On Edge Pair Sum Labeling of Graphs

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Abstract - An injective map \( f : E(G) \rightarrow \{ \pm 1, \pm 2, \ldots, \pm q \} \) is said to be an edge pair sum labeling of a graph \( G(p, q) \) if the induced vertex function \( f^* : V(G) \rightarrow \mathbb{Z} - \{0\} \) defined by \( f^*(v) = \sum_{e \in E} f(e) \) is one – one, where \( E_v \) denotes the set of edges in \( G \) that are incident with a vertex \( v \) and \( f^*(V(G)) \) is either of the form \( \{ \pm k_1, \pm k_2, \ldots, \pm k_p \} \) or \( \{ \pm k_1, \pm k_2, \ldots, \pm k_p, \pm k_{p-\frac{1}{2}} \} \cup \{ k_{\frac{p}{2}} \} \) according as \( p \) is even or odd. A graph with a pair sum labeling is called pair sum graph. Analogous to pair sum labeling we define a new labeling called edge pair sum labeling [3]. Let \( G(p, q) \) be a graph. An injective map \( f : E(G) \rightarrow \{ \pm 1, \pm 2, \ldots, \pm q \} \) is said to be an edge pair sum labeling if the induced vertex function \( f^* : V(G) \rightarrow \mathbb{Z} - \{0\} \) is defined by \( f^*(v) = \sum_{e \in E_v} f(e) \) is one – one where \( E_v \) denotes the set of edges in \( G \) that are incident with a vertex \( v \) and \( f^*(V(G)) \) is either of the form \( \{ \pm k_1, \pm k_2, \ldots, \pm k_p \} \) or \( \{ \pm k_1, \pm k_2, \ldots, \pm k_p, \pm k_{p-\frac{1}{2}} \} \cup \{ k_{\frac{p}{2}} \} \) according as \( p \) is even or odd. A graph with an edge pair sum labeling is called an edge pair sum graph. In this paper we prove that some cycle related graphs are edge pair sum graphs.

Keywords: Pair sum labeling, pair sum graph, edge pair sum labeling, edge pair sum graph, ladder graph.

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I. INTRODUCTION

All graphs in this paper are finite, simple and undirected. For standard terminology and notations we follow Gross and Yellen [1]. The symbols \( V(G) \) and \( E(G) \) denote the vertex set and the edge set of a graph. R. Ponraj et al. introduced the concept of pair sum labeling in [7]. An injective map \( f : V(G) \rightarrow \{ \pm 1, \pm 2, \ldots, \pm p \} \) is said to be a pair sum labeling of a graph \( G(p, q) \) if the induced edge function \( f_e : E(G) \rightarrow \mathbb{Z} - \{0\} \) defined by \( f_e(uv) = f(u) + f(v) \) is one-one and \( f_e(E(G)) \) is either of the form \( \{ \pm k_1, \pm k_2, \ldots, \pm k_q \} \) or \( \{ \pm k_1, \pm k_2, \ldots, \pm k_{\frac{p-1}{2}} \} \cup \{ \pm k_{\frac{p}{2}} \} \) according as \( q \) is even or odd. A graph with a pair sum labeling is called a pair sum graph. We established that the path, cycle, star graph, \( P_m \cup K_{1,n} \) , \( C_n \otimes K_m \) if \( n \) is even, triangular snake, bistar, \( K_{1,n} \cup K_{1,m} \), \( C_n \cup C_n \) and complete bipartite graphs \( K_{1,n} \) are edge pair sum graph [3–6]. In this paper we prove that some cycle related graphs are edge pair sum graphs.
II EDGE PAIR SUM GRAPH WITH MANY ODD AND EVEN CYCLES:

In [3] it was shown that the cycle $C_n$ is an edge pair sum graph. The star graph $K_{1,n}$ is an edge pair sum graph if and only if $n$ is even. We would like to consider graphs with many odd and even cycles. Let $P_n(+)N_m$ be a graph with

$$V(P_n(+)N_m) = \{ v_i, v_2, \ldots, v_n, u_1, u_2, \ldots, u_m \}$$

$$E(P_n(+)N_m) = \{ e_i = v_i v_{i+1}: 1 \leq i \leq (n-1), e'_j = v_i u_j \text{ and } e''_j = v_n u_j: 1 \leq j \leq m \}$$

be the vertices and edges of the graph $P_n(+)N_m$. Define the edge labeling

$$f: E(P_n(+)N_m) = \{ \pm 1, \pm 2, \ldots, \pm (n + 2m - 1) \}$$

by considering the following three cases.

