Abstract—Let G be an simple undirected graphs. A subset D of V is said to be a chromatic strong(weak) dominating set if D is a strong(weak) dominating set and \( \chi(<D>) = \chi(G) \). The minimum cardinality of a chromatic strong(weak) dominating set in a graph G is called the chromatic strong(weak) dominating number and is denoted by \( \gamma_c(G) \) (\( \gamma_w(G) \)). A graph G is called \( \gamma_c(G) \)-excellent if every vertex of G belongs to a \( \gamma_c(G) \)-set. We find that the necessary and sufficient condition for some particular graph, of the form \( K^n \cup P_m \), is \( \gamma_c \)-excellent and \( \gamma_w \)-excellent.

Keywords—Chromatic strong domination, Chromatic weak domination, Chromatic strong excellent graph, Chromatic weak excellent graph.

I. INTRODUCTION

Let \( G=(V,E) \) be a simple undirected graph with vertex set V and edge set E. A coloring of a graph G is an assignment of n colors to its vertices so that no two adjacent vertices have the same color. The chromatic number \( \chi(G) \) is defined as the minimum n for which G has an n-coloring. The diameter of a connected graph G is defined by \( \max\{ d(u,v) : u,v \in V(G) \} \) and is denoted by \( \text{diam}(G) \). Note that diameter of a path \( P_n \) is the distance between the end vertices. That is \( \text{diam}(P_n)=n-1 \). For graph theoretic terminology, we refer to [2] and [4]. A subset D of V is a dominating set of G if every vertex in V-D is adjacent to at least one vertex in D. The domination number \( \gamma(G) \) of G is the minimum cardinality of the dominating set of G. A study of domination in graphs and its advanced topics are given in [6]. Prof. E. Sampathkumar and L. Pushpalatha have defined strong(weak) domination in graphs shown in [8]. A subset D of V is called a strong(weak) dominating set of G if for every vertex in V-D there exists \( u \in D \) such that \( uv \in E \) and \( \deg u \geq \deg v \) (\( \deg v \geq \deg u \)). The strong(weak) domination number \( \gamma_s(G) \) (\( \gamma_w(G) \)) of G is the minimum cardinality of a strong(weak) dominating set of G. A study of domination in graphs and its advanced topics are given in [6].

Prof. T. N. Janakiraman and M. Poobalaranjani [7] introduced a new conditional dom chromatic set and Prof. S. Balamurugan et al [3] extended this dom chromatic set to chromatic strong (weak) dominating set. A subset D of G is said to be a chromatic strong(weak) dominating set if D is a strong(weak) dominating set and \( \chi(<D>)=\chi(G) \). The minimum cardinality of a chromatic strong(weak) dominating set in a graph G is called the chromatic strong(weak) dominating number and is denoted by \( \gamma_c(G) \) (\( \gamma_w(G) \)). A chromatic strong dominating set with cardinality \( \gamma_c(G) \) is called \( \gamma_c \)-set (\( \gamma_w \)-set).

Prof. N. Sridharan and M. Yamuna [9] defined some new classes of excellent graphs with respect to \( \gamma \)-set. Prof. CVR Harinarayanan et al [5] extended it to strong (weak) domination excellent graphs. We introduce a chromatic strong (weak) excellent in graphs and find the condition for chromatic strong (weak) very excellent caterpillar in [1].

II. CHROMATIC STRONG (WEAK) EXCELLENT

A. \( \gamma_c \) (\( \gamma_w \))-EXCELLENT [1]

A vertex \( u \) in G is said to be \( \gamma_c \) (\( \gamma_w \))-good if \( u \) belongs to some \( \gamma_c \) (\( \gamma_w \)-set) of G and \( \gamma_c \) (\( \gamma_w \))-bad otherwise. A graph G is called \( \gamma_c \) (\( \gamma_w \))-excellent if every vertex of G is \( \gamma_c \) (\( \gamma_w \)-good. Equivalently, a graph G is said to be excellent with respect to chromatic strong (Weak) domination if each \( u \in V(G) \) is contained in some \( \gamma_c \) (\( \gamma_w \)-set) of G.

