(Sp)$^*$ Closed Sets in Topological Spaces

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Abstract:
In this paper we introduce a new class of sets namely, (sp)$^*$-closed sets and properties of this set are investigated. We introduce (sp)$^*$-continuous maps and (sp)$^*$-irresolute maps.

Keywords: (sp)$^*$-closed sets, (sp)$^*$-continuous and (sp)$^*$-irresolute.

1. INTRODUCTION:

2. PRELIMINARIES:
Throughout this paper $(X, \tau)$ represents a non-empty topological space on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a topological space $(X, \tau)$, cl(A) and int(A) and $\alpha$ Cl(A) denote the closure, interior and $\alpha$ closure of the subset A.
Definition: 2.1

A subset $A$ of a topological space $(X, \tau)$ is said to be a

1. **pre-closed[14]** if $\text{cl}(\text{int}(A)) \subseteq A$.

2. **semi-closed[10]** if $\text{int}(\text{cl}(A)) \subseteq A$.

3. **semi-pre-closed[1]** if $\text{int}(\text{cl}(\text{Int}(A))) \subseteq A$.

4. **$\alpha$-closed[16]** if $\text{cl}(\text{Int}(\text{cl}(A))) \subseteq A$.

5. **g-closed[9]** if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.

6. **gsp-closed[7]** if $\text{spcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open.

7. **$\alpha g$-closed[11]** if $\alphacl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.

8. **$g\alpha$-closed[12]** if $\alphacl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\alpha$-open in $X$.

9. **sg-closed[5]** if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open in $X$.

10. **gp-closed[13]** if $\text{pcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.

11. **$\alpha^*$-closed[18]** if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\alpha^*$-open in $X$.

12. **gs-closed[3]** if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.

13. **$\omega g$-closed[15]** if $\text{cl}(\text{int}(A)) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.

14. **$\omega g$-closed[17]** if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open in $X$.

Definition: 2.2

A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called a

1. **$\alpha$-continuous[16]** if $f^{-1}(V)$ is $\alpha$-closed in $(X, \tau)$ for every closed set $V$ of $(Y, \sigma)$.

2. **g-continuous[4]** if $f^{-1}(V)$ is g-closed in $(X, \tau)$ for every closed set $V$ of $(Y, \sigma)$.

3. **sg-continuous[5]** if $f^{-1}(V)$ is sg-closed in $(X, \tau)$ for every closed set $V$ of $(Y, \sigma)$. 
4. gs-continuous[6] if \( f^{-1}(V) \) is gs-closed in \((X, \tau)\) for every closed set \( V \) of \((Y, \sigma)\).

5. \( \alpha g \)-continuous[8] if \( f^{-1}(V) \) is \( \alpha g \)-closed in \((X, \tau)\) for every closed set \( V \) of \((Y, \sigma)\).

6. \( \alpha g \)-continuous[12] if \( f^{-1}(V) \) is \( \alpha g \)-closed in \((X, \tau)\) for every closed set \( V \) of \((Y, \sigma)\).

7. gsp-continuous[7] if \( f^{-1}(V) \) is gsp- closed in \((X, \tau)\) for every closed set \( V \) of \((Y, \sigma)\).

8. gp-continuous[2] if \( f^{-1}(V) \) is gp-closed in \((X, \tau)\) for every closed set \( V \) of \((Y, \sigma)\).

9. \( \alpha g \)-continuous[15] if \( f^{-1}(V) \) is \( \alpha g \)-closed in \((X, \tau)\) for every closed set \( V \) of \((Y, \sigma)\).

10. \( \alpha^* \)-continuous[18] if \( f^{-1}(V) \) is \( \alpha^* \)-closed in \((X, \tau)\) for every closed set \( V \) of \((Y, \sigma)\).

11. \( \hat{g} \)-continuous[17] if \( f^{-1}(V) \) is \( \hat{g} \)-closed in \((X, \tau)\) for every closed set \( V \) of \((Y, \sigma)\).

3. Basic Properties of \((sp)^*\)-Closed Sets:

We introduce the following definition.

**Definition 3.01**: A subset \( A \) of a topological space \((X, \tau)\) is said to be \((sp)^*\)-closed if \( cl(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is semi-pre-open in \( X \).