Case (i): $n = 2$

Define $f(e_1) = 2, f(e'_1) = -1, f(e''_1) = -3,$

for $1 \leq i \leq \frac{m-1}{2}$

$$f(e'_{1+i}) = (2i + 3), \quad f\left(e''_{m+1+i}\right) = -(2i + 3).$$

The induced vertex labeling are as follows $f^*(v_1) = f(e_1) + f(e'_1) + f(e''_1) + f\left(e''_{m+1+i}\right) = 1.$

Case (ii): $n$ is odd and take $n = 2k+1$.

Sub case (i): $k = 1$ and $k$ is even

for $1 \leq i \leq k$ $f(e_i) = (1 + i)$ and $f(e_{k+i}) = -i,$

$$f(e'_1) = 1, \quad f(e''_1) = -(k + 1), \quad \text{for } 1 \leq i \leq \frac{m-1}{2}$$

$$f(e'_{1+i}) = (k + 1 + 2i), \quad f\left(e''_{m+1+i}\right) = -(k + 1 + 2i).$$

$$f(e''_{1+i}) = (k + m + 2i) \text{ and } f\left(e''_{m+1+i}\right) = -(k + m + 2i).$$
The induced vertex labeling are
\[ f^*(v_1) = f(e_1) + f(e'_1) + f(e'_{1+i}) + f\left(\frac{e_{m+1+i}}{2}\right) = 3 \]
for \( 1 \leq i \leq k-1 \)
\[ f^*(v_{k+i}) = f(e_i) + f(e_{1+i}) = (2i + 3), \]
\[ f^*(v_{k+1+i}) = f(e_k) + f(e_{k+1}) = k, \] for
\( 1 \leq i \leq k-1 \)
f\( f^*(v_{k+1+i}) = f(e_{k+i}) + f(e_{k+1+i}) = -(2i + 1), \)
\[ f^*(v_{2k+1}) = f(e_{2k}) + f(e'_1) = -2k + 1, \]
f\( f^*(u_1) = f(e'_1) + f(e''_1) = -k, \) for
\( 1 \leq i \leq m-1 \)
f\( f^*(u_{1+i}) = f(e'_{1+i}) + f(e''_{1+i}) = (2k + m + 1 + 4i) \) and
\[ f^*(u_{m+1+i}) = f(e'_{m+1+i}) + f(e''_{m+1+i}) = -(2k + m + 1 + 4i). \]
From the above arguments we get
\[ f^*(V(P_n(+))N_m) = \{\pm k, \pm 3, \pm 5, \pm 7, ..., \pm (2k + 1), \pm (2k + m + 5), \pm (2k + m + 9), ..., \pm (2k + 3m - 1)\}. \]
Hence \( f \) is an edge pair sum labeling of \( P_n(+)N_m \). Figure 2 illustrates the edge sum graph labeling \( P_n(+)N_m \) where \( m = n = 3 \).

Sub case (ii): \( k \) is odd and \( k \geq 3 \)
for \( 1 \leq i \leq k \)
f\( f(e_i) = (1 + i), \) \( f(e_{k+1}) = -2, \)
f\( f(e_{k+2}) = -1, \) for \( 1 \leq i \leq k-2 \)
f\( f(e_{k+2+i}) = f(e_{k+3+i}) = f(e_{k+4+i}) = \ldots = f(e_{k+i}) = 1, \) \( f(e_k) = -2, \) \( f(e_{1+i}) = 1, \) \( f(e'_{1+i}) = 1, \) \( f(e''_{1+i}) = -k, \) for \( 1 \leq i \leq m-1 \)
\[ f(e_{m+1+i}) = (k + 2i), \]
\[ f\left(\frac{e_{m+1+i}}{2}\right) = -(k + 2i), \]
\[ f(e_{1+i}) = (k + m - 1 + 2i) \] and \( f\left(\frac{e_{m+1+i}}{2}\right) = -(k + m - 1 + 2i). \)
The induced vertex labeling are
\[ f^*(v_1) = 3, \] for \( 1 \leq i \leq k-1 \)
\[ f^*(v_{k+i}) = f(e_i) + f(e_{1+i}) = (2i + 3), \]
\[ f^*(v_{k+1+i}) = f(e_k) + f(e_{k+1}) = (k - 1), \]
for \( 1 \leq i \leq k \) \( f^*(v_{k+1+i}) = f(e_{k+i}) + f(e_{k+1+i}) = -(2i + 1), \)
\[ f^*(u_1) = f(e'_1) + f(e''_1) = -(k - 1), \]
for \( 1 \leq i \leq m-1 \)
f\( f^*(u_{1+i}) = f(e'_{1+i}) + f(e''_{1+i}) = (2k + m - 1 + 4i) \) and
\[ f^*(u_{m+1+i}) = f(e'_{m+1+i}) + f(e''_{m+1+i}) = -(2k + m - 1 + 4i). \]
From the above arguments we get
\[ f^*(V(P_n(+))N_m) = \{\pm 3, \pm 5, \pm 7, ..., \pm (2k + 1), \pm (k - 1), \pm (2k + m + 3), \pm (2k + m + 7), ..., \pm (2k + 3m - 3)\}. \]
Hence \( f \) is an edge pair sum labeling of \( P_n(+)N_m \).

