A Class Of Diameter Six Trees with Graceful Labeling

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Abstract

Here we denote a diameter six tree by (a₀; a₁, a₂, . . . , aₘ; b₁, b₂, . . . , bₙ; c₁, c₂, . . . , cᵣ), where a₀ is the center of the tree; aᵢ, i = 1, 2, . . . , m; bⱼ, j = 1, 2, . . . , n, and cₖ, k = 1, 2, . . . , r are the vertices of the tree adjacent to a₀; each aᵢ is the center of a diameter four tree, each bⱼ is the center of a star, and each cₖ is a pendant vertex. Here we give graceful labelings to some new classes of diameter six trees (a₀; a₁, a₂, . . . , aₘ; b₁, b₂, . . . , bₙ; c₁, c₂, . . . , cᵣ) in which we find diameter four trees consisting of four different combinations of odd, even, and pendant branches with the total number of branches odd. Here by a branch we mean a star, i.e. we call a star an odd branch if its center has an odd degree and an even branch if its center has an even degree.

Keywords: graceful labeling, diameter six tree, component moving transformation, transfers of the first and second types, BDSTF

1 Introduction

Definition 1.1. A diameter six tree is a tree which has a representation of the form (a₀; a₁, a₂, . . . , aₘ; b₁, b₂, . . . , bₙ; c₁, c₂, . . . , cᵣ), where a₀ is the center of the tree; aᵢ, i = 1, 2, . . . , m; bⱼ, j = 1, 2, . . . , n, and cₖ, k = 1, 2, . . . , r are the vertices of the tree adjacent to a₀; each aᵢ is the center of a diameter four tree, each bⱼ is the center of a star, and each cₖ is a pendant vertex. We observe that in a diameter six tree with above representation m ≥ 2, i.e. there should be at least two (vertices) aᵢ s adjacent to c which are the centers of diameter four trees. Here we use the notation D₆ to denote a diameter six tree. In the literature [1, 3, 5, 7, 6, 8, 9, 10, 17, 23] we find that all trees up to diameter five are graceful. As far as diameter six trees are concerned only banana trees are graceful [2, 4, 5, 7, 9, 11, 12, 13, 19, 20, 17, 22, 21, 24]. From literature [4] a banana tree is a tree obtained by connecting a vertex v to one leaf of each of any number of stars (v is not in any of the stars). Chen et.al. [4] conjectured that banana trees are graceful. Bhat-Nayak and Deshmukh [2], Murugan and Arumugam [13, 12] and Vilfred [22, 21] gave graceful labelings to different classes of banana trees. Sethuraman and Jesintha [9, 11, 19, 20]) proved that all banana trees and extended banana trees (graphs obtained by joining a vertex to one leaf of each of any number of stars by a path of length of at least two) are graceful. Here we give graceful labelings to some new classes of diameter six trees (a₁, a₂, . . . , aₘ; b₁, b₂, . . . , bₙ; c₁, c₂, . . . , cᵣ) in which the branches of some diameter four tree are all even, whereas the branches of the remaining diameter four trees are all odd. Here by a branch we mean a star, i.e. we call a star an odd branch if its center has an odd degree and an even branch if its center has an even degree.

2 Preliminaries

Definition 2.1. [14, 15, 16] For an edge e = {u, v} of a tree T, we define u(T) as that connected component of T − e which contains the vertex u. Here we say u(T) is a component incident on the vertex v. If a and b are vertices of a tree T, u(T) is a component incident on a, and b ∉ u(T) then deleting the edge {a, u} from T and making b and u adjacent is termed as the component u(T) has been transferred or moved from a to b. In this paper by the label of the
component \( u(T) \) we mean the label of the vertex \( u \). Let \( T \) be a tree and \( a \) and \( b \) be two vertices of \( T \). By \( a \to b \) transfer we mean that some components from \( a \) have been moved to \( b \). If we consider successive transfers \( a_1 \to a_2, a_2 \to a_3, a_3 \to a_4, \ldots \) we simply write \( a_1 \to a_2 \to a_3 \to a_4 \ldots \) transfer. In the transfer \( a_1 \to a_2 \to \ldots \to a_{n-1} \to a_n \), each vertex \( a_i, i = 1, 2, \ldots, n - 1 \) is called a vertex of transfer. Let \( T \) be a labelled tree with a labeling \( f \). We consider the vertices of \( T \) whose labels form the sequence \((a, b, a - 1, b + 1, a - 2, b + 2)\) (respectively, \((a, b, a + 1, b - 1, a + 2, b - 2)\)). Let \( a \) be adjacent to some vertices having labels different from the above labels. The \( a \to b \) transfer is called a transfer of the first type if the labels of the transferred components constitute a set of consecutive integers. The \( a \to b \) transfer is called a transfer of the second type if the labels of the transferred components can be divided into two segments, where each segment is a set of consecutive integers. A sequence of eight transfers of the first type \( a \to b \to a - 1 \to b + 1 \to a \to b \to a - 1 \to b + 1 \to a - 2 \) (respectively, \( a \to b \to a + 1 \to b - 1 \to a \to b \to a + 1 \to b - 1 \to a + 2 \)), is called a backward double 8 transfer of the first type or BD8TF \( a \to a - 2 \) (respectively, \( a \to a + 2 \)).

(a) By making a transfer \( a \to b \) of first type we can keep an odd number of components at \( a \) from the set \( A \) and move the rest to \( b \), and the resultant tree thus formed will be graceful.

