Integral involving a generalized multiple-index Mittag-Leffler function, Bessel functions, a class of polynomials multivariable Aleph-function and multivariable I-function

F.Y. AYANT

1 Teacher in High School, France

1. Introduction and preliminaries.

The function Aleph of several variables generalize the multivariable I-function recently study by C.K. Sharma and Ahmad [5], itself is an generalisation of G and H-functions of multiple variables. The multiple Mellin-Barnes integral occurring in this paper will be referred to as the multivariables Aleph-function throughout our present study and will be defined and represented as follows.

We define:

\[ N(z_1, \ldots, z_r) = \prod_{j=1}^{r} \gamma(\alpha_j^{(1)}, \ldots, \alpha_j^{(r)}; \beta_j^{(1)}, \ldots, \beta_j^{(r)}; M_1, N_1; \ldots, M_r, N_r) \]

\[ \psi(s_1, \ldots, s_r) = \prod_{k=1}^{r} \theta_k(s_k) g_k(s_1) \cdots s_r \]

\[ \frac{1}{(2\pi i)^r} \int_{L_1} \int_{L_r} \psi(s_1, \ldots, s_r) \prod_{k=1}^{r} \theta_k(s_k) g_k(s_1) \cdots s_r \]
Suppose, as usual, that the parameters 

$$b_j, j = 1, \ldots, Q, a_j, j = 1, \ldots, P;$$

$$c_{j,i}^{(k)}, j = n_k + 1, \ldots, P_{i}; c_{j}^{(k)}, j = 1, \ldots, N_k;$$

$$d_{j,i}^{(k)}, j = M_k + 1, \ldots, Q_{i}; d_{j}^{(k)}, j = 1, \ldots, M_k;$$

with 

$$k = 1 \ldots, r, i = 1, \ldots, R, i^{(k)} = 1, \ldots, R^{(k)}$$

are complex numbers, and the \(\alpha's, \beta's, \gamma's\) and \(\delta's\) are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)} = \sum_{j=1}^{N} \alpha_j^{(k)} + \tau_i \sum_{j=N+1}^{P_i} \alpha_j^{(k)} + \sum_{j=1}^{N_k} \gamma_j^{(k)} + \tau_i \sum_{j=n_k+1}^{P_{i}} \gamma_j^{(k)} - \tau_i \sum_{j=1}^{M_k} \delta_j^{(k)} - \sum_{j=1}^{M_k} \delta_j^{(k)}$$

$$- \tau_i \sum_{j=M_k+1}^{Q_{i}} \delta_j^{(k)} \leq 0$$ (1.4)

The reals numbers \(\tau_i\) are positives for \(i = 1\) to \(R\), \(\tau_i^{(k)}\) are positives for \(i^{(k)} = 1\) to \(R^{(k)}\)

The contour \(L_k\) is in the \(s_k-p\) lane and run from \(-\infty \rightarrow \sigma + i\infty\) where \(\sigma\) is a real number with loop, if necessary, ensure that the poles of \(\Gamma(\delta_j^{(k)} - \delta_j^{(k)} s_k)\) with \(j = 1\) to \(m_k\) are separated from those of \(\Gamma(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(k)} s_k)\) with \(j = 1\) to \(N\) and \(\Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} s_k)\) with \(j = 1\) to \(N_k\) to the left of the contour \(L_k\). The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as:

$$|\arg z_k| < \frac{1}{2} A_i^{(k)} \pi, \text{ where}$$

$$A_i^{(k)} = \sum_{j=1}^{N} \alpha_j^{(k)} - \tau_i \sum_{j=N+1}^{P_i} \alpha_j^{(k)} - \tau_i \sum_{j=1}^{Q_{i}} \beta_j^{(k)} + \sum_{j=1}^{N_k} \gamma_j^{(k)} - \tau_i \sum_{j=n_k+1}^{P_{i}} \gamma_j^{(k)}$$

$$+ \sum_{j=1}^{M_k} \delta_j^{(k)} - \tau_i \sum_{j=M_k+1}^{Q_{i}} \delta_j^{(k)} > 0, \text{ with } k = 1 \ldots, r, i = 1, \ldots, R, i^{(k)} = 1, \ldots, R^{(k)}$$ (1.5)