Case (iii): \( n \) is even and take \( n = 2k \)
Sub case (i): \( k \equiv 0 (mod 3) \)
Define \( f(e_1) = -2, \) for \( 1 \leq i \leq k-1 \)
f\( f(e_{1+i}) = (2 + i), \)
\[ f(e_{k+1}) = -1, \] for \( 1 \leq i \leq k-2 \)
f\( f(e_{k+1+i}) = -(2 + i), \)
f\( f(e'_1) = 1, \)
\[ f(e''_1) = -(k + 1), \] for \( 1 \leq i \leq m-1 \)
f\( f(e'_{1+i}) = (k + 2i), \)
f\( f(e''_{1+i}) = -(k + 2i), \) for \( 1 \leq i \leq m-1 \)
\[ f(e_{m+1+i}) = -(k + 2i), \]
\[ f\left(\frac{e_{m+1+i}}{2}\right) = (k + m - 1 + 2i). \]
Define $f(e_1) = 2$, $f(e_2) = -3$, for $1 \leq i \leq k - 2$ and $f(e_{2+i}) = (3 + i)$, $f(e_{k+1}) = -1$, $f(e_{k+2}) = -2$, for $1 \leq i \leq k - 3$ and $f(e_{k+2+i}) = -(3 + i)$, $f(e_1') = 1$, $f(e_1^*) = -(k + 1)$, for $1 \leq i \leq \frac{m-1}{2}$ and $f(e_{1+i}) = (k + 1 + 2i)$, $f(e_{m+1+i}) = -(k + 1 + 2i)$, $f(e_{1+i}) = (k + m + 2i)$ and $f(e_{m+1+i}) = -(k + m + 2i)$. The induced vertex labeling are $f^*(v_1) = 3$, $f^*(v_2) = -1$, $f^*(v_3) = 1$, for $4 \leq i \leq k$ and $f^*(v_{k+1}) = (2i + 1)$, $f^*(v_{k+2}) = k$, $f^*(v_{k+3}) = -3$, $f^*(v_{k+4}) = -6$, for $4 \leq i \leq k$ and $f^*(v_{k+i}) = -(2i + 1)$, $f^*(v_{1+i}) = f(e_i') + f(e_i^*) = -k$, for $1 \leq i \leq \frac{m-1}{2}$ and $f^*(u_{1+i}) = f(e_{1+i}) + f(e_{1+i}^*) = (2k + m + 1 + 4i)$ and $f^*(\cup_{m+1+i}) = f\left(e_{m+1+i}\right) + f\left(e_{m+1+i}^*\right) = -(2k + m + 1 + 4i)$. From the above arguments we get $f^*\left(V\left(P_n(+)N_m\right)\right) = \{\pm 1, \pm k, \pm 3, \pm 9, ..., \pm(2k + 1), \pm(2k + m + 5), \pm(2k + m + 9), ..., \pm(2k + 3m - 1)\} \cup \{-4\}$. Hence $f$ is an edge pair sum labeling of $P_n(+)N_m$ if $m$ is odd.

Theorem 2.2: The graph $P_n(+)N_m$ is an edge pair sum graph if $m$ is even and $n \geq 3$.

Proof: Let $V\left(P_n(+)N_m\right) = \{v_i, u_j : 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E\left(P_n(+)N_m\right) = \{e_i = v_i v_{i+1} : 1 \leq i \leq n, e_j = v_1 u_j \text{ and } e_j^* = v_n u_j : 1 \leq j \leq m\}$ be the vertices and edges of the graph $P_n(+)N_m$. Define the edge labeling $f : E\left(P_n(+)N_m\right) = \{\pm 1, \pm 2, ..., \pm(n + 2m - 1)\}$ by considering the following four cases.