(b) If \( A \) contains an even number of elements, then by making a sequence of transfers of the second type \( a \to b \to a - 1 \to b + 1 \to a - 2 \to b + 2 \) (respectively, \( a \to b \to a + 1 \to b - 1 \to a + 2 \to b - 2 \to \ldots \)) an even number of elements from \( A \) can be kept at each vertex of the transfer, and the resultant tree thus formed is graceful.

(c) By a BD8TF \( a \to b + 1 \) (respectively, \( b - 1 \)), we can keep an even number of elements from \( A \) at \( a, b, a - 1, \) and \( b + 1 \) (respectively, \( a, b, a + 1, \) and \( b - 1 \)), and move the rest to \( a - 2 \) (respectively, \( a + 2 \)). The resultant tree formed in each of the above cases is graceful.

(d) Consider the transfer \( R : a \to b \to a - 1 \to b + 1 \to \ldots \to z \) (respectively, \( a \to b \to a + 1 \to b - 1 \to \ldots \to z \)), with \( z = a - p_1 \) or \( b + p_2 \) (respectively, \( a + r_1 \) or \( b - r_2 \)), such that \( R \) is partitioned as \( R : T_1 \to T_2 \to T_3 \to \ldots \to T_n \), where each \( T_i, 1 \leq i \leq n \), is either a transfer of the first type or BD8TF. Construct a tree \( T^* \) from \( T \) by carrying out the transfer \( R \) by successively carrying out the transfers \( T_1, T_2, T_3, \ldots, T_n \). The tree \( T^* \) is graceful.

(e) Consider the transfer \( R' : a \to b \to a - 1 \to b + 1 \to \ldots \to \ldots \) (respectively, \( a \to b \to a + 1 \to b - 1 \to \ldots \to \ldots \)), such that \( R' \) is partitioned as \( R' : T'_1 \to T'_2 \), where \( T'_1 \) is sequence of transfers consisting of the transfers of

![](image1.png)

Figure 1: The graceful trees in (b), (c), (d), and (e) are obtained from the graceful tree in (a) by applying transfers of the first type \( 22 \to 1 \), the transfer of second type \( 22 \to 2 \), BD8TF \( 22 \to 20 \), and a sequence of transfers consisting of \( 22 \to 1 \to 21 \) transfers of the first type, followed by the BD8TF \( 21 \to 19 \), followed by \( 19 \to 4 \) transfer of the first type, and finally \( 4 \to 18 \to 5 \) transfers of the second type, respectively.

**Theorem 2.2.** [14, 15, 16] In a graceful labeling \( f \) of a graceful tree \( T \), let \( a \) and \( b \) be the labels of two vertices. Let \( a \) be attached to a set \( A \) of vertices (or components) having labels \( n, n+1, n+2, \ldots, n+p \) (different from the above vertex labels), which satisfy \((n+1+i) + (n+p-i) = a + b, i \geq 0 \) (respectively, \((n+i) + (n+p-1-i) = a + b, i \geq 0 \)). Then the following hold.

(a) By making a transfer \( a \to b \) of first type we can keep an odd number of components at \( a \) from the set \( A \) and move the rest to \( b \), and the resultant tree thus formed will be graceful.

(b) If \( A \) contains an even number of elements, then by making a sequence of transfers of the second type \( a \to b \to a-1 \to b+1 \to a-2 \to b+2 \to \ldots \) (respectively, \( a \to b \to a+1 \to b-1 \to a+2 \to b-2 \to \ldots \)), an even number of elements from \( A \) can be kept at each vertex of the transfer, and the resultant tree thus formed is graceful.

(c) By a BD8TF \( a \to b+1 \) (respectively, \( b-1 \)), we can keep an even number of elements from \( A \) at \( a, b, a-1, \) and \( b+1 \) (respectively, \( a, b, a+1, \) and \( b-1 \)), and move the rest to \( a-2 \) (respectively, \( a+2 \)). The resultant tree formed in each of the above cases is graceful.

(d) Consider the transfer \( R : a \to b \to a-1 \to b+1 \to \ldots \to z \) (respectively, \( a \to b \to a+1 \to b-1 \to \ldots \to z \)), with \( z = a-p_1 \) or \( b+p_2 \) (respectively, \( a+r_1 \) or \( b-r_2 \)), such that \( R \) is partitioned as \( R : T_1 \to T_2 \to T_3 \to \ldots \to T_n \), where each \( T_i, 1 \leq i \leq n \), is either a transfer of the first type or BD8TF. Construct a tree \( T^* \) from \( T \) by carrying out the transfer \( R \) by successively carrying out the transfers \( T_1, T_2, T_3, \ldots, T_n \). The tree \( T^* \) is graceful.

(e) Consider the transfer \( R' : a \to b \to a-1 \to b+1 \to \ldots \to \ldots \) (respectively, \( a \to b \to a+1 \to b-1 \to \ldots \to \ldots \)), such that \( R' \) is partitioned as \( R' : T'_1 \to T'_2 \), where \( T'_1 \) is sequence of transfers consisting of the transfers of
the first type and $BDSTF$ and $T'_2$ is a sequence of transfer of the second type. The tree $T^{**}$ obtained from $T$ by making the transfer $R'$ is graceful.

