An Excursion Through Some Characterizations of Hypersemigroups By Normal Hyperideals

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Abstract — *In this paper, we introduce normal hyperideal and bi-hyperideal in normal hypersemigroups. We study (normal)hypersemigroup and normal regular hypersemigroup based on bi-hyperideal proving some equivalent conditions. In particular, we prove, among the other results, that if I_1, I_2 are any two normal hyperideals of a hypersemigroup H, then their product $I_1 \circ I_2$ and $I_2 \circ I_1$ are also normal hyperideals of S and $I_1 \circ I_2 = I_2 \circ I_1$. We also prove that the minimal normal hyperideal of a hypersemigroup H is a hypergroup.*

Keywords — *hypersemigroup, bihyperideal, normal hyperideal, bi-hyperideal*

I. INTRODUCTION AND PRELIMINARIES

Ideals play an important role in higher studies and applications of algebraic structures. Generalization of ideals in various algebraic structures is necessary for further development of algebraic structures. Lots of mathematicians obtained significanly large number of results and characterizations of algebraic structures by applying this notion and the properties of generalization of ideals in different algebraic structures as is clear from the vast literature available on the subject matter [2], [3], [16], [17], [18], [19], [20], [21]. It is a well known fact that the concept of one sided ideal of any algebraic structure is a generalization of concept of an ideal. The quasi ideals are generalization of left ideal and right ideal whereas the bi-ideals are generalization of quasi ideals. The notion of bi-ideals was introduced by Good and Hughes [27] for semigroups. The concept of bi-ideals in rings and semigroups were introduced by Lajos and Szasz [31], [33]. Lajos introduced bi-ideals in semigroups [29], [30].

The applications of hyperstructures in various disciplines, for example in informatics, is the prime motivation and purpose that the mathematicians and algebraists of all over the world are active in the study and continuous enrichment of the theory. In other algebraic structures, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Recently, a book by eminent algebraist Davvaz [9] titled "Semihypergroup Theory" is the first book devoted to the semihypergroup theory that didcusses fundamental results related to semigroup theory and algebraic hyperstructures highlighting the most general algebraic context in which reality can be modeled. Lots of books and research articles have been written on different branches of hyperstructure [11], [12], [14], [15], [22], [23], [25], [26], [28], [34]. Bonansinga and Corsini [24], Davvaz [11], Hila et al. [15], Salvo et al. [22] and other algebraists deeply studied the theory of hyperstructures. Recently, Basar et al. investigated some results in hyperstrcutures [1], [4], [5], [6], [7], [8].

Let H be a nonempty set, then the mapping $\circ: H \times H \rightarrow H$ is called hyperoperation or join operation on H, where $P^*(H) = P(H)\backslash\{0\}$ is the set of all nonempty subsets of H. Let A and B be two nonempty sets. Then, a hypergroupoid $(H,\circ)$ is called a semihypergroups if for every $x, y, z \in H$

$$x \circ (y \circ z) = (x \circ y) \circ z.$$

For subsets A, B of semihypergroup H, the product set $A \circ B$ of the pair (A, B) relative to H is defined as $A \circ B = \{a \circ b: a \in A, b \in B\}$ and for $A \subseteq H$, the product set $A \circ A$ relative to H is defined as $A^2 = A \circ A = A \circ A$. Note that A^0 is an identity operator. That is, $A^0 \circ H = H = H \circ A^0$. Also, $(A] = \{h \in H: h \leq a forsome a \in A\}$.

If there is no ambiguity, we identify hypersemigroup $(H,\circ)$ by H. A sub-hypersemigroup T of a hypersemigroup H is called normal if $h \circ T = T \circ h$ for all $h \in H$. A hypersemigroup H is called left (right) regular if for every element $h \in H$, there exists an element $a \in H$ such that $\{h\} = a \circ h \circ h \circ (\{h\} = h \circ h \circ a)$. A hypersemigroup H is called intra-regular if for any $h \in H$, there exist elements $a, b \in H$ such that $\{h\} = a \circ h \circ h \circ b$. A hypersemigroup H is called completely regular if for any element $h \in H$, there exists an element $a \in H$ such that $\{h\} = h \circ a$ and $h \circ a = a \circ h$. We denote and define the principal left hyperideal, right hyperideal and bi-hyperideal of H generated by $h \in H$ as follows: $(h)_L = h \cup H \circ h$, $(h)_R = h \cup h \circ H$, $(h)_B = h \cup h^2 \cup h \circ H \circ h$. An element a of H is called a zero element of H if $h \circ a = a \circ h = \{a\}$ for $h \in H$.