The complex numbers \(z_k\) are not zero. Throughout this document, we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the asymptotic expansion in the following convenient form:

$$N(\alpha_1, \ldots, \alpha_r) = 0( |z_1|^{\alpha_1}, \ldots, |z_r|^{\alpha_r} ), \max( |z_1|, \ldots, |z_r| ) \rightarrow 0$$

$$N(\beta_1, \ldots, \beta_r) = 0( |z_1|^{\beta_1}, \ldots, |z_r|^{\beta_r} ), \min( |z_1|, \ldots, |z_r| ) \rightarrow \infty$$
where, with \( k = 1, \ldots, r \) : \( \alpha_k = \min[Re(d_j^{(k)}/\delta_j^{(k)})] \), \( j = 1, \ldots, M_k \) and 
\[ \beta_k = \max[Re((c_j^{(k)} - 1)/\gamma_j^{(k)})] \], \( j = 1, \ldots, N_k \).

Serie representation of Aleph-function of several variables is given by
\[
N(y_1, \cdots, y_r) = \sum_{G_1, \cdots, G_r = 0}^{\infty} \sum_{g_1 = 0}^{M_1} \cdots \sum_{g_r = 0}^{M_r} \frac{(-)^{G_1 + \cdots + G_r}}{\delta_{g_1} G_1 \cdots \delta_{g_r} G_r} \psi(\eta_{G_1, g_1}, \cdots, \eta_{G_r, g_r}) \]
\[ \times \theta_1(\eta_{G_1, g_1}) \cdots \theta_r(\eta_{G_r, g_r}) y_1^{-\eta_{g_1, 1}} \cdots y_r^{-\eta_{g_r, r}} \tag{1.6} \]

Where \( \psi(\cdot, \cdots, \cdot) \), \( \theta_i(\cdot) \), \( i = 1, \cdots, r \) are given respectively in (1.2), (1.3) and
\[
\eta_{G_1, g_1} = \frac{d_{g_1}^{(1)} + G_1}{\delta_{g_1}^{(1)}}, \cdots, \eta_{G_r, g_r} = \frac{d_{g_r}^{(r)} + G_r}{\delta_{g_r}^{(r)}}
\]
which is valid under the conditions \( \delta_{g_i}^{(i)}[d_j^{(r)} + \pi] \neq \delta_{g_i}^{(i)}[d_j^{(r)} + G_i] \)

for \( j \neq M_1, M_i = 1, \cdots, \eta_{G_1, g_1}; P_i, N_i = 0, 1, 2, \cdots; y_i \neq 0, i = 1, \cdots, r \) \( \tag{1.8} \)

In the document, we will note:
\[
G(\eta_{G_1, g_1}, \cdots, \eta_{G_r, g_r}) = \phi(\eta_{G_1, g_1}, \cdots, \eta_{G_r, g_r}) \theta_1(\eta_{G_1, g_1}) \cdots \theta_r(\eta_{G_r, g_r}) \tag{1.9} \]

where \( \phi(\eta_{G_1, g_1}, \cdots, \eta_{G_r, g_r}) \), \( \theta_1(\eta_{G_1, g_1}), \cdots, \theta_r(\eta_{G_r, g_r}) \) are given respectively in (1.2) and (1.3).

We will note the Aleph-function of \( r \) variables \( N_{u,v}^{0, N_{u,w}} \)
\[
\begin{pmatrix} z_1 \\ \cdots \\ z_r \end{pmatrix}
\tag{1.10} \]

