Degree of Approximation of a Function Belonging to $W(L_r, \xi(t))(r > 1)$-Class by $(E, 1)(C, 2)$ Product Summability Transform

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Abstract. The field of approximation theory is so vast that it plays an increasingly important role in applications in pure and applied mathematics. The present study deals with a theorem concerning the degree of approximation of a function $f$ belonging to $W(L_r, \xi(t))(r > 1)$-class by using $(E, 1)(C, 2)$ of its Fourier series.

Key Words and Phrases : Degree of approximation, $W(L_r, \xi(t))(r > 1)$-class of function, $(E, 1)$ summability, $(C, 2)$ summability, $(E, 1)(C, 2)$ product summability, Fourier series, Lebesgue integral.

1 Introduction

The degree of approximation of a function belonging to the various classes $Lip_{\alpha}$, $Lip(\alpha, r)$, $Lip(\xi(t), r)$ using different summability methods have been determined by several investigators like Alexits [2], Sahney and Goel [13], Quershi and Neha [11], Qureshi [9, 10], Chandra [1], Khan [4], Liendler [5], Mishra et al. [7] and Rhoades [12]. Recently Nigam [8] has obtained the degree of approximation of a function belonging to $Lip(\xi(t), r)$ class by $(E, 1)(C, 2)$ summability method. In the present paper, a theorem on degree of approximation of a function $f$ belonging to $W(L_r, \xi(t))(r \geq 1)$-class by $(E, 1)(C, 2)$ product summability transform of Fourier series has been obtained which in turn generalizes the result of Nigam [8].

2 Preliminaries

Let $f(x)$ be periodic with period $2\pi$ and integrable in the sense of Lebesgue. The Fourier series associated with $f$ at a point $x$ is defined as
\[ f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \] (2.1)

with \( n^{th} \) partial sums \( s_n(f;x) \).

\( L_\infty \)-norm of a function \( f : R \to R \) is defined by

\[ \|f\|_\infty = \sup \{|f(x)| : x \in R\} \] (2.2)

\( L_r \)-norm of a function is defined by

\[ \|f\|_r = \left( \int_0^{2\pi} |f(x)|^r \, dx \right)^{\frac{1}{r}}, \quad r \geq 1. \] (2.3)

The degree of approximation of a signal \( f : R \to R \) by a trigonometric polynomial \( t_n \) of degree \( n \) under sup norm \( \| \|_\infty \) is defined as

\[ \|t_n - f\|_\infty = \sup \{|t_n(x) - f(x)| : x \in R\} \] (Zygmund [14]) (2.4)

and \( E_n(f) \) of a function \( f \in L_r \) is given by

\[ E_n(f) = \min \|t_n - f\|_r \] (2.5)

This method of approximation is called trigonometric Fourier approximation (TFA).

A function \( f \in Lip_{\alpha} \) if

\[ f(x + t) - f(x) = O(|t|^\alpha) \text{ for } 0 < \alpha \leq 1. \] (2.6)

\( f \in Lip(\alpha, r) \) for \( 0 \leq x \leq 2\pi \), if

\[ \left( \int_0^{2\pi} |f(x + t) - f(x)|^r \, dx \right)^{\frac{1}{r}} = O(|t|^\alpha), \quad 0 < \alpha \leq 1, \quad r \geq 1. \] (2.7)

(definition 5.38 of Mc Fadden [6], 1942).

Given a positive increasing function \( \xi(t) \) and an integer \( r \geq 1 \), \( f \in Lip(\xi(t), r) \) if
If \( \xi(t) = t^\alpha \) then \( \text{Lip}(\xi(t),r) \) class reduces to the \( \text{Lip}(\alpha,r) \) and if \( r \to \infty \) then \( \text{Lip}(\alpha,r) \) class reduces to the \( \text{Lip}_\alpha \) class.

and that \( f \in W(L_r,\xi(t)) \) if

\[
\left( \int_0^{2\pi} \left| f(x+t) - f(x) \right|^r dx \right)^{\frac{1}{r}} = O(\xi(t)), \quad \text{for } 0 < \alpha \leq 1, \ r \geq 1. \tag{2.8}
\]

where \( \xi(t) \) is a positive increasing function of \( t \).