Abstract—Let G be an simple undirected graphs. A subset D of V is said to be a chromatic strong(weak) dominating set if D is a strong(weak) dominating set and \( \chi(<D>) = \chi(G) \). The minimum cardinality of a chromatic strong(weak) dominating set in a graph G is called the chromatic strong(weak) dominating number and is denoted by \( \gamma_c(G) \) (\( \gamma_w(G) \)). A graph G is called \( \gamma_c(G) \)-excellent if every vertex of G belongs to a \( \gamma_c(G) \)-set. We find that the necessary and sufficient condition for some particular graph, of the form \( K^n \cup P_m \), is \( \gamma_c \)-excellent and \( \gamma_w \)-excellent.

Keywords—Chromatic strong domination, Chromatic weak domination, Chromatic strong excellent graph, Chromatic weak excellent graph.
B. $\gamma_c^c (\gamma_{w}^c)$ – Just Excellent [1]

A graph G is said to be just excellent with respect to chromatic strong (Weak) domination if each $u \in V(G)$ is contained in a unique $\gamma_c^c (\gamma_{w}^c)$ - set of G. We also says that G is $\gamma_c^c (\gamma_{w}^c)$ - just excellent graph.

C. $\gamma_c^c (\gamma_{w}^c)$ – Very Excellent [1]

A graph G is said to be very excellent with respect to chromatic strong (Weak) domination if there is a $\gamma_c^c (\gamma_{w}^c)$ - set of G such that to each vertex $v \in V(G)$ there exists a vertex $v \in D$ such that $(D - \{v\}) \cup \{u\}$ is $\gamma_c^c (\gamma_{w}^c)$ - set of G. We also says that G is $\gamma_c^c (\gamma_{w}^c)$ - very excellent graph.

D. $\gamma_c^c (\gamma_{w}^c)$ – Rigid Very Excellent [1]

Let G be a very excellent graph and D be a very excellent $\gamma_c^c (\gamma_{w}^c)$ - set of G. To each $u \in D$, let E(u,D) be the set of vertices of D which are exchangeable with u.

If |E(u,D)| = 1, then D is said to be rigid very excellent $\gamma_c^c (\gamma_{w}^c)$ - set of G. If G has at least one rigid very excellent $\gamma_c^c (\gamma_{w}^c)$ - set then G is said to be rigid very excellent.

III. CHROMATIC STRONG EXCELLENT GRAPHS

A. Theorem

Let $G = K_m \cup P_n$ be a complete graph where $K_m$ is the complete graph with $m(>3)$ vertices and $P_n$(n≥2) is the path with the vertex set {1,2,3,…,n}. Let $X=\{x_1,x_2,x_3,…,x_i\}$ be a non empty set where $x_i$ is the $i^{th}$ vertex of $P_n$ such that $x_i \in V(K_m)$. Then G is $\gamma_c^c$ - excellent graph if and only if the following hold

i. V(K_m) is a subset of every chromatic strong dominating set of G.

ii. In $P_n$, for $t \in \mathbb{N}$,

a. $d(x_1,1) = d(x_n,n) = 3t -1$

b. $d(x_i,1+x_i) = 3t -2$, for all $i=1,2,…,k -1$

Proof:

Given $G=K_m \cup P_n$. Clearly, the chromatic number of G is m. Let D be $\gamma_c^c$ - set of G. If G is $\gamma_c^c$ - excellent, then clearly, $V(K_m) \subseteq V(G)$ is a subset of every chromatic strong dominating set, D of G. Therefore (i) holds.

Case : 1

If $k=1$, i.e., $V(K_m) \cap V(P_n)$={$x_1$}. Then, we have to prove that both $d(x_1,1)$ and $d(x_1,n)$ is of the form 3t-1, $t \in \mathbb{N}$, in $P_n$. Suppose $d(x_1,1)$≠3t-1 if $x_i$≠ n. If $d(x_1,n)$=3t, then the vertices $x_1+1,x_1+4,x_1+7,…,n$-n belong to no $\gamma_c^c$ - set of G. Otherwise, $D \not\supset \gamma_c^c$. If $d(x_1,n)$=3t+1, then $D=V(K_m) \cup \{x_1+3,x_1+6,…,n-1\}$ is a unique $\gamma_c^c$ - set of G. Otherwise, $D \not\supset \gamma_c^c$. Since both sub cases lead to contradiction, $d(x_1,1)=3t-1$. Similarly, $d(x_1,n)=3t-1$.