The multivariable I-function is defined in term of multiple Mellin-Barnes type integral:
\[
I(z_1, z_2, \ldots z_s) = \int_{L_1}^{0,n_2;0,n_3;\cdots;0,n_r;M,n',n'';\cdots;m(s),n(s)} \int_{L_2}^{0,p_2,q_2,q_3;\cdots;p_r,q_r;p',q';\cdots;p(s),q(s)} \frac{z_1}{(a_2j_2, \alpha_2j_2, \alpha_2j_2')_1,p_2;\cdots; (a_2j_2, \alpha_2j_2)_{1,p'}} \]
\[
(b_2j_2', \beta_2j_2', \beta_2j_2')_{1,q_2;\cdots; (b_2j_2', \beta_2j_2')_{1,q'}} \tag{1.11} \]

\[
= \frac{1}{(2\pi i)^s} \int_{L_1} \cdots \int_{L_s} \xi(t_1, \cdots, t_s) \prod_{i=1}^{s} \phi_i(t_i) z_i^t \text{d}t_1 \cdots \text{d}t_s \tag{1.12} \]
The defined integral of the above function, the existence and convergence conditions, see Y.N Prasad [3]. Throughout the present document, we assume that the existence and convergence conditions of the multivariable I-function.

The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as:

$$|\arg z_k| < \frac{1}{2} \Omega_i^{(k)} \pi,$$

where

$$\Omega_i^{(k)} = \sum_{k=1}^{n_i} \alpha_k^{(i)} - \sum_{k=n_i+1}^{p_i} \alpha_k^{(i)} + \sum_{k=1}^{m_i} \beta_k^{(i)} - \sum_{k=m_i+1}^{q_i} \beta_k^{(i)} + \left( \sum_{k=1}^{r_2} \alpha_{2k}^{(i)} - \sum_{k=r_2+1}^{q_2} \alpha_{2k}^{(i)} \right) + \cdots$$

$$= \left( \sum_{k=1}^{n_s} \alpha_{sk}^{(i)} - \sum_{k=n_s+1}^{p_s} \alpha_{sk}^{(i)} \right) - \left( \sum_{k=1}^{q_2} \beta_{2k}^{(i)} + \sum_{k=1}^{q_3} \beta_{3k}^{(i)} + \cdots + \sum_{k=1}^{q_s} \beta_{sk}^{(i)} \right)$$

(1.13)

where \( i = 1, \ldots, s \).

The complex numbers \( z_k \) are not zero. Throughout this document, we assume the existence and absolute convergence conditions of the multivariable I-function.

We may establish the asymptotic expansion in the following convenient form:

$$I(z_1, \ldots, z_s) = 0( |z_1|^{\alpha_1}, \ldots, |z_s|^{\alpha_s}) \cdot \max( |z_1|, \ldots, |z_s|) \to 0$$

$$I(z_1, \ldots, z_s) = 0( |z_1|^{|\beta_1^{(s)}}, \ldots, |z_s|^{|\beta_s^{(s)}}) \cdot \min( |z_1|, \ldots, |z_s|) \to \infty$$

where \( k = 1, \ldots, z : \alpha_k = \min[\text{Re}(b_j^{(k)}/\beta_j^{(k)})], j = 1, \ldots, m_k \) and

$$\beta_k = \max[\text{Re}((a_j^{(k)} - 1)/\alpha_j^{(k)}), j = 1, \ldots, n_k$$

We will use these following notations in this paper:

$$U = p_2, q_2, p_3, q_3, \ldots, p_{s-1}, q_{s-1}; \quad V = 0, n_2; 0, n_3; \ldots; 0, n_{s-1}$$

(1.14)

$$W = (p', q'); \ldots; (p'(s), q'(s)); \quad X = (m', n'); \ldots; (m'(s), n'(s))$$

(1.15)

$$A = (a_2, a_2', a_2''); \ldots; (a_{(s-1)k}, a_{(s-1)k}', a_{(s-1)k}''); \ldots; (a_{(s-1)k}, a_{(s-1)k}''', a_{(s-1)k}''''; \ldots; a_{(s-1)k}^{(s-1)})$$

(1.16)

$$B = (b_2, b_2', b_2''; \ldots; (b_{(s-1)k}, b_{(s-1)k}', b_{(s-1)k}''); \ldots; b_{(s-1)k}^{(s-1)})$$