**Theorem 3.02**: Every closed set is \((sp)^*\)-closed.

Proof follows from the definition.

**Theorem 3.03**: Every \((sp)^*\)-closed set is gsp-closed.

**Proof**: Let \( A \) be \((sp)^*\)-closed. Let \( A \subseteq U \) and \( U \) be open. Then \( A \subseteq U \) and \( U \) is semi-pre-open and \( cl(A) \subseteq U \), since \( A \) is \((sp)^*\)-closed. Then \( spcl(A) \subseteq cl(A) \subseteq U \). Therefore \( A \) is gsp-closed.

The converse of the above theorem is not true as seen in the following example.

**Example 3.04**: Let \( X=\{a,b,c\}, \ \tau = \{\emptyset, \{a\}, \{b,c\}, X\} \) \( A=\{a,b\} \) is gsp-closed but not \((sp)^*\)-closed in \((X, \tau)\).

**Theorem 3.05**: Every \((sp)^*\)-closed set is \( g \)-closed.

Proof follows from the definition.
The converse of the above theorem is not true as seen in the following example.

**Example 3.06**: Let $X=\{a, b, c\}$, $\tau = \{\phi, \{a\}, \{b, c\}, X\}$. $A=\{a, c\}$ is g-closed but not (sp)$^*$-closed in $(X, \tau)$

**Theorem 3.07**: Every (sp)$^*$-closed set is gs-closed.

**Proof**: Let $A$ be (sp)$^*$-closed. Let $A \subseteq U$ and $U$ be open. Then $A \subseteq U$ and $U$ is semi-pre-open and $\text{cl}(A) \subseteq U$, since $A$ is (sp)$^*$-closed. Then $\text{scl}(A) \subseteq \text{cl}(A) \subseteq U$. Hence $A$ is (sp)$^*$-closed.

The converse of the above theorem is not true always as seen in the following example.

**Example 3.08**: Let $X=\{a, b, c\}$, $\tau = \{\phi, \{a\}, \{b, c\}, X\}$. $A=\{c\}$ is gs-closed but not a (sp)$^*$-closed set in $(X, \tau)$

**Theorem 3.09**: Every (sp)$^*$-closed set is gp-closed.

**Proof**: Let $A$ be (sp)$^*$-closed. Let $A \subseteq U$ and $U$ be open. Then $A \subseteq U$ and $U$ is semi-pre-open and $\text{cl}(A) \subseteq U$, since $A$ is (sp)$^*$-closed. Then $\text{pcl}(A) \subseteq \text{cl}(A) \subseteq U$. Hence $A$ is gp-closed.

The converse of the above Theorem is not true always as seen in the following example.

**Example 3.10**: Let $X=\{a, b, c\}$, $\tau = \{\phi, \{a\}, \{b, c\}, X\}$. $A=\{a, c\}$ is gp-closed but not (sp)$^*$-closed in $(X, \tau)$.

**Theorem 3.11**: Every (sp)$^*$-closed set is sg-closed.

**Proof**: Let $A$ be (sp)$^*$-closed. Let $A \subseteq U$ and $U$ be semi-pre-open. Then $A \subseteq U$ and $U$ is semi-pre-open and $\text{cl}(A) \subseteq U$ since $A$ is (sp)$^*$-closed. Then $\text{scl}(A) \subseteq \text{cl}(A) \subseteq U$. Hence $A$ is sg-closed.

The converse of the above theorem is not true in general as it can be seen from the following example.

**Example 3.12**: Let $X=\{a, b, c\}$, $\tau = \{\phi, \{a\}, X\}$. $A=\{c\}$ is sg-closed but not (sp)$^*$-closed in $(X, \tau)$.
Theorem 3.13: Every (sp)*-closed set is $\hat{g}$-closed.

Proof follows from the definition.

The converse of the above theorem need not be true in general as it can be seen from the following example.

Example 3.14: Let $X=\{a,b,c\}$, $\tau = \{\phi, \{b, c\}, X\}$. $A=\{a,c\}$ is $\hat{g}$-closed but not (sp)*-closed in $(X, \tau)$

Theorem 3.15: Every (sp)*-closed set is $\alpha g$-closed.