Case : 2

If $k≠1$. If $x_i$≠ 1, then by case : 1, the result.$d(x_1,1)=3t-1$ is true. Similarly, $d(x_1,n)=3t-1$ if $x_i$≠ n. Hence, (ii)-(a) holds. Let $S_i$ be the set of all vertices lies between $x_i$ and $x_{i+1}$ in $P_n$. That is $S_i=\{s \in P_n | x_i<s<s_{i+1}\}$. Let $S_i=\{s_{i1},s_{i2},…,s_{i(n)}\}$. Suppose that $d(x_{i-n},s_{i1})≠3t-2$ in $P_n$. If $d(x_{i-n},s_{i1})=3t$, then $D=V(K_m) \cup \{s_{i1},s_{i2},…,s_{i(3t-1)}\}$ is a unique $\gamma_c^c$ - set of G. If $d(x_{i-n},s_{i1})=3t-1$, then the vertices $s_{i1},s_{i2},…,s_{i(3t-1)}$ belong to no $\gamma_c^c$ - set of G. Otherwise, $D \not\supset \gamma_c^c$. Since both sub cases lead to contradiction, $d(x_{i-n},s_{i1})=3t-2$. Hence (ii)-(b) holds. Conversely,

Now, we assume that the given graph $G=K_m \cup P_n$ satisfies the condition (i) and (ii). Suppose G is not a $\gamma_c^c$-excellent. Let D be any $\gamma_c^c$ - set of G. Then there exists a vertex x in V(G) such that no $\gamma_c^c$ - set, D of G containing x. Since by (i), xnotin V(K_m). Hence $x \in V(P_n)$-V(K_m).

Case : 1

If x lies between 1 and $x_1$ then, Let $S_b$ be the set of all vertices lies between $x_1$ and 1 including $x_1$ and 1. Let $R=S_b \cap D$={$r_1,r_2,…,r_q$}. Clearly, $x \not\in R$ and x not in R. If $d(r_{j+1},r_j)=3$, for all $1 \leq j < q$, then, $d(x_1,1)=3t+1, (t \in \mathbb{N})$, contradicts (ii)-(a). Otherwise, If $d(r_{j+1},r_j)=2$, for a unique j, then $d(x_1,1)=3t, (t \in \mathbb{N})$, contradicts (ii)-(a).
It is clear that the end vertex of \( P_{m-i} \) is adjacent to \( x \). Let \( r \) be the adjacent vertex of \( r_{j+1} \) other than \( x \) as shown in the following figure.

\[
\begin{array}{c}
\ldots \quad r_j \quad r_{j+1} \quad r_{j+2} \quad \cdots \\
\end{array}
\]

Then, clearly, \( D-\{r_j, r_{j+1}\} \cup \{r,x\} \) is a \( \gamma^c_x \) set of \( G \) containing \( x \), which is contradiction. If \( d(r_j, r_{j+1}) = 2 \), for more than two \( j \), then, \(|D| > \gamma^c_x \). Hence \( G \) is \( \gamma^c_x \) excellent.

**Case : 2**

If \( x \) lies between \( x_k \) and \( x \). It is similar to case : 1. Hence, by case : 1, \( G \) is \( \gamma^c_x \) excellent.

**Case : 3**

If \( x \) lies between \( x_i \) and \( x_{i+1} \), then. Let \( S \) be the set of all vertices lies between \( x_i \) and \( x_{i+1} \) including \( x_i \) and \( x_{i+1} \). And let \( T = S \cup D = \{r_1, r_2, \ldots, r_p\} \). Clearly, \( x_i \in T \), \( x_{i+1} \in T \) and \( x \) not in \( T \). If \( d(t_i, t_{i+1}) = 3 \), for all \( i \leq \text{p} \), then, \( d(x_i, x_{i+1}) = 3t_0 (\text{e} \ N) \). Which is contradiction to (ii)-(b). Otherwise, If \( d(t_i, t_{i+1}) = 2 \), for a unique \( j \), then \( d(x_i, x_{i+1}) = 3t_1 (\text{e} \ N) \). Which is contradiction to (ii)-(b). If \( d(t_i, t_{i+1}) = 2 \), for any two \( j \), then, \( |D| > \gamma^c_x \). Hence \( G \) is \( \gamma^c_x \) excellent.

