β# - I- Continuous Multifunctions

Adiya K. Hussein
Department of Mathematics, College of Basic Education
Al- Mustansiriya University, Iraq

Abstract. In this paper, we study a new type of continuity of multifunctions by using β# − I-open sets. Several characterizations and some properties of these multifunctions are obtained. Relationships with other kinds of I− continuity of multifunctions are investigated.

Keywords: β − open set, β# − I-open set, Multifunctions, β# − I-continuous multifunctions,

I. INTRODUCTION

In [1] C. Berge introduced the theory of multifunctions. A multifunction is a set-valued function. The concept of multifunctions has applications in functional analysis and fixed point theory. The notion of ideal in topological space was first introduced by Kuratowski[2] and Vaidyanathswamy [3]. In 1990, D. Janković and T. R. Hamlett [4] introduced the notion of I−open sets in topological spaces. In 1992, Abd El-Monsef et al. [5] further investigated I−open sets and I−continuous functions. Dontchev [6] introduced the concept of pre-I−open sets and obtained a decomposition of I−continuity. Hatir and Noiri [7] introduced the notion of semi−I−open sets and α−I−open sets to obtain decomposition of continuity. In [8] the notion of weakly semi-I-open (Which we called β#-I-open) sets was introduced by Hatir and Jafari. Akdag [9] introduced and study the I−continuous multifunction. In [10], the concepts of upper (lower) α−I−continuous multifunctions on ideal topological spaces are studied. The notion of semi−I−continuous multifunctions was studied in [11]. In [12] Adiya introduced and study the concepts of upper (lower) β − I−continuous multifunctions on ideal topological spaces. In the present paper, we introduce and study the concepts of β# − I−continuous multifunctions on ideal topological spaces. Some characterizations and properties are obtained. Also, we investigate its relationships with other types of I−continuities of multifunctions.

II. PRELIMINARIES

An ideal is a nonempty collection I of subsets of X satisfying the following two conditions:

(1) A ∈ I and B ⊆ A implies B ∈ I. (2) If A ∈ I and B ∈ I, then A ∪ B ∈ I.

An ideal topological space is a topological space (X, τ) with an ideal I on X and is denoted by (X, τ, I). An operator (·)∗ : P(X) → P(X), is called the local function[6] of I on X with respect to τ and I is defined as follows:

For A ⊆ X, A∗ (I, τ) = {x ∈ X : G ∈ I(I) A ∈ I and B ⊆ A implies B ∈ I(2) If A ∈ I and B ∈ I, then A ∪ B ∈ I. An ideal topological space is a topological space (X, τ) with an ideal I on X and is denoted by (X, τ, I). An operator (·)∗ : P(X) → P(X), is called the local function[6] of I on X with respect to τ and I is defined as follows:

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For A ⊆ X, A∗ (I, τ) = {x ∈ X : G ∈ I(2) If A ∈ I and B ∈ I, then A ∪ B ∈ I. Moreover, cl∗ (A) = A ∪ A∗ defines a Kuratowski closure operator for a topology τ∗ (I, τ) which is finer than τ. For any ideal space (X, τ, I), the collection {U \ G : U ∈ τ, G ∈ I} is a base for τ∗ (I, τ).

A multifunction of a set X into Y will be denoted by F: X ↦ Y. For a multifunction F, the upper and lower inverse set of a set B of Y will be denoted by F+(B) and F−(B) respectively that is F+(B) = {x ∈ X : F(x) ⊆ B} and F−(B) = {x ∈ X : F(x) ∩ B = φ} . A multifunction F: X ↦ Y is said to be upper semi continuous (briefly u. s. c) at a point x ∈ X if for each open set V in Y with F(x) ⊆ V, there exists an open set U containing x such that F(U) ⊆ V; lower semi continuous (briefly l. s. c) at a point x ∈ X if for each open set V in Y with F(x) ∩ V = φ; there exists an open set U containing x such that F(z) ∩ V = φ for every z ∈ U [13]. Throughout this paper, A∗ denote the complement of A. Spaces X and Y mean topological spaces and int(A) and cl(A) denote the interior and closure of A respectively.
III. $\beta^* - I$–Continuity of Multifunctions