Proof: Let $A$ be (sp)*-closed. Let $A \subseteq U$ and $U$ be open. Then $A \subseteq U$ and $U$ is semi-pre-open and $cl(A) \subseteq U$, since $A$ is (sp)*-closed. Then $\alpha cl(A) \subseteq cl(A) \subseteq U$. Hence $A$ is $\alpha g$-closed.

The following example supports that the converse of the above theorem is not true.

Example 3.16: Let $X=\{a,b,c\}$, $\tau = \{\phi, \{a\}, \{b, c\}, X\}$. $A=\{b\}$ is $\alpha g$-closed but not (sp)*-closed in $(X, \tau)$.

Theorem 3.17: Every (sp)*-closed set is $g\alpha$-closed.

Proof: Let $A$ be (sp)*-closed. Let $A \subseteq U$ and $U$ be open. Then $A \subseteq U$ and $U$ is semi-pre-open and $cl(A) \subseteq U$, since $A$ is (sp)*-closed. Then $\alpha cl(A) \subseteq cl(A) \subseteq U$. Hence $A$ is $g\alpha$-closed.

The converse of the above theorem is not true always as seen in the following example.

Example 3.18: Let $X=\{a,b,c\}$, $\tau = \{\phi, \{a\}, \{b, c\}, X\}$. $A=\{c\}$ is $g\alpha$-closed but not (sp)*-closed in $(X, \tau)$.

Theorem 3.19: Every (sp)*-closed set is $\omega g$-closed.

Proof: Let $A$ be (sp)*-closed. Let $A \subseteq U$ and $U$ be open. Then $A \subseteq U$ and $U$ is semi-pre-open and $cl(A) \subseteq U$, since $A$ is (sp)*-closed. Then $\alpha cl(int(A)) \subseteq cl(A) \subseteq U$. Hence $A$ is $\omega g$-closed.

The converse of the above theorem is not true always as seen in the following example.
Example 3.20: Let $X=\{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$. $A = \{b\}$ is $\omega$-closed but not $(sp)^*$-closed in $(X, \tau)$.

Theorem 3.21: Every $(sp)^*$-closed set is $\alpha^*$-closed.

Proof follows from the definition.

The converse of the above theorem is not true as seen in the following example.

Example 3.22: Let $X=\{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$. $A = \{c\}$ is $\alpha^*$-closed but not $(sp)^*$-closed in $(X, \tau)$.

Theorem 3.23: If $A$ and $B$ are $(sp)^*$-closed, then $A \cup B$ is also $(sp)^*$-closed.

Proof: Let $A$ and $B$ be $(sp)^*$-closed sets. Let $A \cup B$ where $U$ is semi-pre-open.

$cl(A \cup B) = cl(A) \cup cl(B) \subseteq U$. Hence $A \cup B$ is $(sp)^*$-closed.

Theorem 3.24: If $A$ is $(sp)^*$-closed set $\ni A \subseteq B \subseteq cl(A)$ then, $B$ is also $(sp)^*$-closed set.

Proof: Let $A$ be $(sp)^*$-closed set and $A \subseteq B \subseteq cl(A)$. Let $B \subseteq U$ where $U$ is semi-pre-open.

$B \subseteq cl(A)$, $cl(B) \subseteq cl(A) \subseteq U$. Hence $B$ is $(sp)^*$-closed.

Theorem 3.25: $A$ is a $(sp)^*$-closed set of $(X, \tau)$ if and only if $cl(A) \setminus A$ does not contain any non-empty semi-pre-closed set.

Proof: Necessity: Let $F$ be a semi-pre-closed set of $(X, \tau)$ such that $F \subseteq cl(A) \setminus A$. Then $A \subseteq X \setminus F$. $A$ is $(sp)$-closed and $X \setminus F$ is semi-pre-open, $cl(A) \subseteq X \setminus F$. Since $F \subseteq X \setminus cl(A)$.

So, $F \subseteq (X \setminus cl(A)) \cap (cl(A) \setminus A) = \emptyset$, Therefore $F = \emptyset$.