**B. Corollary**

Let \( G = K_m \cup P \) be a graph where \( K_m \) is the complete graph with \( m(>3) \) vertices and \( P \) is the union of disjoint paths \( P_j (j \geq 2) \) with the vertex set \{1, 2, 3, \ldots, n\}. Let \( X^0 = \{x_1^0, x_2^0, x_3^0, \ldots, x_k^0\} \) be a non empty set, where \( x_i^0 \) is the \( i^{th} \) vertex of \( P \) such that \( x_i^0 \in V(K_m) \). Then \( G \) is \( \gamma^c_x \) excellent graph if and only if the following hold

1. \( V(K_m) \) is a subset of every chromatic strong dominating set of \( G \).
2. \( \text{In } P_j \) for each \( j \) and for \( \text{e} \ N \)
   - \( d(1, x_i^0) = d(x_i^0, n) = 3t_1 \)
   - \( d(x_i^0, x_i^0) = 3t_2 \), for all \( i = 1, 2, \ldots, k \)

**C. Theorem**

Let \( G = K_m \cup C_n \) be a connected graph where \( K_m \) is the complete graph with \( m(>3) \) vertices and \( C_n \) is the cycle with \( (n \geq 3) \) vertices. Let \( H = \{ V(G)-V(K_m) \} \) and let \( H_1, H_2, \ldots, H_p \) be components of \( H \). Then \( G \) is \( \gamma^c_x \) excellent graph if and only if the following hold

1. \( V(K_m) \) is a subset of every chromatic strong dominating set of \( G \).
2. \( \text{diam}(H_t) = 3t_1, \text{e} \ N \), for each \( i = 1, 2, \ldots, p \)

**Proof:**

Given \( G = K_m \cup C_n \) is a connected graph. Then \( H \) is a disjoint union of paths. ie., each \( H_i \) is a path. Let \( H_i = P_{k_i} \) be a path with the vertex set \{1, 2, \ldots, k_i\}. It is clear that the chromatic number and clique number of the graph \( G \) is \( m \). Let \( D \) be \( \gamma^c_x \) set of \( G \). If \( G \) is \( \gamma^c_x \) excellent, clearly, \( V(K_m) \subseteq V(G) \) is a subset of every chromatic strong dominating set, \( D \) of \( G \). Therefore (i) holds. Now we have to prove that \( \text{diam}(H_t) = 3t_1, \text{e} \ N \), for each \( i = 1, 2, \ldots, p \). It is enough to prove that \( ki = 3t_1 \), for each \( i \). It is clear that the end vertex of \( P_{k_i} \) is adjacent to vertex of \( D \),
since \( V(K_m) \subset D \). Suppose \( ki \neq 3t \). If \( ki=3t+1 \), then the vertices \( h_1, h_2, h_3, \ldots, h_{3t+1} \) belongs to no \( \gamma^c_x \)- set of \( G \). Otherwise, \([D] \not\supset \gamma^c_x \). If \( ki=3t-1 \), then \( D=V(K_m) \cup \{ h_1, h_2, \ldots, h_{3t} \} \) is a unique \( \gamma^c_x \)- set of \( G \). Since both cases lead to contradiction, \( ki=3t \).

Hence (ii) holds.

Conversely,

Now, we assume that the given graph \( G=K_m \cup C \) satisfies the condition (i) and (ii). Suppose \( G \) is not \( \gamma^c_x \)- excellent. Let \( D \) be any \( \gamma^c_x \)- set of \( G \). Then there exists a vertex \( x \) in \( V(G) \) such that no \( \gamma^c_x \)- set \( D \) of \( G \) containing \( x \). Since by (i), \( x \) not in \( V(K_m) \). Hence \( x \in H \) implies \( x \notin H_i \) for some \( i \). Let \( p \) and \( q \) be the vertices of \( V(K_m) \) which is also adjacent to the pendant vertex of \( H_i \). Let \( S=(H_i \cap D) \cup \{ p, q \} \) and let \( S=\{ s_1, s_2, \ldots, s_t, q \} \) (say). Clearly, \( x \notin S \). If \( d(s_i, s_{i+1})=3 \), for all \( 1 \leq i < r \), then \( d(p, q)=3t \), \(( \forall \in N) \) implies \( \text{diam}(H_i)=3t-2 \), \(( \forall \in N) \) contradicts (ii). Otherwise, if \( d(s_i, s_{i+1})=2 \), for a unique \( j \), then \( d(p, q)=3t-1 \), \(( \forall \in N) \) implies \( \text{diam}(H_i)=3t \), \(( \forall \in N) \) contradicts (2). If \( d(s_j, s_{j+1})=2 \), for any two \( j \), \( j=1, 2, \ldots, r-1 \), \( d(s_1, s_{j+1})=d(s_2, s_{j+2})=2 \) then in particular, let \( j=1+1 \) and \( x \) is adjacent to both \( s_j \) and \( s_{j+1} \) as shown in the following figure.

\[ \ldots s_j \ldots x \ldots s_{j+1} \ldots \]

Then, clearly, \( (S-S_{j+1}) \cup \{ x \} \) is a \( \gamma^c_x \)- set of \( G \) containing \( x \) which is also contradiction. If \( d(s_j, s_{j+1})=2 \), for more than two \( j \), then \( D \not\supset \gamma^c_x \).