If \( \beta = 0 \) then \( W(L_r,\xi(t)) \) reduces to the class \( \text{Lip}(\xi(t),r) \) and if \( \xi(t) = t^\alpha \) then \( \text{Lip}(\xi(t),r) \) class coincides with the class \( \text{Lip}(\alpha,r) \) and if \( r \to \infty \) then \( \text{Lip}(\alpha,r) \) class reduces to the class \( \text{Lip}_\alpha \).

We observe that

\[
\text{Lip}_\alpha \subseteq \text{Lip}(\alpha,r) \subseteq \text{Lip}(\xi(t),r) \subseteq W(L_r,\xi(t)) \quad \text{for } 0 < \alpha \leq 1, \ r \geq 1.
\]

Let \( \sum_{n=0}^\infty u_n \) be a given infinite series with sequence of its \( n^{th} \) partial sum \( \{s_n\} \).

The \((E,1)\) transform is defined as the \( n^{th} \) partial sum of \((E,1)\) summability and is given by

\[
(E,1) = E_n^1 = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} s_k \to s \text{ as } n \to \infty \tag{2.10}
\]

then the infinite series \( \sum_{n=0}^\infty u_n \) is said to be \((E,1)\) summable to a definite number \( s \) (Hardy [3]).

The \((C,2)\) transform is defined as the \( n^{th} \) partial sum of \((C,2)\) summability and is given by

\[
t_n = \frac{2}{(n+1)(n+2)} \sum_{k=0}^{n} (n-k+1) s_k \to s \text{ as } n \to \infty \tag{2.11}
\]

then the infinite series \( \sum_{n=0}^\infty u_n \) is \((C,2)\) summable to a definite number \( s \).

The \((E,1)\) transform of the \((C,2)\) transform defines \((E,1)(C,2)\) transform and we denote it by \( E_n^1C_n^2 \).
Thus if
\[ E_n^1 C_n^2 = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} C_k^2 \rightarrow s \text{ as } n \rightarrow \infty. \] (2.12)
then the series \( \sum_{n=0}^{\infty} u_n \) is said to be summable by \((E, 1) (C, 2)\) summability transform to a definite number \(s\).

We use the following notations:
\[
\phi (t) = f (x + t) + f (x - t) - 2f (x)
M_n (t) = \frac{1}{2^n \pi} \sum_{k=0}^{n} \left\{ \binom{n}{k} \frac{1}{(k+1)(k+2)} \sum_{\nu=0}^{k} (k - \nu + 1) \frac{\sin (\nu + \frac{1}{2}) t}{\sin \frac{t}{2}} \right\}
\]

### 3 Main Theorem

If \( f \) is a \( 2\pi \)-periodic function, Lebesgue integrable on \((0, 2\pi)\), belonging to \( W (L_r, \xi (t)) \) class then its degree of approximation by \((E, 1) (C, 2)\) summability transform of its Fourier series is given by
\[
\left\| E_n^1 C_n^2 - f \right\|_r = O \left( \frac{1}{(n+1)^{\beta+\frac{1}{r}}} \xi \left( \frac{1}{n+1} \right) \right) \] (3.1)
provided \( \xi (t) \) satisfies the following conditions:
\[
\left\{ \int_{\frac{1}{n+1}}^{\frac{1}{n-1}} \left( t \frac{\phi (t)}{\xi (t)} \right)^r \sin^{\beta r} \, dt \right\}^{\frac{1}{r}} = O \left( \frac{1}{n+1} \right) \] (3.2)
\[
\left\{ \int_{\frac{1}{n+1}}^{1} \left( t^{-\delta} \frac{\phi (t)}{\xi (t)} \right)^r \, dt \right\}^{\frac{1}{r}} = O \left( (n+1)^{\delta} \right) \] (3.3)
and
\[
\left\{ \frac{\xi (t)}{t} \right\} \text{ is non-increasing in } t, \] (3.4)
where \( \delta \) is an arbitrary number such that \( s (1 - \delta) - 1 > 0, \frac{1}{r} + \frac{1}{s} = 1, 1 \leq r \leq \infty, \) conditions (3.2) and (3.3) hold uniformly in \( x \) and \( E_n^1 C_n^2 \) is \((E, 1) (C, 2)\) means of the series (2.1).
Note 3.1 For $\beta = 0$, our theorem reduces to the theorem of Nigam [8] and thus generalizes it.