Sufficiency: Let $A$ be a subset of $(X, \tau)$ such that $cl(A) \setminus A$ does not contain any non-empty semi-pre-closed set. Let $U$ be a semi-pre-open set of $(X, \tau)$ such that $A \subseteq U$. If $cl(A) \not\subseteq U$, then $cl(A) \cap U^c \neq \emptyset$ and $cl(A) \cap U^c$ is semi-pre-closed. Therefore $\emptyset \neq cl(A) \cap U^c \subseteq cl(A) \setminus A$. Therefore $cl(A) \setminus A$ contains a non-empty semi-pre-closed set, which is a contradiction. Therefore $cl(A) \subseteq U$. Therefore $A$ is a $(sp)^*$-closed set.
**Theorem 3.26:** If $A$ is both semi-pre-open and $(sp)^*$-closed, then $A$ is closed.

**Proof:** Let $A$ be both semi-pre-open and $(sp)^*$-closed. Let $A \subseteq A$, where $A$ is semi-pre-open. Then $cl(A) \subseteq A$, since $A$ is $(sp)^*$-closed. Therefore $A$ is closed.

The above results can be represented as the following diagram.

![Diagram](image)

where $A \rightarrow B$ represents $A$ implies $B$, but not $B$ implies $A$.

**4.(sp)$^*$-continuous And (sp)$^*$-irresolute Maps**

We introduce the following definition.

**Definition 4.01:** A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called $(sp)^*$-continuous if $f^{-1}(V)$ is a $(sp)^*$-closed set of $(X, \tau)$ for every closed set $V$ of $(Y, \sigma)$. 

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Theorem 4.02: Every continuous map is (sp)*-continuous.

Theorem 4.03: Every (sp)*-continuous map is gsp-continuous.

Proof: Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be (sp)*-continuous. Let $V$ be closed set of $(Y, \sigma)$. Then $f^{-1}(V)$ is a (sp)*-closed, since $f$ is (sp)*-continuous and hence by theorem 3.03, it is gsp-closed in $(X, \tau)$. Therefore $f$ is gsp-continuous.

The converse of the above theorem is not true as seen in the following example.

Example 4.04: Let $X=Y=\{a,b,c\}$, $\tau=\{X, \phi, \{a\}, \{b,c\}\}$, $\sigma=\{Y, \phi, \{b\}\}$. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be defined by an identity mapping. $f^{-1}\{a,c\}=\{a,c\}$ is gsp-closed but not (sp)*-closed.

Theorem 4.05: Every (sp)*-continuous map is g-continuous.

Proof: Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be (sp)*-continuous. Let $V$ be closed set of $(Y, \sigma)$. Then $f^{-1}(V)$ is a (sp)*-closed set of $(X, \tau)$, since $f$ is (sp)*-continuous and hence by theorem 3.5, $f^{-1}(V)$ is g-closed in $(X, \tau)$. Therefore $f$ is g-continuous.

The converse of the above theorem is not true as seen in the following example.

Example 4.06: Let $X=\{a,b,c\}=Y$, $\tau=\{\phi, \{a\}, \{b,c\}, X\}$, $\sigma=\{\phi, \{c\}, Y\}$. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be defined by an identity mapping. $f^{-1}\{a,c\}=\{a,c\}$ is g-closed but not (sp)*-closed.

Theorem 4.07: Every (sp)*-continuous map is gs-continuous.

Proof: Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be (sp)*-continuous. Let $V$ be closed set of $(Y, \sigma)$. Then $f^{-1}(V)$ is a (sp)*-closed set of $X$, since $f$ is (sp)*-continuous and hence by theorem 3.7, $f^{-1}(V)$ is gs-closed in $(X, \tau)$. Therefore $f$ is gs-continuous.

The converse of the above theorem is not true in general as it can be seen in the following example.

Example 4.08: Let $X=\{a,b,c\}=Y$, $\tau=\{\phi, \{a\}, \{b,c\}, X\}$, $\sigma=\{\phi, \{b\}, Y\}$. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be defined by an identity mapping. $f^{-1}\{a,c\}=\{a,c\}$ is gs-closed but not (sp)*-closed.
Theorem 4.09: Every \((sp)^*\)-continuous map is gp-continuous.

Proof: Let \(f: (X, \tau) \rightarrow (Y, \sigma)\) be \((sp)^*\)-continuous. Let \(V\) be closed set of \((Y, \sigma)\). Then \(f^{-1}(V)\) is a \((sp)^*\)-closed set of \((X, \tau)\), since \(f\) is \((sp)^*\)-continuous and hence by theorem-3.9, \(f^{-1}(V)\) is gp-closed in \((X, \tau)\). Therefore \(f\) is gp-continuous.