(1.17)

$$\mathfrak{a} = (a_{sk}; a_{sk}, a_{sk}', a_{sk}''; \ldots, a_{sk}^{(s)}); \quad \mathfrak{b} = (b_{sk}; b_{sk}, b_{sk}', b_{sk}''; \ldots, b_{sk}^{(s)}$$

(1.18)

$$A' = (a_k', a_k''; \ldots; (a_k^{(s)}, a_k^{(s)})_{1,p^{(s)}}); \quad B' = (b_k', b_k''; \ldots; (b_k^{(s)}, b_k^{(s)})_{1,q^{(s)}}$$

(1.19)

The multivariable I-function write:
The generalized polynomials defined by Srivastava [6], is given in the following manner:

\[
I(z_1, \ldots, z_s) = I^{V;0,n_1,X}_{U;\rho_1,q_1,W}\left( \begin{array}{c} z_1 \\ \vdots \\ z_s \\ A; A' \\ \mathfrak{A}; \mathfrak{A}' \\ B; B' \\ \mathfrak{B}; \mathfrak{B}' \end{array} \right) 
\]  

(1.20)

Where \( n_1, \rho_1, q_1, W \) are arbitrary positive integers and the coefficients \( A, A', \mathfrak{A}, \mathfrak{A}', B, B', \mathfrak{B}, \mathfrak{B}' \) are arbitrary constants, real or complex.

In the present paper, we use the following notation

\[
S_{N_1', \ldots, N_{r'}'}^{M_1', \ldots, M_{r'}'}[y_1, \ldots, y_r] = \sum_{K_1=0}^{[N_1'/M_1']} \cdots \sum_{K_{r'}=0}^{[N_{r'}/M_{r'}']} \frac{(-N_1')_{M_1'K_1} \cdots (-N_{r'}')_{M_{r'}'K_{r'}}}{K_1! \cdots K_{r'}!} A[N_1', K_1; \ldots; N_{r'}, K_{r'}]y_1^{K_1} \cdots y_r^{K_{r'}}
\]

(1.21)

Where \( M_1', \ldots, M_{r'}' \) are arbitrary positive integers and the coefficients \( A[N_1', K_1; \ldots; N_{r'}, K_{r'}] \) are arbitrary constants, real or complex. In the present paper, we use the following notation

\[
a_1 = \frac{(-N_1')_{M_1'K_1}}{K_1!} \cdots \frac{(-N_{r'}')_{M_{r'}'K_{r'}}}{K_{r'}!} A[N_1', K_1; \ldots; N_{r'}, K_{r'}]
\]

(1.22)

2. Generalized multiple-index Mittag-Leffler function

A further generalization of the Mittag-Leffler functions is proposed recently in Paneva-Konovska [2]. These are \( 3m \)-parametric Mittag-Leffler type functions generalizing the Prabhakar [3] \( 3 \)-parametric function , defined as:

\[
E_{(\gamma_i)}^{(\alpha_i)},(\beta_i) (z) = \sum_{k=0}^{\infty} \frac{(\gamma_1)_k \cdots (\gamma_m)_k}{\Gamma(\alpha_1 k + \beta_1) \cdots \Gamma(\alpha_m k + \beta_m)} \frac{z^k}{k!}
\]

(2.1)

where \( \alpha_i, \beta_i, \gamma_i \in \mathbb{C}, i = 1, \ldots, m, \text{Re}(\alpha_i) > 0 \)

3. Required formula

See see Gradshteyn and Ryzhik ([1], 6653 5 page1097)

\[
\int_0^\infty x^{\lambda-1} e^{-ax^2} J_\mu(bx) J_\nu(bx) dx = 2^{\mu-\nu-1} \alpha^{-\nu-\lambda} \beta^{\mu+\nu} \Gamma\left(\frac{\lambda+\mu+\nu+1}{2}\right) \Gamma(\mu+1) \Gamma(\nu+1)
\]

\[
\times _2F_2\left( \begin{array}{c} \frac{\mu+\nu+1}{2}, \frac{\mu+\nu+1}{2} + 1, \frac{\mu+\nu+\lambda}{2} \\ \nu + 1, \mu + 1 \end{array} ; -\frac{b^2}{a^2} \right)
\]