**Definition 3.1** A subset $A$ of an ideal topological space $(X, \tau, I)$ is said to be

1. $I$–open if $A \subseteq \text{int} \left( A^* \right)$[5]
2. $\alpha - I$–open if $A \subseteq \text{Int} (\text{Cl}^I (\text{int} (A)))$[7]
3. $\beta - I$–open if $A \subseteq \text{Int} (\text{Cl} (\text{int} (A)))$[16]
4. $\text{pre} - I$–open if $A \subseteq \text{Int} (\text{Cl}^I (A))$[6]
5. $\text{semi} - I$–open if $A \subseteq \text{Cl}^I (\text{int} (A))$[7]
6. $\text{pre} -$open if $A \subseteq \text{Int} (\text{Cl} (A))$[14]
7. $\text{semi} -I$–open if $A \subseteq \text{Int} (\text{Cl} (A))$[15]
8. $\beta^*$–$I$– open (or weakly semi-$I$– open relative to $[8]$) set if $A \subseteq \text{Cl}^I (\text{Int} (\text{Cl}(A)))$

Using [12] and [8, Remarks 2.1 and 2.2], we have the following diagram,

$$\text{open} \Rightarrow \alpha - I \text{– open} \Rightarrow \text{semi} - I \text{– open} \Rightarrow \text{semi} - \text{open} \Rightarrow \beta - I \text{– open}$$

$$I - \text{open} \Rightarrow \text{pre} - I \text{– open} \Rightarrow \text{pre} - \text{open} \Rightarrow \beta^* - I \text{– open} \Rightarrow \beta - \text{open}$$

We introduce the following definition.

**Definition 3.2** A multifunction $F : (X, \tau, I) \rightarrow (Y, \sigma)$ is said to be upper (resp. lower) $\beta^*$–$I$–continuous iff for each $x \in X$ and each open set $V$ in $Y$ with $F(x) \subseteq V$ (resp. $F(x) \cap V \neq \emptyset$), there exists a $\beta^*$–$I$–open set $U$ containing $x$ such that $F(U) = \bigcup \{ F(u) : u \in U \} \subseteq V$ (resp. if $u \in U$, then $F(u) \cap V \neq \emptyset$). We say that $F$ is $\beta^*$–$I$–continuous if it is upper and lower $\beta^*$–$I$–continuous.

Now we introduce the following characterizations.

**Theorem 3.3** Let $F : (X, \tau, I) \rightarrow (Y, \sigma)$ be a multifunction, then the following statements are equivalent:

1. $F$ is upper (resp. lower) $\beta^*$–$I$–continuous.
2. For each $x \in X$ and each open set $V$ in $Y$ with $x \in F^*(V)$ (resp. $x \in F^{-*} (V)$), there exists a $\beta^*$–$I$–open set $U$ containing $x$ such that $U \subseteq F^*(V)$ (resp. $U \subseteq F^{-*} (V)$).
3. For every open set $V$ in $Y$, $F^*(V)$ (resp. $F^{-*} (V)$) is a $\beta^*$–$I$–open set in $X$.
4. For every closed set $V$ in $Y$, $F^*(V)$ (resp. $F^{-*} (V)$) is a $\beta^*$–$I$–closed set in $X$.
5. $\text{int}^I (\text{Cl} (\text{int} ( F^{-*} (V)))) \subseteq F^{-*} (\text{Cl}(V))$ (resp. $\text{int}^I (\text{Cl} (\text{int} ( F^*(V)))) \subseteq F^*(\text{Cl}(V))$ for any subset $V$ of $Y$.
6. $F(\text{int}^I (\text{Cl} (\text{int}(U)))) \subseteq \text{Cl} ( F(U))$, for each subset $U$ of $X$.