Lemma 2.3. [10] If $g$ is a graceful labeling of a tree $T$ with $n$ edges then the labeling $g_n$ defined as $g_n(x) = n - g(x)$, for all $x \in V(T)$, called the inverse transformation of $g$ is also a graceful labeling of $T$.

3 Results

Theorem 3.1. (a) $D_6 = \{a_0; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n; c_1, c_2, \ldots, c_r\}$, with degree of $a_i$, $b_j$ are even $i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$. If the branches incident on the center $a_i$, $i = 1, 2, \ldots, m_1, m_1 < m$ are odd and branches incident on the center $a_i, i = m_1 + 1, m_1 + 2, \ldots, m$ are even branches. Then $D_6$ has a graceful labeling.

(b) $D_6 = \{a_0; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n\}$, with degree of $a_i, b_j$ are even for $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$. If the branches incident on the center $a_i$, $i = 1, 2, \ldots, m_1, m_1 < m$ are odd and branches incident on the center $a_i, i = m_1 + 1, m_1 + 2, \ldots, m$ are even branches. Then $D_6$ has a graceful labeling.

(c) $D_6 = \{a_0; a_1, a_2, \ldots, a_m; c_1, c_2, \ldots, c_r\}$, with $m$ odd, degree of $a_i$ are even for $i = 1, 2, \ldots, m$. If the branches incident on the center $a_i$, $i = 1, 2, \ldots, m_1, m_1 < m$ are odd and branches incident on the center $a_i, i = m_1 + 1, m_1 + 2, \ldots, m$ are even branches. Then $D_6$ has a graceful labeling.

(d) $D_6 = \{a_0; a_1, a_2, \ldots, a_m\}$, with either $m$ is odd or $m_o \geq 2$, degree of $a_i$ are even for $i = 1, 2, \ldots, m$. If the branches incident on the center $a_i$, $i = 1, 2, \ldots, m_1, m_1 < m$ are odd and branches incident on the center $a_i, i = m_1 + 1, m_1 + 2, \ldots, m$ are even branches. Then $D_6$ has a graceful labeling.

Proof: (a) Case - I Let $m + n$ be odd. Let $|E(D_6)| = q$ and $deg(a_0) = m + n = 2k + 1$. Let us remove the pendant vertices adjacent to $a_0$ and represent the new graceful tree by $D_6^{(1)}$. Consider the graceful tree $G$ as represented in Figure 2.

Let $A = \{k + 1, k + 2, \ldots, q - k - r - 1\}$. Observe that $(k + i) + (q - r - k - i) = q - r$. Consider the sequence of transfer $T_1 : q - r \rightarrow 1 \rightarrow q - r - 1 \rightarrow 2 \rightarrow q - r - 2 \rightarrow \ldots \rightarrow k \rightarrow q - r - k \rightarrow k + 1$ of the first type of the vertex levels in the set $A$. Observe that the transfer $T_1$ and the set $A$ satisfy the properties of Theorem 2.2. We execute the transfer $T_1$ by keeping an odd number of elements of $A$ at each vertex of the transfer. In the transfer $T_1$, the first $m$ vertices are designated as the vertices $a_1, a_2, \ldots, a_m$, respectively, and the remaining $n$ vertices are designated as the vertices $b_1, b_2, \ldots, b_n$. Observe that

$$a_i = \begin{cases} q - r - \frac{i - 1}{2} & \text{if } i \text{ is odd} \\ \frac{i}{2} & \text{if } i \text{ is even} \end{cases}$$

and $b_j = \begin{cases} q - r - \frac{m+j-1}{2} & \text{if } j \text{ is odd if } m \text{ is even} \\ \frac{m+j}{2} & \text{if } j \text{ is even if } m \text{ is odd} \end{cases}$

Let $A_1$ be the set of vertex labels of $A$ which have come to the vertex $k+1$ after the transfer $T_1$. Since each transfer in $T_1$ is a transfer of 1st type, the elements of $A_1$ are the consecutive integers. Next consider the transfer $T_2 : k+1 \rightarrow q-r-k-1 \rightarrow k+2 \rightarrow q-r-k-2 \rightarrow k+3 \rightarrow q-r-k-3 \rightarrow \ldots \rightarrow r$ where

$$r = \begin{cases} k+k_1+1 & \text{if } m \text{ is odd} \\ q-r-k_1 & \text{if } m \text{ is even} \end{cases}$$

and $k_1 = \sum_{i=1}^{m} deg(a_i)$

Observe that the vertices of transfer $T_2$ and the elements of $A_1$ satisfy the hypothesis of Theorem 2.2. Excluding $a_0$ let the sum of number of neighbours of $a_1, a_2, \ldots, a_m$ be $s_1$ and the sum of number of neighbours of $a_{m+1}, a_{m+2}, \ldots, a_m$ be $s_2$. As per Theorem 2.2 the first $s_1$ vertices of $T_2$ lie on $a_i, 1 \leq i \leq m_1$ and they are the centers of odd branches and the remaining $s_2-1$ vertices of $T_2$ lie on $a_i, m_1 + 1 \leq i \leq m$ and they are the centers of
vertices. So $T_2$ consists of $s_1$ transfers of the first type (for keeping desired odd number of vertices of $A_1$ at each vertex of transfer) followed by $s_2 - 1$ transfers of the second type (for keeping desired even number of vertices of $A_1$ at each vertex of transfer) so that we get back the tree $D_6^{(1)}$ and by Theorem 2.2 it is graceful. Attach the vertices $c_1, c_2, \ldots, c_r$ to $a_0$ and assign them the labels $q - r + 1, q - r + 2, q - r + 3, \ldots, q$ so as to get back $D_6$ with a graceful labeling.