(3.1)

where \( \text{Re}(\nu + \lambda + \mu) > 0, \text{Re}(\alpha) > 0 \)

4. Main integral
Let \( b_k = \frac{(\gamma_1)_k \cdots (\gamma_m)_k}{\Gamma(\alpha_1 k + \beta_1) \cdots \Gamma(\alpha_m k + \beta_m)} \), we have the following integral,

\[
\int_0^\infty x^{\lambda-1} e^{-ax^2} J_\mu(bx) J_\nu(bx) E^{(\gamma_1), m}_{(\alpha_1), (\beta_1)} \left( \frac{y_1 x^{\gamma_1}}{\ldots} \right) S_{N_1, \ldots, N_r}^{\alpha_1, \ldots, \alpha_r} \left( \frac{y_2 x^{\gamma_r}}{\ldots} \right) \left( \frac{z_1 x^{\alpha_1}}{\ldots} \right) \left( \frac{z_r x^{\alpha_r}}{\ldots} \right) \, dx
\]

\[
\frac{(-1)^{G_1 + \cdots + G_r}}{G_1! \cdots G_r!} \left( \frac{a_1}{(\mu + 1)(\nu + 1)} \right) \sum_{K_1=0}^{[N_1'/M'_1]} \cdots \sum_{K_r=0}^{[N_r'/M'_r]} \sum_{n=0}^{\infty} \sum_{k=0}^{G_1} \cdots \sum_{g_{r}=0}^{G_r} M_1 \cdots M_r
\]

\[
y_1^{K_1} \cdots y_t^{K_t} a^{\frac{\gamma_1}{2}} \left( k + \sum_{i=1}^{t} K_i \gamma_i \right) a^{\sum_{i=1}^{r} \eta_i g_i} X^{(\alpha_1, \beta_1)}_{U; ps + 1; x} \left( \frac{a^{\eta_1} Z_1}{\ldots} \right) \left( \frac{a^{\eta_r} Z_r}{\ldots} \right)
\]

\[
A: (1-n-\frac{1}{2}(\lambda + \mu + v + \alpha k + \sum_{i=1}^{t} K_i \gamma_i + \sum_{i=1}^{r} \eta_i g_i) + \frac{\eta_1}{2}, \ldots, \frac{\eta_r}{2}), \mathfrak{B}, A'
\]

\[
\mathfrak{B}; \mathfrak{B}'; A'
\]

Provided that

a) \( \min\{\gamma_j, \alpha_k, \eta_l\} > 0 \) \( j = 1, \ldots, t, k = 1, \ldots, r, l = 1, \ldots, s, \) \( Re(\nu + \lambda + \mu) > 0. \)

b) \( Re(a + \alpha k) + \sum_{i=1}^{r} \alpha_i \min_{1 \leq j \leq M_i} Re\left( \frac{\delta_j^{(i)}}{\delta_j^{(i)}} \right) + \sum_{i=1}^{s} \eta_i \min_{1 \leq j \leq m(i)} Re\left( \frac{\beta_j^{(i)}}{\beta_j^{(i)}} \right) > 0 \)

c) \( |argz_k| < \frac{1}{2} A_i^{(k)} \pi, \) where \( A_i^{(k)} \) is defined by (1.5) \( i = 1, \ldots, r \)

d) \( |argZ_k| < \frac{1}{2} \Omega_i^{(k)} \pi, \) where \( \Omega_i^{(k)} \) is defined by (1.11) \( i = 1, \ldots, s \)

e) The multiple serie occurring on the right-hand side of (3.1) is absolutely and uniformly convergent.

f) \( \bar{\alpha}_i, \bar{\beta}_i, \bar{\gamma}_i \in \mathbb{C}, i = 1, \ldots, m, \) \( Re(\bar{\alpha}_i) > 0 \)