Notation 1. We denote by $\mathscr{B}(H)$ the set of all bi-hyperideals of H and $\mathfrak{B}(H)$ the set of nonempty subsets of the hypersemigroup H.

Definition 1.1 *A hypersemigroup H is called normal if $h \circ H = H \circ h$ for all $h \in H$.*

Definition 1.2 *A hyperideal of a hypersemigroup is called normal if $h \circ I = I$ for all $h \in$. H..*

II. HYPERIDEALS OF NORMAL HYPERSEMIGROUPS

We study this part by investigating some fundamental results based on (normal) hypersemigroups and thereafter, characterizing the (normal) hypersemigroups through (normal) hyperideals and bi-hyperideals.

PROPOSITION 2.1 Suppose I is any hyperideal of a hypersemigroup H. Then, we have the following:

(i). $I \circ (h)_B = I \circ (h)_L = I \circ h$ for all $h \in H$,

(ii). $(h)_B \circ I = (h)_R \circ I = h \circ I$ for all $h \in H$.

Proof. Suppose $h \in H$. Then, we have the following:

$$I \circ (h)_L = I \circ (h \cup H \circ h)$$

$$= I \circ h \cup I \circ (H \circ h)$$

$$= I \circ h \cup (I \circ H) \circ h$$

$$\subseteq I \circ h \subseteq I \circ (h)_L,$$

and

$$I \circ (h)_B = I \circ (h \cup h^2 \cup h \circ H \circ h)$$

$$= I \circ h \cup I \circ h^2 \cup I \circ (h \circ H \circ h)$$

$$= I \circ h \cup (I \circ h) \circ h \cup (I \circ h \circ H) \circ h$$

$$\subseteq I \circ h \subseteq I \circ (h)_B.$$

So, $I \circ (h)_B = I \circ (h)_L = I \circ h$ for all $h \in H$.

In a similar way, we can prove that

$$(h)_B \circ I = (h)_R \circ I = h \circ I \text{ for all } h \in H.$$

THEOREM 2.1 The following assertions are equivalent for a hyperideal I of H:

(i). I is normal;

(ii). $Y \circ I = I \circ Y$ for all $Y \in \mathcal{B}(H)$;

(iii). $(h)_B \circ I = I \circ (h)_B$ for all $h \in H$;

(iv). $(h)_B \circ I = I \circ (h)_L$ for all $h \in H$;

(v). $(h)_B \circ I = I \circ h$ for all $h \in H$;

(vi). $(h)_R \circ I = I \circ (h)_B$ for all $h \in H$;

(vii). $(h)_R \circ I = I \circ (h)_L$ for all $h \in H$;

(viii). $(h)_R \circ I = I \circ h$ for all $h \in H$;

(ix). $h \circ I = I \circ (h)_B$ for all $h \in H$;

(x). $h \circ I = I \circ (h)_L$ for all $h \in H$.

PROOF. $(i) \Rightarrow (ii)$. Let I be normal. Suppose Y is any nonempty subset of H and for $y \in Y, x \in I$, $y \circ x \subseteq Y \circ I$. Then, we obtain that $y \circ x \subseteq y \circ I = I \circ y \subseteq I \circ Y$, and therefore, $Y \circ I \subseteq I \circ Y$. Similarlly, we can observe that $I \circ Y \subseteq Y \circ I$ for all $Y \in \mathcal{B}(H)$.

$(ii) \Rightarrow (iii), (iii) \Rightarrow (iv)$. Straightforward.

$(iv) \Rightarrow (v), (v) \Rightarrow (vi), (vi) \Rightarrow (vii), (vii) \Rightarrow (viii), (viii) \Rightarrow (ix), (ix) \Rightarrow (x)$ are consequences of Proposition 2.1.