Case (i): $n = 3$

Define $f(e_1) = -2$, $f(e_2) = 1$, for $1 \leq i \leq \frac{m}{2}$ and $f(e_{1+i}) = (1 + 2i)$, $f\left(e_{m+1+i}\right) = -(1 + 2i)$, $f\left(e_{1+i}\right) = (m + 1 + 2i)$ and $f\left(e_{m+1+i}\right) = -(m + 1 + 2i)$. The
induced vertex labeling are \( f^*(v_1) = -2 \), \( f^*(v_2) = -1 \), \( f^*(v_3) = 1 \), for \( 1 \leq i \leq \frac{m}{2} \)

\[ f^*(v_1) = f^*(v_2) = 1, \]

\[ f^*(u_1) = f^*(u_2) = f^*(u_3) = \ldots = f^*(u_{\frac{m}{2}}) = (m + 6 + 4i). \]

From the above arguments we get

\[ f^*(V(P_n(+)N_m)) = \{ \pm 1, \pm (m + 6), \pm (m + 10), \ldots, \pm (3m + 2) \cup \{-2\} \}. \]

Hence \( f \) is an edge pair sum labeling of \( P_n(+)N_m \). Figure 4 shows that \( P_n(+)N_m \) is an edge pair sum graph labeling for \( m = n = 4 \).

Case (i): \( n = 2 \) and \( m = 3 \).

Define \( f(e_i) = \begin{cases} 
-2 & \text{if } i = k - 1 \\
-1 & \text{if } i = k \\
3 & \text{if } i = k + 1 
\end{cases} \)

\[ \begin{aligned}
 f(e_1) &= (2k + 1 - 2i) & \text{if } 1 \leq i \leq k - 2 \\
 f(e_2) &= (2k - 1 - 2i) & \text{if } k + 2 \leq i \leq 2k - 1 
\end{aligned} \]

for \( 1 \leq i \leq \frac{m}{2} \)

\[ f(e_i) = \begin{cases} 
(2k + 1 - 2i) & \text{if } 1 \leq i \leq k - 2 \\
(2k - 1 - 2i) & \text{if } k + 2 \leq i \leq 2k - 1 
\end{cases} \]

\[ \begin{aligned}
 f(e_i) &= (2k + 1 - 2i) & \text{if } 1 \leq i \leq k - 2 \\
 f(e_i) &= (2k - 1 - 2i) & \text{if } k + 2 \leq i \leq 2k - 1 
\end{aligned} \]

The induced vertex labeling are as follows

\[ f^*(v_1) = f(e_1) + \]

Figure 3

Figure 4

Case (iii): \( n \) is even and take \( n = 2k, k \geq 3 \)
\[ f(e_i) + f\left( e_{m+i}^m \right) = (2k - 1), \quad f^*(v_n) = f(e_{2k-1}) + f\left( e_{m+i}^m \right) = -(2k + m - 1 + 2i). \] The induced vertex labeling are as follows \( f^*(v_1) = f(e_1) + f\left( e_{m+i}^m \right) = -(2k + 1), \quad f^*(v_n) = f(e_{2k}) + f\left( e_{m+i}^m \right) = (2k + 1), \) for \( 2 \leq i \leq k - 2 \)

\[ f^*(v_{k-1}) = f(e_{i-1}) + f(e_i) = 4(k + 1 - i), \quad f^*(v_k) = 3, \quad f^*(v_{k+1}) = -2, \text{ for } k + 3 \leq i \leq 2k - 1 \]

\[ f^*(v_{i-1}) + f(e_i) = 4(k - i), \text{ for } 1 \leq i \leq \frac{m}{2} \]

From the above arguments we get \( f^*(V(P_n(+)N_m)) = \{ \pm 2, \pm 3, \pm 12, \pm 16, \ldots, \pm 4(k - 1), \pm (2k - 1), \pm (4k + m + 2), \pm (4k + m + 6), \ldots, \pm (4k + 3m - 2) \}. \) Hence \( f \) is an edge pair sum labeling of \( P_n(+)N_m. \)

Case (iv): \( n \) is odd and take \( n = 2k+1, k \geq 2 \)