**Proof.**

1. $\Rightarrow$ (2): Let $x \in X$ and $V$ be any open set in $Y$ with $x \in F^*(V)$ (resp. $x \in F^{-*} (V)$). Then $F(x) \subseteq V$ (resp. $F(x) \cap V \neq \emptyset$). Since $F$ is upper (lower) $\beta^*$–$I$–continuous, there exists a $\beta^*$–$I$–open set $U$ containing $x$ such that $F(U) \subseteq V$ (resp. if $u \in U$, then $F(u) \cap V \neq \emptyset$). Thus $U \subseteq F^*(V)$ (resp. $U \subseteq F^{-*} (V)$).
2. $\Rightarrow$ (3): Let $V$ be any open set in $Y$ and let $x \in F^*(V)$ (resp. $x \in V$). Then by (2), there exists a $\beta^*$–$I$–open set $U$ containing $x$ such that $F(U) \subseteq V$ (resp. if $u \in U$, then $F(u) \cap V \neq \emptyset$). Since the union of $\beta^*$–$I$–open sets is a $\beta^*$–$I$–open set [8], $F^*(V) = \bigcup U_i$ (resp. $F^{-*} (V) = \bigcup U_i$) is a $\beta^*$–$I$–open set in $X$.
3. $\Rightarrow$ (4): Let $V$ be a closed set in $Y$. Hence $Y - V$ is an open set in $Y$. Then by (3), $F^*(Y - V) = X - F^{-*} (V)$ (resp. $F^{-*} (Y - V) = X - F^*(V)$) is a $\beta^*$–$I$–open set in $X$. So $F(V)$ (resp. $F^{-*} (V)$) is a $\beta^*$–$I$–closed set in $X$.
4. $\Rightarrow$ (5): Let $V \subseteq$ any subset of $Y$. Since $\text{Cl} (V)$ is closed set in $Y$. By (4), $( F^{-*} (\text{Cl}(V)))$ (resp. $F(\text{Cl} (V))$) is $\beta^*$–$I$–closed set in $X$. Thus $F^{-*} (\text{Cl}(V)))$ is a $\beta^*$–$I$–closed set in $X$. Therefore $( F^{-*} (\text{Cl}(V)))^{\beta^*} \subseteq \text{int}^I (\text{Cl}(F^{-*} (\text{Cl}(V))))^{\beta^*} = ((\text{int} (\text{Cl}(F^{-*} (\text{Cl}(V)))))^{\beta^*}$
Hence $F^{-}(V) = (\text{int}(F^{-}(V)))^{\circ}$. Hence $\text{int}(F^{-}(V)) \subseteq F^{-}(V)$.

$(5) \Rightarrow (6)$: Let $U$ be any subset of $X$. By $(5)$, we have $\text{int}(\text{Cl}(\text{int}(U))) \subseteq \text{int}(\text{Cl}(\text{F}(U))) \subseteq (F^{-}\text{Cl}(F(U)))(\text{resp. } \text{int}(\text{Cl}(\text{int}(U))) \subseteq \text{int}(\text{Cl}(\text{F}(\text{F}(U)))) \subseteq F^{-}(\text{Cl}(F(U)))$.

$(6) \Rightarrow (1)$: Let $V$ be any open subset of $Y$. Then by $(6)$, $F(\text{int}(\text{Cl}(\text{int}(V)))) \subseteq F(\text{F}(\text{F}(V)))) \subseteq \text{Cl}(\text{F}(V))$. So $\text{int}(\text{Cl}(\text{int}(V)))) \subseteq (F^{-}(V))^{\circ}$. Hence $F^{-}(V) \subseteq \text{Cl}(\text{F}(V)))$.

**Remark 3.5** The converses of the above implications is not true in general, for the converse of implication $(2)$ is not true in general.

**Theorem 3.6** Let $F: (X, \tau, I) \rightarrow (Y, \sigma, J)$ be a multifunction and $\{U_{\lambda} : \lambda \in \Delta\}$ be an open cover of $X$. If the restriction functions $F_{\lambda_{\lambda}}$ is upper $\beta^{*}$ -- I--continuous for each for each $\lambda \in \Delta$, then $F$ is upper $\beta^{*}$ -- I--continuous.

**Proof.** Let $V$ be any open subset of $Y$. Since $F_{\lambda_{\lambda}}$ is upper $\beta^{*}$ -- I--continuous for each $\lambda \in \Delta$, hence
In this section we provide some applications on $\beta^*-I$-continuous multifunctions.

**Definition 4.1** [18] An ideal topological space $(\mathbb{X}, \tau, I)$ is said to be $I$-compact if for every $I$-open cover \[\{W_{\lambda} : \lambda \in \Delta\}\], there exists a finite subset $\Delta_0$ of $\Delta$ such that $(X - \cup \{W_{\lambda} : \lambda \in \Delta_0\}) \in I$.

We introduce the following definition.

**Definition 4.2** An ideal topological space $(\mathbb{X}, \tau, I)$ is said to be $\beta^*-I$-compact if for every $\beta^*-I$-open cover \[\{W_{\lambda} : \lambda \in \Delta\}\] of $X$, there exists a finite subset $\Delta_0$ of $\Delta$ such that $(X - \cup \{W_{\lambda} : \lambda \in \Delta_0\}) \in I$.