THEOREM 2.2 Suppose I_1, I_2 are any two normal hyperideals of H. Then, their product $I_1 \circ I_2$ and $I_2 \circ I_1$ are also normal hyperideals of H and $I_1 \circ I_2 = I_2 \circ I_1$.

PROOF. We have $I_1 \circ \Gamma \circ I_2 = I_2 \circ I_1$ by Theorem 2.1. For $h \in H$, we have $h \circ (I_1 \circ I_2) = (h \circ I_1) \circ I_2 = (I_1 \circ h) \circ I_2 = I_1 \circ (h \circ I_2) = I_1 \circ (I_2 \circ h) = (I_1 \circ I_2 \circ) \circ h$. Hence, $I_1 \circ I_2$ is normal.

THEOREM 2.3 Let I be a hyperideal of a regular hypersemigroup H. Then, the following assertions are equivalent:

(i). I is normal; and for all idempotents $e \in H$;

(ii). $e \circ I = I \circ e$;

(iii). $(e)_B \circ I = I \circ (e)_B$;

(iv). $(e)_B \circ I = I \circ (e)_L$;

(v). $(e)_B \circ I = I \circ e$;

(vi). $(e)_R \circ I = I \circ (e)_B$;

(vii). $(e)_R \circ I = I \circ (e)_L$;

(viii). $(e)_R \circ I = I \circ e$;

(ix). $e \circ I = I \circ (e)_B$;

(x). $e \circ I = I \circ (e)_L$.

PROOF. $(i) \Rightarrow (ii)$. It is obvious. Furthermore, the equivalence of (ii) to (x) can be shown similar to the equivalence of (i) and (v) to (x) in the proof of Theorem 2.2.

$(ii) \Rightarrow (i)$. SUPPOSE $h \in H$. AS H is regular, there exists $x \in H$ such that $\{h\} = h \circ x \circ h$ and $x \circ h$ is idempotent. It follows that

$h \circ I = (h \circ x \circ h) \circ I = h \circ ((x \circ h) \circ I) = h \circ (I \circ (x \circ\circ h)) = (h \circ I \circ x) \circ h \subseteq I \circ h$.
Similarly, we can show the reverse inclusion relation $I \circ h \subseteq h \circ I$. Consequently, we obtain
$h \circ I = I \circ h$ for all $h \in H$.

PROPOSITION 2.2 Suppose I is any normal hyperideal of H. Then, we have that $h \circ I$ is a hyperideal of H for any $h \in H$.

Proof. Suppose I is a normal hyperideal of H and $h \in H$. Then, it follows that
$(h \circ I) \circ H = h \circ (I \circ H) \subseteq h \circ I$ and $H \circ (h \circ I) = H \circ (I \circ h) = (H \circ I) \circ h \subseteq I \circ h$.
Hence, $h \circ I$ is a hyperideal of H.

THEOREM 2.4 [32] The product of a bi-ideal and a nonempty subset of a semigroup H is also a bi-ideal of H.

THEOREM 2.5 Any minimal hyperideal of a hypersemigroup H is a zero element of $\mathcal{B}(H)$.

Proof. Suppose I is a minimal hyperideal of H. Then, it is obvious that $I \in \mathcal{B}(H)$. Suppose B is any bi-hyperideal of H. Then, we obtain $B \circ I \subseteq H \circ I \subseteq I$. Therefore, by hypersemigroup analogue of Theorem 2.4 and the minimality of I, we have $B \circ I = I$. Similarly, we can show that $I \circ B = I$ for all $B \in \mathcal{B}(H)$.

THEOREM 2.6 Any minimal normal hyperideal of a hypersemigroup H is a hypergroup.

Proof. Suppose I is a minimal normal hyperideal of a hypersemigroup H and $h \in H$. Then, we obtain
$I \circ h = h \circ I \subseteq H \circ I \subseteq I$. Then, by Proposition 2.2 and the minimality of I, we obtain
$I \circ h = h \circ I = I$. This shows that $I \circ h = h \circ I = I$ for all $h \in H$. Hence, I is a hypergroup.