Case - II Let $m + n$ be even. Let us construct a tree $G_6$ from $D_6$ by removing the vertices $b_n, c_1, c_2, \ldots, c_r$. Obviously $G_6$ is a diameter six tree with center $a_0$ having odd degree. Let $|E(G_6)| = q_1$. Repeat the procedure in the proof for Case - I by replacing $n$ with $n - 1$ and $q - r$ with $q_1$ and give a graceful labeling to $G_6$. Observe that the vertex $a_0$ in the graceful tree $G_6$ gets the label 0. Attach $b_n, c_1, c_2, \ldots, c_r$ to $a_0$ and assign the labels $q_1 + 1, q_1 + 2, \ldots, q_1 + r, q_1 + r + 1$ to them. Obviously, the tree $G_6 \cup \{b_n, c_1, c_2, \ldots, c_r\}$ is graceful with a graceful labeling, say $g$. Apply inverse transformation $g_{q_1+r+1}$ to $G_6 \cup \{b_n, c_1, c_2, \ldots, c_r\}$ so that the label of the vertex $b_n$ becomes 0. By Lemma 2.3, $g_{q_1+r+1}$ is a graceful labeling of $G_6 \cup \{b_n, c_1, c_2, \ldots, c_r\}$. Let there be $p$ pendant vertices adjacent to $b_n$ in $D_6$. Now attach these vertices to $b_n$ and assign labels $q_1 + r + 2, q_1 + r + 3, \ldots, q_1 + r + p + 1$ to them. Observe that we finally form the tree $D_6$ and the labeling mentioned above is a graceful labeling of $D_6$.

(b) Proof follows if we set $r = 0$ in the proof involving part (a).

(c) Proof follows if we set $n = 0$ in the proof involving part (a).

(d) Case - I If $m$ is odd then the proof follows immediately from the same involving part (b) by setting $n = 0$.

Case - II Let $m$ be even. Let us designate the vertices $a_1, a_2, \ldots, a_{m_1}$ such that $\deg(a_1) \leq \deg(a_2) \geq \deg(a_3) \geq \ldots \geq \deg(a_{m_1})$, i.e. the degree $a_{m_1}$ is minimum among all the neighbours of $a_0$ which are the centers of diameter four trees containing only odd branches. Excluding $a_0$ let there be $2p_i + 1$ neighbours of $a_i, i = 1, 2, \ldots, m_1$ in $D_6$. Remove $a_{m_1}$ and all the components incident on it, i.e. construct the tree $D_6 \setminus \{a_{m_1}\}$. Make any $2p_{m_1}$ neighbours of $a_{m_1}$ adjacent to the vertex $a_2$. The resultant tree thus formed from $D_6$ is obviously a diameter six tree and let it be denoted by $G_6$. Let $|E(G_6)| = q_1$. Repeat the procedure in the proof involving Case - I of part (a) by setting $n = 0$ and $r = 0$ and replacing $m_1$ with $m_1 - 1$ and $q - r$ with $q_1$ and give a graceful labeling to $G_6$. We observe that the vertex $a_2$ gets label 1, and the $2(p_2 + p_{m_1}) + 1$ neighbours of $a_2$ get the labels $q_1 - x, x + 1 + i, q_1 - x - i, x = k + p_1 + 1, i = 1, 2, \ldots, p_2 + p_{m_1}$. While labeling $G_6$ we allot labels $x + i + 2, q_1 - x - i, i = 1, 2, \ldots, p_{m_1}$ to $2p_{m_1}$ neighbours of $a_{m_1}$ that were shifted to $a_2$ while constructing $G_6$. Next we attach the vertex $a_{m_1}$ to $a_0$ and assign label $q_1 + 1$ to $a_{m_1}$. Now we move the vertices $x + i + 2, q_1 - x - i, i = 1, 2, \ldots, p_{m_1}$ to $a_{m_1}$. Since $(x + i + 2) + (q_1 - x - i) = q_1 + 2 = 1 + (1 + 1)$, for $i = 1, 2, \ldots, p_{m_1}$, by Theorem 2.2 the resultant tree, say $G_1$ thus formed is graceful with a graceful labeling, say $g$. Apply inverse transformation $g_{q_1+1}$ to $G_1$ so that the label of the vertex $a_{m_1}$ becomes 0. By Lemma 2.3, $g_{q_1+1}$ is a graceful labeling of $G_1$. Now attach one remaining vertex to $a_{m_1}$ and assign the label $q_1 + 2$ to it. Let this graceful labeling of the new tree, say $G_2$ thus formed be $g_1$. Let there be $p$ neighbours of $q_1 + 2$ in $D_6$. Apply inverse transformation $g_{q_1+2}$ to $G_2$ so that the label of the vertex $q_1 + 2$ of $G_2$ becomes 0. By Lemma 2.3, $g_{q_1+2}$ is a graceful labeling of $G_2$. Now attach the $p$ pendant vertices adjacent to the vertex labelled 0 and assign them the labels $q_1 + 3, q_1 + 4, \ldots, q_1 + p + 2$. Observe that we finally form the tree $D_6$ and the labeling mentioned above is a graceful labeling of $D_6$.