The following example supports that the converse of the above theorem is not true.

Example 4.10: Let \(X=\{a, b, c\}=Y, \quad \tau = \{\phi, \{a\}, \{b, c\}, X\}, \quad \sigma = \{\phi, \{c\}, Y\}\). Let \(f: (X, \tau) \rightarrow (Y, \sigma)\) be defined by an identity mapping. \(f^{-1}\{a, b\}=\{a, b\}\) is gp-closed but not \((sp)^*\)-closed.

Theorem 4.11: Every \((sp)^*\)-continuous map is sg-continuous.

Proof: Let \(f: (X, \tau) \rightarrow (Y, \sigma)\) be \((sp)^*\)-continuous. Let \(V\) be closed set of \((Y, \sigma)\). Then \(f^{-1}(V)\) is a \((sp)^*\)-closed set of \((X, \tau)\), since \(f\) is \((sp)^*\)-continuous, and hence by theorem-3.11, \(f^{-1}(V)\) is sg-closed in \((X, \tau)\). Therefore \(f\) is sg-continuous.

The converse of the above theorem is not true always as seen in the following example.

Example 4.12: Let \(X=\{a, b, c\}=Y, \quad \tau = \{\phi, \{a\}, \{b, c\}, X\}, \quad \sigma = \{\phi, \{c\}, Y\}\). Let \(f: (X, \tau) \rightarrow (Y, \sigma)\) be defined by an identity mapping. \(f^{-1}\{b\}=\{b\}\) is sg-closed but not \((sp)^*\)-closed.

Theorem 4.13: Every \((sp)^*\)-continuous map is \(\hat{g}\)-continuous.

Proof: Let \(f: (X, \tau) \rightarrow (Y, \sigma)\) be \((sp)^*\)-continuous. Let \(V\) be closed set of \((Y, \sigma)\). Then \(f^{-1}(V)\) is a \((sp)^*\)-closed set of \((X, \tau)\), since \(f\) is \((sp)^*\)-continuous and hence by theorem-3.13, \(f^{-1}(V)\) is \(\hat{g}\)-closed in \((X, \tau)\). Therefore \(f\) is \(\hat{g}\)-continuous.

The following example supports that the converse of the above theorem is not true.

Example 4.14: Let \(X=\{a, b, c\}=Y, \quad \tau = \{\phi, \{b, c\}, X\}, \quad \sigma = \{\phi, \{a\}, Y\}\). Let \(f: (X, \tau) \rightarrow (Y, \sigma)\) be defined by an identity mapping. \(f^{-1}\{b, c\}=\{b, c\}\) is \(\hat{g}\)-closed but not \((sp)^*\)-closed.
Theorem 4.15: Every (sp)*-continuous map is \( \alpha g \)-continuous.

Proof: Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be (sp)*-continuous. Let \( V \) be closed set of \( (Y, \sigma) \). Then \( f^{-1}(V) \) is a (sp)*-closed set of \( (X, \tau) \), since (sp)*-continuous and hence by theorem 3.15, \( f^{-1}(V) \) is \( \alpha g \)-closed in \( (X, \tau) \). Therefore \( f \) is \( \alpha g \)-continuous.

The converse of the above theorem is not true as seen in the following example.

Example 4.16: Let \( X=\{a, b, c\} \) = \( Y \), \( \tau = \{\phi, \{a\}, \{b, c\}, X\} \), \( \sigma = \{\phi, \{c\}, Y\} \).

Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be defined by an identity mapping. \( f^{-1}\{a, b\} = \{a, b\} \) is \( \alpha g \)-closed but not (sp)*-closed.

Theorem 4.17: Every (sp)*-continuous map is \( g \alpha \)-continuous.

Proof: Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be (sp)*-continuous. Let \( V \) be closed set of \( (Y, \sigma) \). Then \( f^{-1}(V) \) is a (sp)*-closed set of \( (X, \tau) \), since (sp)*-continuous and hence by theorem 3.17, \( f^{-1}(V) \) is \( g \alpha \)-closed in \( (X, \tau) \). Therefore \( f \) is \( g \alpha \)-continuous.

The converse of the above theorem is not true in general it can be seen from the following example.

