and several variables and simpler special functions of one and several variables. Hence the formulae derived in this paper are most general in character and may prove to be useful in several interesting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics.

**REFERENCES.**