Example 3.2. The diameter six tree in Figure 3(a) is a graceful diameter six of the type in Theorem 3.1(b). Here $q = 79, m = 6$, and $n = 3$.

Figure 3: A diameter six tree of the type in Theorem 3.1(b) with a graceful labeling.
Notation 3.3. Let \( D_6 = \{a_0; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n; c_1, c_2, \ldots, c_r\} \) be a diameter six tree. For next couple of results we will consistently use the following notations.

\( n_e = \) Number of stars adjacent to \( a_0 \) with center having odd degree.

\( n_o = \) Number of stars adjacent to \( a_0 \) with center having even degree, i.e. \( n = n_e + n_o \).

\( m_o^o = \) Number of diameter four trees adjacent to \( a_0 \) containing only odd branches and centers having even degree.

\( m_e^o = \) Number of diameter four trees adjacent to \( a_0 \) containing only even branches and centers having even degree.

\( m_o^e = \) Number of diameter four trees adjacent to \( a_0 \) containing only odd branches and centers having odd degree.

\( m_e^e = \) Number of diameter four trees adjacent to \( a_0 \) containing only even branches and centers having odd degree, i.e. \( m = m_o^o + m_e^o + m_o^e + m_e^e \).

Theorem 3.4. If \( m_o^o \equiv 0 \mod 4 \), \( m_e^e \equiv 0 \mod 4 \), and \( n_e \equiv 0 \mod 4 \), \( m + n \) is odd then

(a) \( D_6 = \{a_0; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n; c_1, c_2, \ldots, c_r\} \) has a graceful labeling.

(b) \( D_6 = \{a_0; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n\} \) has a graceful labeling.

Proof: Let us first prove part (a). Let \( |E(D_6)| = q \) and \( \deg(a_0) = m + n = 2k + 1 \). Remove \( r \) pendant vertices adjacent to \( a_0 \) and denote the new diameter six tree thus formed by \( D_6^{(1)} \).

Form the graceful tree as \( G \) (Figure 2), the set \( A \), and the transfer \( T_1 \) as in the proof involving Case-1 of Theorem 3.1(a). Suppose that the vertices \( a_1, a_2, \ldots, a_{m_o^o} \) are the centers of \( m_o^o \) diameter four trees adjacent to \( a_0 \) with even degree and each is attached to an odd branches, \( a_{m_o^o+1}, a_{m_o^o+2}, \ldots, a_{m_o^o} \) are the centers of \( m_o^o \) diameter four trees adjacent to \( a_0 \) with even degree and each is attached to odd branches, and \( a_{m_o^o+m_e^e+1}, a_{m_o^o+m_e^e+2}, \ldots, a_m \) are the centers of \( m_e^e \) diameter four trees adjacent to \( a_0 \) with odd degree and each is adjacent to odd branches. Suppose that the centers \( b_1, b_2, \ldots, b_{n_e} \) are the centers of stars adjacent to \( a_0 \) with odd degree and

\( b_{n_e+1}, b_{n_e+2}, \ldots, b_n \) are the centers of stars adjacent to \( a_0 \) with even degree. Here \( T_1 \) consists of \( m_o^o \) successive transfers of the first type, followed by \( \frac{m_e^e+m_o^e}{4} \) successive BDSTF, followed by \( m_o^e+n_o \) successive transfers of the first type, and finally \( \frac{n}{2} \) successive BDSTF. Carry out the transfer \( T_1 \) by keeping desired number of elements of \( A \) at each vertex of the transfer. Then repeat the remaining procedure in the proof of Theorem 3.1 and get the results. The proof for part (b) follows by setting \( r = 0 \) in the proof involving part (a).

Example 3.5. The diameter six tree in Figure 4 (a) is a diameter six of the type in Theorem 3.4. Here \( q = 127 \), \( m = 10 \), and \( n = 5 \). We first form the graceful diameter six tree \( G_6 \) as in Figure (b) by removing all the pendant vertices adjacent to \( a_0 \). Finally, the graceful tree \( D_6 \) in Figure (c) which is obtained from the graceful tree in (b) by attaching two pendant vertices to \( a_0 \) and assigning them the labels 126 and 127.
Theorem 3.6. If $m^o_e \cong 0$ mod 4, $m^e_e \cong 0$ mod 4, $n_e \cong 0$ mod 4, $m + n$ is even, and $n_e \geq 1$ then
(a) $D_6 = \{a_0; a_1, a_2, \ldots , a_m; b_1, b_2, \ldots , b_n; c_1, c_2, \ldots , c_r\}$ has a graceful labeling.
(b) $D_6 = \{a_0; a_1, a_2, \ldots , a_m; b_1, b_2, \ldots , b_n\}$ has a graceful labeling.

Theorem 3.7. If $m^o_e \cong 0$ mod 4, $m^e_e \cong 0$ mod 4, and $n_e \cong 1$ mod 4, and $m + n$ is even then
(a) $D_6 = \{a_0; a_1, a_2, \ldots , a_m; b_1, b_2, \ldots , b_n; c_1, c_2, \ldots , c_r\}$ has a graceful labeling.
(b) $D_6 = \{a_0; a_1, a_2, \ldots , a_m; b_1, b_2, \ldots , b_n\}$ has a graceful labeling.