Example 4.18: Let \( X=\{a, b, c\} \) = \( Y \), \( \tau = \{\phi, \{a\}, \{b, c\}, X\} \), \( \sigma = \{\phi, \{c\}, Y\} \).

Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be defined by an identity mapping. \( f^{-1}\{a, b\} = \{a, b\} \) is \( g \alpha \)-closed but not (sp)*-closed.

Theorem 4.19: Every (sp)*-continuous map is \( \omega g \)-continuous.

Proof: Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be (sp)*-continuous. Let \( V \) be closed set of \( (Y, \sigma) \). Then \( f^{-1}(V) \) is a (sp)*-closed set of \( (X, \tau) \), since \( f \) is (sp)*-continuous and hence by theorem 3.19, \( f^{-1}(V) \) is \( \omega g \)-closed in \( (X, \tau) \). Therefore \( f \) is \( \omega g \)-continuous.

The converse of the above theorem is not true always as seen in the following example.

Example 4.20: Let \( X=\{a, b, c\} \) = \( Y \), \( \tau = \{\phi, \{a\}, \{b, c\}, X\} \), \( \sigma = \{\phi, \{c\}, Y\} \).

Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be defined by an identity mapping. \( f^{-1}\{a, b\} = \{a, b\} \) is \( \omega g \)-closed but not (sp)*-closed.
**Theorem 4.21:** Every (sp)$^*$-continuous map is $\alpha^*$-continuous.

**Proof:** Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be (sp)$^*$-continuous. Let $V$ be closed set of $(Y, \sigma)$. Then $f^\dagger(V)$ is a (sp)$^*$-closed set of $(X, \tau)$, since (sp)$^*$-continuous and hence by theorem-3.21, $f^\dagger(V)$ is $\alpha^*$-closed in $(X, \tau)$. Therefore $f$ is $\alpha^*$-continuous.

The converse of the above theorem is not true in general it can be seen from the following example.

**Example 4.22:** Let $X=\{a,b,c\}=Y$, $\tau = \{\phi, \{a\}, \{b,c\}, X\}$, $\sigma = \{\phi, \{c\}, Y\}$.

Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be defined by an identity mapping. $f^\dagger\{a,b\}=\{a\}$ is $\alpha^*$-closed but not (sp)$^*$-closed.

**Definition 4.23:** A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called (sp)$^*$-irresolute if $f^\dagger(V)$ is a (sp)$^*$-closed set of $(X, \tau)$ for every (sp)$^*$-closed set $V$ of $(Y, \sigma)$.

**Theorem 4.24:** Every (sp)$^*$-irresolute function is (sp)$^*$-continuous.

The converse of the above theorem is not true as seen in the following example.

**Example 4.25:** Let $X=\{a,b,c\}=Y$, $\tau = \{\phi, \{a\}, \{b,c\}, X\}$, $\sigma = \{\phi, \{a,c\}, Y\}$.

Define $f: (X, \tau) \rightarrow (Y, \sigma)$ by $f(a)=c$, $f(b)=a$ and $f(c)=b$. $f^\dagger\{b\}=\{a\}$ is (sp)-closed in $(X, \tau)$. Therefore $f$ is (sp)$^*$-continuous. $\{b,c\}$ is (sp)$^*$-closed in $Y$. $f^\dagger\{b,c\}=\{a,b\}$ is not (sp)$^*$-closed in $(X, \tau)$. Therefore $f$ is not (sp)$^*$-irresolute.

**Theorem 4.26:** Every (sp)$^*$-irresolute function is gsp-continuous.

The converse of the above theorem is not true as seen in the following example.

**Example 4.27:** Let $X=\{a,b,c\}=Y$, $\tau = \{\phi, \{a\}, \{b,c\}, X\}$, $\sigma = \{\phi, \{a\}, Y\}$.

Define $f: (X, \tau) \rightarrow (Y, \sigma)$ by $f(a)=c$, $f(b)=b$ and $f(c)=a$. $f^\dagger\{b,c\}=\{c\}$ is gsp-closed in $(X, \tau)$. Therefore $f$ is gsp-continuous. $\{b,c\}$ is (sp)$^*$-closed in $Y$. $f^\dagger\{b,c\}=\{a,b\}$ is not (sp)$^*$-closed in $(X, \tau)$. Hence $f$ is not (sp)$^*$-irresolute.