THEOREM 2.7 The following propositions based on a hypersemigroup H are equivalent:

(i). H is normal;

(ii). $B \circ H = H \circ B$ for all $B \in \mathcal{B}(H)$;

(iii). $(h)_B \circ H = H \circ (h)_B$ for all $h \in H$;

(iv). $(h)_B \circ H = H \circ (h)_L$ for all $h \in H$;

(v). $(h)_B \circ H = H \circ h$ for all $h \in H$;

(vi). $(h)_R \circ H = H \circ (h)_B$ for all $h \in H$;

(vii). $(x)_R \circ H = H \circ (h)_L$ for all $h \in H$;

(viii). $(h)_R \circ H = H \circ h$ for all $h \in H$;

(ix). $h \circ H = H \circ (h)_B$ for all $h \in H$;

(x). $h \circ H = H \circ (h)_L$ for all $h \in H$;

(xi). $\mathcal{B}(H)$ is normal;

(xii). $(h)_B \circ \mathcal{B}(H) = \mathcal{B}(H) \circ (h)_B$ for all $h \in H$;

(xiii). $(h)_B \circ \mathcal{B}(H) = \mathcal{B}(H) \circ (h)_L$ for all $h \in H$;

(xiv). $(h)_B \circ \mathcal{B}(H) = \mathcal{B}(H) \circ h$ for all $h \in H$;

(xv). $(h)_R \circ \mathcal{B}(H) = \mathcal{B}(H) \circ (h)_B$ for all $h \in H$;

(xvi). $(h)_R \circ \mathcal{B}(H) = \mathcal{B}(H) \circ (h)_L$ for all $h \in H$;

(xvii). $(h)_R \circ \mathcal{B}(H) = \mathcal{B}(H) \circ h$ for all $h \in H$;

(xviii). $h \circ \mathcal{B}(H) = \mathcal{B}(H) \circ (h)_B$ for all $h \in H$;

(xix). $h \circ \mathcal{B}(H) = \mathcal{B}(H) \circ (h)_L$ for all $h \in H$;

(xx). $h \circ \mathcal{B}(H) = \mathcal{B}(H) \circ h$ for all $h \in H$.

PROOF. As the hypersemigroup H is a hyperideal of itself, we obtain that from (i) to (x) are equivalent by Theorem 2.1.

$(i) \Rightarrow (xi)$. Suppose (i) is true, then we show (xi). Let I and B be any two bi-hyperideals of H and $x \in I$. Therefore, we have $x \circ B \subseteq x \circ H = H \circ x \subseteq H \circ I \subseteq \mathcal{B}(H) \circ I$ and hence, $I \circ \mathcal{B}(H) \subseteq \mathcal{B}(H) \circ I$. In a similar way, we can prove that the reverse inclusion is correct. Therefore, we have that $I \circ \mathcal{B}(H) = \mathcal{B}(H) \circ I$ for all $I \in \mathcal{B}(H)$ and also that $\mathcal{B}(H)$ is normal.

$(xi) \Rightarrow (xii)$. Straightforward.

$(xii) \Rightarrow (i)$. Suppose $h \in H$. Then, for some $I \in \mathcal{B}(H)$, WE OBTAIN $h \circ H \subseteq (h)_B \circ H = I \circ (h)_B \subseteq H \circ (h)_B \subseteq H \circ h$. In a similar way, we can see that the reverse inclusion relation is true. Therefore, H is normal. The remaining part of the proof is straightforward.

COROLLARY 2.1 Every one-sided hyperideal of a normal hypersemigroup H is a hyperideal of H.

III. CONCLUSIONS

In this article, we studied some semigroup-theoretic results in the context of hypersemigroup. Our results provided correspondence for normal hyperideals and bi-hyperideals of semigroups in [10] giving the description of the characterization of hypersemigroup in terms of normal hyperideals and particularly bi-hyperideals. In fact, the class of normal hyperideals in normal hypersemigroups is a generalization of the class of the normal ideals in normal semigroups. Our work opens up a new direction of further research work for other researchers in other algebraic structures.

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