Proof of Theorems 3.6 and 3.7: Let us designate the vertex $b_n$ as the center of a star with even degree if $D_6$ is a diameter six tree in Theorem 3.6 and as the center of a star with odd degree if $D_6$ is a diameter six tree in Theorem 3.7. Construct a tree $G_6$ from $D_6$ by removing the vertices $c_1, c_2, \ldots , c_r$, and the star with center $b_n$ as we have done in the proof of Case - II of Theorem 3.1(a). Then we proceed as in the proof involving Case - II of Theorem 3.1(a) so as to get back $D_6$ with a graceful labeling. ■

Theorem 3.8. If $m^o_e \cong 1$ mod 4, $m^e_e \cong 0$ mod 4, and $n_e \cong 0$ mod 4, $m + n$ is even, $m^o_e + m^e_e \geq 3$, and the degree of center of at least one diameter four tree whose center has odd degree consists of only odd branches $\geq 4$, then
(a) $D_6 = \{a_0; a_1, a_2, \ldots , a_m; b_1, b_2, \ldots , b_n; c_1, c_2, \ldots , c_r\}$ has a graceful labeling.
(b) $D_6 = \{a_0; a_1, a_2, \ldots , a_m; b_1, b_2, \ldots , b_n\}$ has a graceful labeling.

Proof: Let us first prove the part (a). Let us designate the vertices $a_1, a_2, \ldots , a_m$ in such a way that $a_2$ is the center of a diameter four tree which contains only odd branches and $\text{deg}(a_2) \geq 4$. Excluding $a_0$ let there be $2p_i + 1$ neighbours of $a_i$, for $i = 1, 2, \ldots , m_e^o$ and there be $2p_i$ neighbours of $a_i$, for $i = m_e^o + 1, m_e^o + 2, \ldots , m_e^o + m_e^e$ in $D_6$. Remove $c_1, c_2, \ldots , c_r$, and the diameter four tree with center $a_{m_e^o + m_e^e}$, i.e. construct the tree $D_6 \setminus \{c_1, c_2, \ldots , c_r, a_{m_e^o + m_e^e}\}$. Make all, say $2p_m^{m_o} + m_e^o$ neighbours of $a_{m_e^o + m_e^e}$ adjacent to the vertex $a_2$. The resultant tree thus formed from $D_6$ is obviously a diameter six tree and let it be denoted by $G_6$. Let $|E(G_6)| = q_1$. Repeat the procedure in the proof of Theorem 3.4 (a) by replacing $m_e^o$ with $m_e^o + 1$ and $q$ with $q_1$ and give a graceful labeling to $G_6$. Observe that the vertex $a_2$ gets label 1, and the $2(p_2 + p_m^{m_o + m_e^o} + 1)$ (or $2(p_2 + p_m^{m_o + m_e^o})$) neighbours of $a_2$ get the labels $q_1 - x, x + 1 + i, q_1 - x - i, x = k + p_1 + 1, i = 1, 2, \ldots , p_2 + p_m^{m_o + m_e^o}$ (or $q_1 - x, x + 1 + i, q_1 - x - i, x = k + p_1 + 1, i = 1, 2, \ldots , p_2 + p_m^{m_o + m_e^o}$, and one more vertex). While labeling $G_6$ we allot labels $x + i + 2, q_1 - x - i, i = 1, 2, \ldots , p_m^{m_o + m_e^o}$ to $2p_m^{m_o + m_e^o}$ neighbours of $a_{p_m^{m_o} + m_e^o}$ that were shifted to $a_2$ while constructing $G_6$. Next we attach the vertex $a_{p_m^{m_o} + m_e^o}$ to $a_0$ and assign label $q_1 + 1$ to $a_{p_m^{m_o} + m_e^o}$. Now we move the vertices $x + i + 2, q_1 - x - i, i = 1, 2, \ldots , p_m^{m_o + m_e^o}$ to $a_{p_m^{m_o} + m_e^o}$. Since $(x + i + 2) + (q_1 - x - i) = q_1 + 2 = 1 + (q_1 + 1)$, for $i = 1, 2, \ldots , p_m^{m_o + m_e^o}$, by Theorem 2.2 the resultant tree, say $G_1$ thus formed is graceful with a graceful labeling, say $g$. Finally, attach the pendant vertices $a_1, a_2, \ldots , a_m$ to $a_0$ and assign them the labels $q_1 + 2, q_1 + 3, \ldots , q_1 + r + 1$. Observe that we finally form the tree $D_6$ and the labeling mentioned above is a graceful labeling of $D_6$. The proof of part (b) follows if we set $r = 0$ in the proof of part (b). ■

Example 3.9. The diameter six tree in Figure 5 (a) is a diameter six of the type in Theorem 3.8. Here $q = 149$, $m = 12$, and $n = 7$. We first form the graceful diameter six tree $G_6$ as in Figure (b) by removing all the pendant vertices and one star adjacent to $a_0$. Figure (c) represents the tree obtained from the graceful tree in (b). Here we attach five vertices to $a_0$ and assign them the labels 142, 143, 144, 145, and 146. Shift the components 11 and 131 incident on 1 to the vertex whose label is 142. The graceful tree in Figure (d) is obtained by applying inverse transformation to the graceful tree in Figure (c). Finally, the graceful tree $D_6$ in Figure (e) is obtained from the graceful tree in Figure (d) when we attach three vertices to the vertex labelled 0 and assign them the labels 147, 148, and 148.
Figure 5: A diameter six tree of the type in Theorem 3.8 with a graceful labeling.