**Theorem 4.28:** Every (sp)$^*$-irresolute function is g-continuous.

The converse of the above theorem is not true as seen in the following example.
Example 4.29: Let \( X=\{a,b,c\}=Y, \ \tau = \{\phi, \{a\}, \{b,c\}, X \} \) \( \sigma = \{\phi, \{a\}, Y\} \).

Define \( f: (X, \tau) \to (Y, \sigma) \) by \( f(a)=c, f(b)=b \) and \( f(c)=a \). \( f^{-1}\{b,c\}=\{c,a\} \) is g-closed in \( (X, \tau) \). Therefore \( f \) is \( g \)-continuous. \( \{b,c\} \) is \( (sp)^* \)-closed set in \( Y \). \( f^{-1}\{b,c\}=\{a,b\} \) is not \( (sp)^* \)-closed in \( (X, \tau) \). Hence \( f \) is not \( (sp)^* \)-irresolute.

Theorem 4.30: Every \( (sp)^* \)-irresolute function is \( gs \)-continuous.

The following example supports that the converse of the above theorem is not true always.

Example 4.31: Let \( X=\{a,b,c\}=Y, \ \tau = \{\phi, \{a\}, \{b,c\}, X \} \) \( \sigma = \{\phi, \{a\}, Y\} \).

Define \( f: (X, \tau) \to (Y, \sigma) \) by \( f(a)=c, f(b)=a \) and \( f(c)=b \). \( f^{-1}\{b,c\}=\{a,b\} \) is gs-closed in \( (X, \tau) \).

Therefore \( f \) is gs-continuous. \( \{b,c\} \) is \( (sp)^* \)-closed set in \( Y \). \( f^{-1}\{b,c\}=\{a,b\} \) is not \( (sp)^* \)-closed in \( (X, \tau) \). Hence \( f \) is not \( (sp)^* \)-irresolute.

Theorem 4.32: Every \( (sp)^* \)-irresolute function is \( gp \)-continuous.

The converse of the above Theorem is not true always as seen in the following example.

Example 4.33: Let \( X=\{a,b,c\}=Y, \ \tau = \{\phi, \{a\}, \{b,c\}, X \} \) \( \sigma = \{\phi, \{a\}, Y\} \).

Define \( f: (X, \tau) \to (Y, \sigma) \) by \( f(a)=b, f(b)=a \) and \( f(c)=c \). \( f^{-1}\{b,c\}=\{a,c\} \) is gp-closed in \( (X, \tau) \).

Therefore \( f \) is gp-continuous. \( \{b,c\} \) is \( (sp)^* \)-closed set in \( Y \). \( f^{-1}\{b,c\}=\{a,c\} \) is not \( (sp)^* \)-closed in \( (X, \tau) \). Hence \( f \) is not \( (sp)^* \)-irresolute.

Theorem 4.34: Every \( (sp)^* \)-irresolute function is \( sg \)-continuous.

The converse of the above theorem is not true as seen in the following example.

Example 4.35: Let \( X=\{a,b,c\}=Y, \ \tau = \{\phi, \{a\}, X \} \) \( \sigma = \{\phi, \{b,c\}, Y\} \).

Define \( f: (X, \tau) \to (Y, \sigma) \) by \( f(a)=b, f(b)=c \) and \( f(c)=a \). \( f^{-1}\{a\}=\{b\} \) is sg-closed in \( (X, \tau) \).

Therefore \( f \) is sg-continuous. \( \{a\} \) is \( (sp)^* \)-closed set in \( Y \). \( f^{-1}\{a\}=\{b\} \) is not \( (sp)^* \)-closed in \( (X, \tau) \). Hence \( f \) is not \( (sp)^* \)-irresolute.

Theorem 4.36: Every \( (sp)^* \)-irresolute function is \( g^* \)-continuous.

The converse of the above theorem is not true as seen in the following example.
Example 4.37: Let $X=\{a,b,c\}=Y$, $\tau = \{\emptyset, \{b,c\}, X\}$ $\sigma = \{\emptyset, \{a\}, \{b,c\}, Y\}$.