Proof
First, expressing the generalized multiple-index Mittag-Leffler function $E_{(\gamma_i),(\beta_i)}^{(r_i,m)}(z; x, \alpha)$ in series with the help of equation (2.1), the Aleph-function of $r$ variables in series with the help of equation (1.6), the general class of polynomial of several variables $S_{N_1, \ldots, N_t}$ with the help of equation (1.22) and the Prasad’s multivariable I-function of $s$ variables in Mellin-Barnes contour integral with the help of equation (1.9), changing the order of integration and summation (which is easily seen to be justified due to the absolute convergence of the integral and the summations involved in the process) and then evaluating the resulting integral with the help of equation (3.1) and expressing the generalized hypergeometric function $_2F_1$ in series. Use several times the following relation $\Gamma(a)/\Gamma(a)_n = \Gamma(a+n)$ with $Re(a) > 0$. Finally interpreting the result thus obtained with the Mellin-barnes contour integral, we arrive at the desired result.

The quantities $U, V, W, X, A, B, a_i, b_i, A', B'$ are defined by the equations (1.14) to (1.19)

5. Particular case

If $U = V = A = B = 0$, the multivariable I-function defined by Prasad degenerate in multivariable H-function defined by Srivastava et al [7]. We have the following result.

\[
\int_0^\infty z^{\lambda-1}e^{-z^2}J_\mu(bz)J_\nu(bz)E_{(\gamma_i),(\beta_i)}^{(r_i,m)}(z; x, \alpha)S^{M_1, \ldots, M_t}_{N_1, \ldots, N_t}\left(\begin{array}{c}
y_1x^{\gamma_1} \\
y_2x^{\gamma_2} \\
\vdots \\
y_tx^{\gamma_t}
\end{array}\right)N_0^0, N_0^0, \ldots, N_t^0, \ldots, N_t^0, w^0, \ldots, z^0, x^{\alpha_r}\]

\[
H_{\lambda, \mu, \eta, \xi, \omega}^{0, n_s; X}
\left(\begin{array}{c}
Z_1x^{\gamma_1} \\
Z_2x^{\gamma_2} \\
\vdots \\
Z_tx^{\gamma_t}
\end{array}\right)dx = \frac{2^{-v-1-\alpha-\mu-\beta+v+\mu}}{\Gamma(\mu+1)\Gamma(\nu+1)}\sum_{K_1=0}^{[N_1'/M_1'^{\nu}]} \cdots \sum_{K_t=0}^{[N_t'/M_t'^{\nu}]} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{g_1=0}^{M_1} \cdots \sum_{g_t=0}^{M_t} \sum_{g_r=0}^{M_r} \delta_{g_1} G_1! \cdots \delta_{g_r} G_r! G(\gamma_1, \ldots, \gamma_t, \beta_1, \ldots, \beta_t, \alpha_1, \ldots, \alpha_t) a_1^k b_k z^k
\]

\[
y_1^{K_1} \cdots y_t^{K_t} a_n^{K_n} = \frac{1}{(\lambda + \mu + \nu + \alpha k + \sum_{i=1}^{t} K_i \gamma_i + \sum_{i=1}^{r} \eta g_i + \alpha_i)} H_{\lambda, \mu, \eta, \xi, \omega}^{0, n_s+1; X}
\left(\begin{array}{c}
\begin{array}{c}
a_n^{-\frac{1}{2}} Z_1 \\
a_n^{-\frac{1}{2}} Z_2 \\
\vdots \\
a_n^{-\frac{1}{2}} Z_t
\end{array}
\end{array}\right)
\]

(5.1)

under the same notations and conditions that (4.1) with $U = V = A = B = 0$

6. Conclusion
In this paper we have evaluated a generalized finite integral involving the product of two Bessel functions, the multivariable Aleph-function, the generalized multiple-index Mittag-Leffler function, a class of polynomials of several variables a sequence of functions and the multivariable I-function defined by Prasad. The integral established in this paper is of very general nature as it contains Multivariable Aleph-function, which is a general function of several variables studied so far. Thus, the integral established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

REFERENCES


Personal adress: 411 Avenue Joseph Raynaud
Le parc Fleuri, Bat B
83140, Six-Fours les plages
Tel: 06-83-12-49-68
Department: VAR
Country: FRANCE