4 Lemma

In order to prove our theorem, we need following lemmas:

**Lemma 4.1.** For $0 \leq t \leq \frac{1}{n+1}$,

$$|M_n(t)| = O(n)$$ (4.1)

**Proof.** For $0 \leq t \leq \frac{1}{n+1}$, $\sin nt \leq n \sin t$,

$$|M_n(t)| \leq \frac{1}{2n\pi} \sum_{k=0}^{n} \left\{ \binom{n}{k} \frac{1}{(k+1)(k+2)} \sum_{0}^{k} \frac{(k-\nu + 1) \sin \left(\frac{\nu + \frac{1}{2}}{2}\right) t}{\sin \frac{\nu}{2}} \right\}$$

$$\leq \frac{1}{2n\pi} \sum_{k=0}^{n} \left\{ \binom{n}{k} \frac{1}{(k+1)(k+2)} \sum_{0}^{k} \frac{(k-\nu + 1) (2\nu + 1) \sin \frac{\nu}{2} t}{\sin \frac{\nu}{2}} \right\}$$

$$\leq \frac{1}{2n\pi} \sum_{k=0}^{n} \left\{ \binom{n}{k} \frac{1}{(k+1)(k+2)} \left(2k+1\right) \sum_{0}^{k} (k-\nu + 1) \right\}$$

$$= \frac{1}{2^{n+1}\pi} \sum_{k=0}^{n} \binom{n}{k} (2k+1)$$

$$= \frac{1}{2^{n+1}\pi} \{2^n (n+1)\}$$

$$= O(n)$$

Lemma 4.2. For $\frac{1}{n+1} \leq t \leq \pi$,

$$|M_n(t)| = O\left(\frac{1}{t}\right)$$ (4.2)
Proof. For \( \frac{1}{n+1} \leq t \leq \pi \), \( \sin \frac{t}{2} \geq \frac{t}{\pi} \) and \( \sin nt \leq 1 \)

\[
|M_n(t)| \leq \frac{1}{2^n \pi} \left| \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) \frac{1}{(k+1)(k+2)} \sum_{\nu=0}^{k} (k - \nu + 1) \frac{\sin \left( \frac{\nu + 1}{2} \right) t}{\sin \frac{t}{2}} \right|
\]

\[
\leq \frac{1}{2^n \pi} \left| \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) \frac{1}{(k+1)(k+2)} \sum_{\nu=0}^{k} (k - \nu + 1) \left( \frac{1}{\pi} \right) \right|
\]

\[
\leq \frac{1}{2^n t} \left| \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) \frac{1}{(k+1)(k+2)} \sum_{\nu=0}^{k} (k - \nu + 1) \right|
\]

\[
\leq \frac{1}{2^n t} \left| \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) \frac{1}{(k+1)(k+2)} \frac{(k+1)(k+2)}{2} \right|
\]

\[
= \frac{1}{2^{n+1} t} \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right)
\]

\[
= \frac{1}{2^{n+1} t} \{2^n\}
\]

\[
= O\left(\frac{1}{t}\right)
\]

\[\square\]

5 Proof of Main Theorem

Let \( s_n(x) \) denote the partial sum of the series (2.1), then we have

\[
s_n(x) - f(x) = \frac{1}{2\pi} \int_0^{\pi} \phi(t) \frac{\sin \left( n + \frac{1}{2} \right) t}{\sin \frac{t}{2}} dt
\]

Therefore, the \( E_1^1 C_n^2 \) transform of \( s_n(f; x) \) is given by

\[
E_1^1 C_n^2 - f(x) = \frac{1}{\pi} \left[ \frac{1}{2^n} \left( \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) \frac{1}{(k+1)(k+2)} \right) \int_0^{\pi} \phi(t) \frac{\sin \left( n - \nu + 1 \right) \frac{1}{2} t}{\sin \frac{t}{2}} dt \right]
\]

\[
= \int_0^{\pi} \phi(t) M_n(t) dt
\]
\[
\int_0^{1/n+1} + \int_{1/n+1}^{1} \phi(t) M_n(t) \, dt = I_1 + I_2 \text{ (say)} \tag{5.1}
\]