Define $f: (X, \tau) \rightarrow (Y, \sigma)$ by $f(a)=a$, $f(b)=c$ and $f(c)=b$. $f^{-1}\{a\}=\{a\}$ is $g$-closed in $(X, \tau )$.
Therefore $f$ is $g$-continuous. $\{b,c\}$ is $(sp)^*$-closed sets in $Y$. $f^{-1}\{b,c\}=\{c,b\}=\{b,c\}$ is not $(sp)^*$-closed in $(X, \tau )$. Hence $f$ is not $(sp)^*$-irresolute.

Theorem 4.38: Every $(sp)^*$-irresolute function is $\alpha g$-continuous.

The converse of the above theorem is not true as seen in the following example.

Example 4.39: Let $X=\{a,b,c\}=Y$, $\tau = \{\emptyset, \{a\}, \{b,c\}, X\}$ $\sigma = \{\emptyset, \{a\}, Y\}$.

Define $f: (X, \tau) \rightarrow (Y, \sigma)$ by $f(a)=c$, $f(b)=a$ and $f(c)=b$. $f^{-1}\{b,c\}=\{a,b\}$ is $\alpha g$-closed in $(X, \tau )$.
Therefore $f$ is $\alpha g$-continuous. $\{b,c\}$ is $(sp)^*$-closed sets in $Y$. $f^{-1}\{b,c\}=\{a,b\}$ is not $(sp)^*$-closed in $(X, \tau )$. Hence $f$ is not $(sp)^*$-irresolute.

Theorem 4.40: Every $(sp)^*$-irresolute function is $g\alpha$-continuous.

The converse of the above theorem is not true as seen in the following example.

Example 4.41: Let $X=\{a,b,c\}=Y$, $\tau = \{\emptyset, \{a\}, \{b,c\}, X\}$ $\sigma = \{\emptyset, \{a\}, Y\}$.

Define $f: (X, \tau) \rightarrow (Y, \sigma)$ by $f(a)=c$, $f(b)=b$ and $f(c)=a$. $f^{-1}\{b,c\}=\{c,a\} = \{a,c\}$ is $g\alpha$-closed in $(X, \tau )$.
Therefore $f$ is $g\alpha$-continuous. $\{b,c\}$ is $(sp)^*$-closed set in $Y$. $f^{-1}\{b,c\}=\{a,b\}$ is not $(sp)^*$-closed in $(X, \tau )$. Hence $f$ is not $(sp)^*$-irresolute.

Theorem 4.42: Every $(sp)^*$-irresolute function is $\alpha g$-continuous.

The converse of the above theorem is not true as seen in the following example.

Example 4.43: Let $X=\{a,b,c\}=Y$, $\tau = \{\emptyset, \{a\}, \{b,c\}, X\}$ $\sigma = \{\emptyset, \{b,c\}, Y\}$.

Define $f: (X, \tau) \rightarrow (Y, \sigma)$ by $f(a)=b$, $f(b)=b$ and $f(c)=a$. $f^{-1}\{a\}=\{b\}$ is $\alpha g$-closed in $(X, \tau )$.
Therefore $f$ is $\alpha g$-continuous. $\{a\}$ is $(sp)^*$-closed sets in $Y$. $f^{-1}\{a\}=\{b\}$ is not $(sp)^*$-closed in $(X, \tau )$. Hence $f$ is not $(sp)^*$-irresolute.

Theorem 4.44: Every $(sp)^*$-irresolute function is $\alpha^*$-continuous.

The following example supports that the converse of the above theorem is not true.
Example 4.45: Let \( X=\{a,b,c\}=Y, \tau = \{ \phi, \{a\}, \{b,c\}, X \} \sigma = \{ \phi, \{a\}, Y \} \).

Define \( f: (X, \tau) \to (Y, \sigma) \) by \( f(a)=b, f(b)=a \) and \( f(c)=c \). \( \tau^{-1}\{b,c\} = \{a,c\} \) is g-closed in \( (X, \tau) \).

Therefore \( f \) is \( \alpha^* \)-continuous. \( \{b,c\} \) is \( (sp)^* \)-closed sets in \( Y \). \( \tau^{-1}\{b,c\} = \{a,c\} \) is not \( (sp)^* \)-closed in \( (X, \tau) \). Hence \( f \) is not \( (sp)^* \)-irresolute.

REFERENCES