Define \( f(e_i) = \begin{cases} 1 & \text{if } i = k + 1 \\ 2 & \text{if } i = k \\ -5 & \text{if } i = k - 1 \end{cases} \)

\[ f(e_i) = \begin{cases} 5 & \text{if } i = k + 2 \\ -(2k + 3 - 2i) & \text{if } 1 \leq i \leq k - 2 \\ -2k + 1 + 2i & \text{if } k + 3 \leq i \leq 2k \end{cases} \]

\( f(e_1') = 4, f(e_2') = -4, f(e_1^*) = 6, f(e_2^*) = -6, \) for \( 1 \leq i \leq \frac{m-2}{2} \), \( f(e_{2+i}^{'}) = (2k + 2i + 1), f\left( e_{m+2+i}^{'*} \right) = -(2k + 2i + 1), \) \( f(e_{2+i}^{'}) = (2k + m - 1 + 2i) \) and \( f\left( e_{m+2+i}^{'*} \right) = -(2k + m - 1 + 2i). \)

From the above arguments we get \( f^*(V(P_n(+)N_m)) = \{ \pm 3, \pm 12, \pm 16, \ldots, \pm 4(k + 1), \pm (4k + m + 4), \pm (4k + m + 8), \ldots, \pm (4k + 3m - 4) \} \cup \{ 6 \}. \)

Hence \( f \) is an edge pair sum labeling of \( P_n(+)N_m. \)

Figure 5 shows that \( P_n(+)N_m \) is an edge pair sum graph for \( m = 2 \) and \( n = 6. \)
III. LADDER GRAPH:

Let $L_n = P_n \times P_2$ be a ladder with $V(L_n) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(L_n) = \{u_i u_{i+1}, v_i v_{i+1} : 1 \leq i \leq (n-1)\} \cup \{u_i v_i : 1 \leq i \leq n\}$.

Theorem 3.1: The graph $L_n$ is an edge pair sum graph if $n$ is even.

Proof: Let $V(L_n) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(L_n) = \{e_i = u_i u_{i+1} : 1 \leq i \leq (n-1), e'_i = u_i v_i : 1 \leq i \leq n\}$ are the vertices and edges of the graph $L_n$. Define the edge labeling $f: E(L_n) = \{\pm 1, \pm 2, \ldots, \pm (3n-2)\}$ by considering the following two cases.

Case (i): $n = 2$

Define $f(e_1) = -2$, $f(e'_1) = 1, f(e_2') = -1, f(e_2) = 2$. The induced vertex labeling $f^*(u_1) = f(e'_1) + f(e_1) = -1, f^*(u_2) = f(e_1) + f(e'_2) = -3, f^*(v_1) = f(e'_1) + f(e'_2) = 3$ and $f^*(v_2) = f(e'_1) + f(e'_2) = 1$. From the above arguments we get $f^*(V(L_n)) = \{\pm 1, \pm 3\}$. Hence $f$ is an edge pair sum labeling of $L_n$.

Case (ii) $n = 2k, k \geq 2$

for $1 \leq i \leq \frac{n}{2}-1 \quad f(e_i) = 2 \left(\frac{n}{2} - i\right) + 1, \quad f\left(\frac{e_n}{2}\right) = -2$, for $1 \leq i \leq \frac{n}{2}-1 \quad f\left(\frac{e_{n+i}}{2}\right) = -6i + 2$, for $1 \leq i \leq \frac{n}{2}-1 \quad f\left(e'_i\right) = 6\left(\frac{n}{2} - i\right), \quad f\left(\frac{e'_n}{2}\right) = 1, \quad f\left(\frac{e'_{n+1+i}}{2}\right) = -1, \quad \text{for} \quad 1 \leq i \leq \frac{n}{2}-1 \quad f\left(e''_{n+i}\right) = 2, \quad f\left(\frac{e''_{n}}{2}\right) = 2, \quad f\left(e''_{n+1+i}\right) = -2, \quad \text{for} \quad 1 \leq i \leq \frac{n}{2}-2 \quad f^*(v_n) = f\left(e'_n\right) + f\left(e''_{n-1}\right) = -4n - 7$. From the above labeling we get $f^*(V(L_n)) = \{\pm 2, \pm 7, \pm 10, \pm 14, \pm 14, \pm 24, \pm 34, \ldots, \pm (5n-16), \pm 20, \pm 38, \pm 56, \ldots, \pm (9n-34)\}$. Hence $f$ is an edge pair sum labeling of $L_n$. 

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edge pair sum labeling for $L_n$ if $n$ is even. The example for the edge pair sum graph labeling of $L_n$ for $n = 4$ is shown in Figure 6.

![Figure 6](image-url)

**REFERENCES**