Hence \( G \) is \( \gamma^c_x \)- excellent.

D. Corollary

Let \( G=K_m \cup C \) be a connected graph where \( K_m \) is the complete graph with \( m(>3) \) vertices and \( C \) is the union of disjoint cycles, \( C_j(n \geq 3) \). Let \( H=\left\{ V(G)-V(K_m) \right\} \) and let \( H_1, H_2, \ldots, H_p \) be a components of \( H \). Then \( G \) is \( \gamma^c_x \)- excellent graph if and only if the following hold

i. \( V(K_m) \) is a subset of every chromatic strong dominating set of \( G \).

ii. \( \text{diam}(H_i)=3t-1, \forall \in N \), for each \( i=1, 2, \ldots, p \)

IV. CHROMATIC WEAK EXCELLENT GRAPHS

A. Theorem

Let \( G=K_m \cup P \) be a graph where \( K_m \) is the complete graph with \( m(>3) \) vertices and \( P \) is the union of disjoint paths \( P_j \) with the vertex set \( \{ 1, 2, 3, \ldots, n \} \). Let \( X=\{ x_1, x_2, x_3, \ldots, x_k \} \) is non empty, where \( x_i \) is the \( i^{th} \) vertex of \( P \) such that \( x_i \in V(K_m) \). then \( G \) is \( \gamma^c_x \)- excellent graph if and only if the following hold

i. \( V(K_m) \) is a subset of every chromatic weak dominating set of \( G \).

ii. In \( P_j \), for \( \forall \in N \),

a. \( d(1, x_i)=d(1, n)=1 \) or \( 3t \)

b. \( d(x_i, x_{i+1})=3 \) or \( 3t-1 \), for all \( i=1, 2, \ldots, k-1 \)

B. Corollary

Let \( G=K_m \cup P \) be a graph where \( K_m \) is the complete graph with \( m(>3) \) vertices and \( P \) is the union of disjoint paths \( P_j \) with the vertex set \( \{ 1, 2, 3, \ldots, n \} \). Let \( X=\{ x_1, x_2, x_3, \ldots, x_k \} \) is non empty where \( x_i \) is the \( i^{th} \) vertex of \( P \) such that \( x_i \in V(K_m) \). then \( G \) is \( \gamma^c_x \)- excellent graph if and only if the following hold

i. \( V(K_m) \) is a subset of every chromatic weak dominating set of \( G \).

ii. In \( P_j \), for each \( j \) and for \( \forall \in N \),

a. \( d(1, x_i)=d(1, n)=1 \) or \( 3t \)

b. \( d(x_i, x_{i+1})=3 \) or \( 3t-1 \),

for all \( i=1, 2, \ldots, k-1 \)
C. Theorem

Let $G = K_m \cup C_n$ be a connected graph where $K_m$ is the complete graph with $m(>3)$ vertices and $C_n$ is the cycle with the $(n \geq 3)$ vertices. Let $H=(\ V(G)-V(K_m))$ and let $H_1, H_2, \ldots, H_p$ be a components of $H$ Then $G$ is $\gamma_{wc}^G$- excellent graph if and only if the following hold

i. $V(K_m)$ is a subset of every chromatic weak dominating set of $G$.

ii. $diam(H_i) = 1$ or $3t$, $t \in \mathbb{N}$, for each $i=1,2,\ldots,p$

D. Corollary

Let $G = K_m \cup C$ be a connected graph where $K_m$ is the complete graph with $m(>3)$ vertices and $C$ is the union of disjoint cycles, $C_n(n \geq 3)$. Let $H=(\ V(G)-V(K_m))$ and let $H_1, H_2, \ldots, H_p$ be a components of $H$ Then $G$ is $\gamma_{wc}^G$- excellent graph if and only if the following hold

i. $V(K_m)$ is a subset of every chromatic weak dominating set of $G$.

ii. $diam(H_i) = 1$ or $3t$, $t \in \mathbb{N}$, for each $i=1,2,\ldots,p$

V. REFERENCES