**Theorem 3.10.** If \( m^o_e \cong 0 \mod 4 \), \( m^e_e \cong 0 \mod 4 \), and \( n_e \cong 0 \mod 4 \), \( m + n \) is even, \( m^o_e \geq 1 \), \( m^o_e + m^e_e \geq 3 \), and the degree of the center of at least one diameter four tree consisting of only odd branches \( \geq 4 \), then \( D_6 = \{ a_0; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n \} \) has a graceful labeling.

**Proof:** Let us designate the vertices \( a_1, a_2, \ldots, a_m \) in such a way that \( a_2 \) is the center of a diameter four tree which contains only odd branches and \( \deg(a_2) \geq 4 \). Excluding \( a_0 \) let there be \( 2p_i + 1 \) neighbours of \( a_i \), \( i = 1, 2, \ldots, m^o_e - 1 \) and there be \( 2p_i \) neighbours of \( a_i \), \( i = m^o_e, m^o_e + 1, m^o_e + 2, \ldots, m^o_e + m^e_e - 1 \) in \( D_6 \). Remove one diameter four tree containing only odd branches. Let us designate this vertex as \( a_{m^o_e + m^e_e} \). Excluding \( a_0 \) let there be \( 2q + 1 \) neighbours of \( a_{m^o_e + m^e_e} \). Make any \( 2p_{m^o_e + m^e_e} \) neighbours of \( a_{m^o_e + m^e_e} \) adjacent to the vertex \( a_2 \). The resultant tree thus formed from \( D_6 \) is obviously a diameter six tree and let it be denoted by \( G_6 \). Let \( |E(G_6)| = q_1 \). Repeat the procedure in the proof of Theorem 3.4 (a) by replacing \( m^e_e \) with \( m^o_e - 1 \) and \( q \) with \( q_1 \) and give a graceful labeling to \( G_6 \). We observe that the vertex \( a_2 \) gets label 1, and the \( 2(p_2 + p_{m^o_e + m^e_e}) + 1 \) neighbours of \( a_2 \) get the labels \( q_1 - x \), \( x + 1 + i \), \( q_1 - x - i \), \( x = k + p_1 + 1 \), \( i = 1, 2, \ldots, p_2 + p_{m^o_e + m^e_e} \). While labeling \( G_6 \) we allot labels \( x + i + 2 \), \( q_1 - x - i \), \( i = 1, 2, \ldots, p_{m^o_e + m^e_e} \) to \( 2p_{m^o_e + m^e_e} \) neighbours of \( a_{p_{m^o_e + m^e_e}} \) that were shifted to \( a_2 \) while constructing \( G_6 \). Next we attach the vertex \( a_{p_{m^o_e + m^e_e}} \) to \( a_0 \) and assign label \( q_1 + 1 \) to \( a_{p_{m^o_e + m^e_e}} \). Now we move the vertices \( x + i + 2 \), \( q_1 - x - i \), \( i = 1, 2, \ldots, p_{m^o_e + m^e_e} \), to \( a_{p_{m^o_e + m^e_e}} \). Since \( (x + i + 2) + (q_1 - x - i) = q_1 + 2 = 1 + (q_1 + 1) \),
for \( i = 1, 2, \ldots, p_{p+m^2} \), by Theorem 2.2 the resultant tree, say \( G_1 \) thus formed is graceful with a graceful labeling, say \( g_1 \). Apply inverse transformation \( g_{q_1+1} \) to \( G_1 \) so that the label of the vertex \( a_{p+m^2} \) becomes 0. By Lemma 2.3, \( g_{q_1+1} \) is a graceful labeling of \( G_1 \). Now attach one remaining vertex to \( a_{p+m^2} \) and assign the label \( q_1 + 2 \) to it. Let this graceful labeling of the new tree, say \( G_2 \) thus formed be \( g_1 \). Let there be \( p \) neighbours of \( q_1 + 2 \) in \( D_6 \). Apply inverse transformation \( g_{q_1+2} \) to \( G_2 \) so that the label of the vertex \( q_1 + 2 \) of \( G_2 \) becomes 0. By Lemma 2.3, \( g_{q_1+2} \) is a graceful labeling of \( G_2 \). Now attach the \( p \) pendant vertices adjacent to the vertex labelled 0 and assign them the labels \( q_1 + 3, q_1 + 4, \ldots, q_1 + p + 2 \). Observe that we finally form the tree \( D_6 \) and the labeling mentioned above is a graceful labeling of \( D_6 \).

**Example 3.11.** The diameter six tree in Figure 6 (a) is a diameter six of the type in Theorem 3.10. Here \( q = 140 \), \( m = 10 \), and \( n = 6 \). We first form the graceful diameter six tree \( G_6 \) as in Figure (b) by removing one diameter four tree and one star adjacent to \( a_0 \) and shifting all except one branches of the removed diameter four tree to the vertex labelled 1 in \( G_6 \). Figure (c) represents the tree obtained from the graceful tree in (b) by attaching a new vertex to \( a_0 \), assigning it the label 134, and shifting the branches with centers with labels 10 and 125 from the vertex 1 to the vertex 134. The graceful tree in Figure (d) is obtained by applying inverse transformation to the graceful tree in Figure (c). The graceful tree in Figure (e) is obtained from the graceful tree in Figure (d) when we attach one vertex to the vertex labelled 0 and assign the label 135. The graceful tree in Figure (f) is obtained by applying inverse transformation to the graceful tree in Figure (e). Finally, the graceful tree \( D_6 \) in Figure (g) is obtained from the graceful tree in Figure (f) when we attach five vertices to the vertex labelled 0 and assign them the labels 136, 137, 138, 139, and 148.
Theorem 3.12. If \( m_e^o \equiv 1 \mod 4 \), \( m_e^e \equiv 0 \mod 4 \), \( n_e \equiv 1 \mod 4 \), and \( m+n \) is odd then
(a) \( D_6 \) has a graceful labeling.
(b) \( D_6 = \{a_0; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n; c_1, c_2, \ldots, c_r\} \) has a graceful labeling.