**Lemma 4.3.** [9] For any surjective multifunction $F : (\mathbb{X}, \tau, I) \mapsto (Y, \sigma)$, $F(I)$ is an ideal on $Y$.

**Theorem 4.4.** Let $(\mathbb{X}, \tau, I)$ be $\beta^*-I$-compact space and $F : (\mathbb{X}, \tau, I) \mapsto (Y, \sigma)$ is upper $\beta^*-I$-continuous surjection. Then $(Y, \sigma)$ is $F(I)$-compact.

**Proof.** Let $F : X \mapsto Y$ be a upper $\beta^*-I$-continuous surjection and \[\{V_{\lambda} : \lambda \in \Delta\}\] be an open cover of $Y$. Then \[\{F^+(V_{\lambda}) : \lambda \in \Delta\}\] is a $\beta^*-I$-open cover of $X$. Since $X$ is $\beta^*-I$-compact, there exists a finite subset $\Delta_0$ of $\Delta$ such that $(X - \cup \{F^+(V_{\lambda}) : \lambda \in \Delta_0\}) \in I$. Therefore by lemma 4.3, $F(X - \cup \{F^+(V_{\lambda}) : \lambda \in \Delta_0\}) = (Y - \cup \{V_{\lambda} : \lambda \in \Delta_0\}) \in F(I)$. Hence $(Y, \sigma, F(I))$ is $F(I)$-compact.

**Definition 4.5** An ideal topological space $(\mathbb{X}, \tau, I)$ is called $\beta^*-I$-Hausdorff if for each two distinct points $x \neq y$ there exists disjoint $\beta^*-I$-open sets $U$ and $V$ containing $x$ and $y$ respectively. Then we say that $x$ and $y$ are $\beta^*-I$-separated.

**Theorem 4.6** Let $F : (\mathbb{X}, \tau, I) \mapsto (Y, \sigma, J)$ be upper $\beta^*-I$- continuous multifunction such that $F(x)$ is closed for each $x \in X$. If $Y$ is normal space then $X$ is $\beta^*-I$-Hausdorff where $F(x) \cap F(y) = \emptyset$ for each distinct $x, y \in X$.

**Proof.** Let $x, y \in X$ be distinct. Then $F(x) \cap F(y) = \emptyset$. Since $Y$ is normal space then there exist distinct open sets $U$ and $V$ containing $F(x)$ and $F(y)$ respectively. Thus $F^+(U)$ and $F^+(V)$ are disjoint $\beta^*-I$-open sets containing $x$ and $y$ respectively. Then $X$ is $\beta^*-I$-Hausdorff.

**Definition 4.7** A multifunction $F : (\mathbb{X}, \tau, I) \mapsto (Y, \sigma, J)$ is said to be
(a) upper $\beta^*-I$-irresolute if $F^+(V)$ is $\beta^*-I$-open in $X$ for each $\beta^*-I$-open set $V$ in $Y$.
(b) lower $\beta^*-I$-irresolute if $F^-(V)$ is $\beta^*-I$-open in $X$ for each $\beta^*-I$-open set $V$ in $Y$.
(c) $\beta^*-I$-irresolute if $F$ is upper $\beta^*-I$-irresolute and lower $\beta^*-I$-irresolute.
**Theorem 4.8** (1) If $F$ is upper $\beta^* - I$-irresolute multifunction then $F$ is upper $\beta^* - I$-continuous multifunction.

(2) If $F$ is lower $\beta^* - I$-irresolute multifunction then $F$ is lower $\beta^* - I$-continuous multifunction.

**Proof.** Obvious, since any open set is $\beta^* - I$-open set.

**Theorem 4.9** Let $F : (X, \tau, I) \mapsto (Y, \sigma, J)$ be upper $\beta^* - I$-irresolute multifunction and $G : (Y, \sigma, J) \mapsto (Z, \mu)$ be upper $\beta^* - I$-continuous multifunction then $(G \circ F)$ is upper $\beta^* - I$-continuous multifunction.

**Proof.** Let $V$ be any open set in $Z$. Since $G$ is upper $\beta^* - I$-continuous. Then $G^+(V)$ is $\beta^* - J$-open in $Y$. Since $F$ is upper $\beta^* - I$-irresolute, then $F^+(G^+(V)) = ((G \circ F)^+)(V)$ is $\beta^* - I$-open in $X$. Thus $G \circ F$ is upper $\beta^* - I$-continuous.

**References**