We consider,

\[
|I_1| \leq \int_0^{1/n+1} |\phi(t)||M_n(t)| \, dt
\]

Applying Hölder’s inequality and the fact that \(\phi(t) \in W(L_r, \xi(t))\) due to the fact that \(f \in W(L_r, \xi(t))\), condition (3.2) and Lemma 4.1, we have

\[
|I_1| \leq \left[ \int_0^{1/n+1} \left\{ \frac{t \phi(t) | \sin \beta t}{\xi(t)} \right\}^r \, dt \right]^{1/r} \left[ \int_0^{1/n+1} \left\{ \frac{\xi(t) |M_n(t)|}{t^{1+\beta}} \right\}^s \, dt \right]^{1/s} = O \left( \frac{1}{n+1} \right) \left[ \int_0^{1/n+1} \left\{ \frac{\xi(t) |M_n(t)|}{t^{1+\beta}} \right\}^s \, dt \right]^{1/s}
\]

Since \(\xi(t)\) is positive increasing function and using second mean value theorem for integrals, we have

\[
|I_1| = O \left\{ \xi \left( \frac{1}{n+1} \right) \left[ \int_{\xi}^{1/n+1} \frac{dt}{t^{(1+\beta)s}} \right]^{1/s} \text{ for some } 0 \leq \xi < \frac{1}{n+1} \right\} = O \left[ \xi \left( \frac{1}{n+1} \right) \left\{ \frac{t^{-(1+\beta)s+1}}{-(1+\beta)s+1} \right\}^{1/n+1} \right]^{1/2}
\]

\[
= O \left\{ \xi \left( \frac{1}{n+1} \right) (n+1)^{1+\beta-1/2} \right\} = O \left\{ (n+1)^{\beta+1/2} \xi \left( \frac{1}{n+1} \right) \right\} = O \left\{ (n+1)^{\beta+1/2} \xi \left( \frac{1}{n+1} \right) \right\} \text{ since } \frac{1}{r} + \frac{1}{s} = 1, \, 1 \leq r \leq \infty. \tag{5.2}
\]

Now we consider,

\[
|I_2| \leq \int_{1/n+1}^{1} |\phi(t)||M_n(t)| \, dt
\]
Using Hölder’s inequality, $|\sin t| < 1$ and $\sin t \geq \left(\frac{2t}{\pi}\right)$,

$$|I_2| \leq \left[ \int_{\frac{1}{n+1}}^{\pi} \left\{ \frac{t^{-\delta} \phi(t) |\sin^\beta t|}{\xi(t)} \right\}^r dt \right]^{\frac{1}{r}} \left[ \int_{\frac{1}{n+1}}^{\pi} \left\{ \frac{\xi(t)}{t^{\delta-\beta}} |M_n(t)| \right\}^s dt \right]^{\frac{1}{s}}$$

$$= O\left\{ (n+1)^\delta \right\} \left[ \int_{\frac{1}{n+1}}^{\pi} \left\{ \frac{\xi(t)}{t^{\delta-\beta}} \right\}^s dt \right]^{\frac{1}{s}} \text{ by (3.3)}$$

$$= O\left\{ (n+1)^\delta \right\} \left[ \int_{\frac{1}{n+1}}^{\pi} \left\{ \frac{\xi(t)}{t^{\delta-\beta}} \right\}^s dt \right]^{\frac{1}{s}} \text{ by Lemma 4.2}$$

Now putting $t = \frac{1}{y}$,

$$I_2 = O\left\{ (n+1)^\delta \right\} \left[ \int_{\frac{1}{n+1}}^{\pi} \left\{ \frac{\xi\left(\frac{1}{y}\right)}{y^{\delta-1-\beta}} \right\}^s \frac{dy}{y^2} \right]^{\frac{1}{s}}$$