Theorem 3.13. If \( m_e^o \equiv 1 \mod 4 \), \( m_e^e \equiv 0 \mod 4 \), \( n_o \geq 1 \), and \( m+n \) is odd then
(a) \( D_6 = \{a_0; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n; c_1, c_2, \ldots, c_r\} \) has a graceful labeling.
(b) \( D_6 = \{a_0; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n\} \) has a graceful labeling.

Proof of Theorems 3.12 and 3.13: Let us designate the vertex \( a_2 \) as the center of diameter four tree whose degree \( \geq 4 \). Let there be \( 2p_m \) branches adjacent to \( a_m \). Construct a tree \( G_6 \) from \( D_6 \) by removing the vertices \( c_1, c_2, \ldots, c_r \), one star with center \( b_n \), where \( \deg(b_n) \) is odd (respectively, even) for Theorem 3.13 (respectively, for Theorem 3.12), and one diameter four tree with center \( a_m \) such that degree of \( a_m \) is odd and the branches incident on it are all odd branches. Obviously \( G_6 \) is a diameter six tree with center \( a_0 \) having odd degree. Let \( |E(G_6)| = q_1 \). Repeat the procedure in the proof of Theorem 3.4 (a) by replacing \( m_e^o \) with \( m_e^o - 1 \) (i.e. \( m \) with \( m-1 \)), \( n \) with \( n-1 \) and \( q \) with \( q_1 \) and give a graceful labeling to \( G_6 \). Observe that the vertex \( a_0 \) in the graceful tree \( G_6 \) gets the label 0 and the vertex \( a_2 \) gets label 1, and the \( 2(p_2+p_m)+1 \) neighbours of \( a_2 \) get the labels \( q_1-x, x+1+i, q_1-x-i, x = k+p_1+1, i = 1,2,\ldots,p_2+p_m-1 \), and one more vertex if \( m_o \leq 1 \). While labeling \( G_6 \) we allot labels \( x+i+2, q_1-x-i, i = 1,2,\ldots,p_m \) to \( 2p_m \) neighbours of \( a_m \) that were shifted to \( a_2 \) while constructing \( G_6 \). Attach vertices \( c_1, c_2, \ldots, c_r, b_n \), and and \( a_m \) to \( a_0 \) and assign the labels \( q_1+2, q_1+3, \ldots, q_1+r+1, q_1+r+2, q_1+1 \), respectively. Obviously, the tree \( G_6 \cup \{a_m, c_1, c_2, \ldots, c_r, b_n\} \) is graceful with a graceful labeling, say \( g \). Now we move the vertices \( x+i+2, q_1-x-i, i = 1,2,\ldots,p_m \) to \( a_m \). Since \( (x+i+2) \mod 4, (q_1-x-i) \mod 4, q_1+2 = 1+(q_1+1) \), for \( i = 1,2,\ldots,p_m \), by Theorem 2.3 the resultant tree, say \( G_1 \) thus formed is graceful with a graceful labeling, say \( g \). Apply inverse transformation \( g_{q_1+r+2} \) to \( G_1 \) so that the label of the vertex \( b_n \) becomes 0. By Lemma 2.3, \( g_{q_1+r+2} \) is a graceful labeling of \( G_2 \). Let there be \( p \) pendant vertices adjacent to \( b_n \) in \( D_6 \). Now attach these vertices to \( b_n \) and assign labels \( q_1+r+3, q_1+r+4, \ldots, q_1+r+p+2 \) to them. Observe that we finally form the tree \( D_6 \) and the labeling mentioned above is a graceful labeling of \( D_6 \). Proof of part (b) follows if we set \( r = 0 \) in the proof of part (a).

Example 3.14. The diameter six tree in Figure 7 (a) is a diameter six of the type in Theorem 3.13. Here \( q = 139, m = 11, \) and \( n = 6 \). We first form the graceful diameter six tree \( G_6 \) as in Figure (b) by removing all the pendant vertices and one diameter four tree adjacent to \( a_0 \). Figure (c) represents the tree obtained from the graceful tree in (b). First we attach a vertex to \( a_0 \) and assign the label 128 and shift the components incident on the vertices 118 and 11 adjacent to the vertex labeled 1 in \( G_6 \) to the vertex 128. Then we attach four pendant vertices to \( a_0 \) and assign them the labels 129, 130, 131, 132. Next we apply inverse transform and attach the pendant vertices to the vertex labeled 0 which acts as the center of the star which was removed while constructing \( G_0 \) from \( D_6 \). Assign labels 133, 134, 135, 136, 137, and 138 to the vertices adjacent to 0.
Figure 7: A diameter six tree of the type in Theorem 3.13 with a graceful labeling.

References


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