Since $\xi(t)$ is a positive increasing function and $\frac{\xi\left(\frac{1}{y}\right)}{y^s}$ is also increasing function and using second mean value theorem for integrals,

$$I_2 = O\left\{ (n+1)^\delta (n+1) \xi\left(\frac{1}{n+1}\right) \right\} \left[ \int_{\frac{1}{n+1}}^{\pi} \frac{dy}{y^{\delta s+2-\beta s}} \right]^{\frac{1}{s}} \text{, for some } \frac{1}{n+1} \leq \eta \leq n+1$$

$$= O\left\{ (n+1)^{\delta+1} \xi\left(\frac{1}{n+1}\right) \right\} \left[ \int_{\frac{1}{n+1}}^{\pi} \frac{dy}{y^{\delta s+2-\beta s}} \right]^{\frac{1}{s}} \text{, for some } \frac{1}{n+1} \leq \eta \leq n+1$$

$$= O\left\{ (n+1)^{\delta+1} \xi\left(\frac{1}{n+1}\right) \right\} \left[ \left\{ \frac{y^{-\delta s+2-\beta s}}{-s \delta + \beta s - 1} \right\}^{n+1} \right]^{\frac{1}{s}}$$

$$= O\left\{ (n+1)^{\delta+1} \xi\left(\frac{1}{n+1}\right) \right\} \left[ (n+1)^{-\delta-\frac{1}{2}+\beta} \right]$$

$$= O\left\{ \xi\left(\frac{1}{n+1}\right) \right\} \left\{ (n+1)^{1-\frac{1}{2}+\beta} \right\}$$

$$= O\left\{ (n+1)^{\beta+\frac{1}{r}} \xi\left(\frac{1}{n+1}\right) \right\} \text{ since } \frac{1}{r} + \frac{1}{s} = 1 \quad (5.3)$$

Now combining (5.1), (5.2) and (5.3), we get

$$|E_n^1 C_n^2 - f| = O\left\{ (n+1)^{\beta+\frac{1}{r}} \xi\left(\frac{1}{n+1}\right) \right\}$$
Now using $L_r$-norm, we get

$$\| E_n^1 C_n^2 - f \|_r = \left\{ \int_0^{2\pi} |E_n^1 C_n^2 - f|^r dx \right\}^{\frac{1}{r}}$$

$$= \left[ \int_0^{2\pi} \left\{ (n+1)^{\frac{\beta+\frac{1}{r}}{2} + \frac{1}{2}} \xi \left( \frac{1}{n+1} \right) \right\}^r dx \right]^{\frac{1}{r}}$$

$$= \left\{ (n+1)^{\frac{\beta+\frac{1}{r}}{2} + \frac{1}{2}} \xi \left( \frac{1}{n+1} \right) \right\} \left\{ \int_0^{2\pi} dx \right\}^{\frac{1}{r}}$$

$$= \left\{ (n+1)^{\frac{\beta+\frac{1}{r}}{2} + \frac{1}{2}} \xi \left( \frac{1}{n+1} \right) \right\} .$$

This completes the proof of the main theorem.

6 Applications

Following corollaries can be derived from our main theorem:

6.1 Corollary

If $\xi(t) = t^\alpha$, $0 < \alpha \leq 1$, then weighted class $W(L_r, \xi(t))$, $r \geq 1$, reduces to the class $Lip(\alpha, r)$ and the degree of approximation of $2\pi$- periodic function $f$, belonging to the class $Lip(\alpha, r)$, $\frac{1}{r} < \alpha < 1$ is given by

$$\| E_n^1 C_n^2 - f \| = O \left\{ \frac{1}{(n+1)^{\frac{\alpha-1}{2}}} \right\}$$

(6.1)

Proof. The result follows by setting $\beta = 0$ in (3.1).

6.2 Corollary

If $\xi(t) = t^\alpha$ for $0 < \alpha < 1$ and $r = \infty$ in corollary 6.1, then $f \in Lip_\alpha$. In this case, using (6.1), we have

$$\| E_n^1 C_n^2 - f \| = O \left\{ \frac{1}{(n+1)^{\alpha}} \right\}$$
Proof. For \( r = \infty \), we get

\[
\|E^1C^2_n - f\|_{\infty} = \sup_{0 \leq x \leq 2\pi} |E^1C^2_n - f| = O\left\{ \frac{1}{(n+1)^{\alpha}} \right\}
\]

that is,

\[
|E^1C^2_n - f| = O\left\{ \frac{1}{(n+1)^{\alpha}} \right\}
\]

\( \square \)

References


