Some Results on Super Harmonic Mean Graphs

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Abstract - Let G be a graph with p vertices and q edges. Let f: V(G) \rightarrow \{1, 2, \ldots, p + q\} be a injective function. For a vertex labeling f, the induced edge labeling f(e = uv) is defined by f(e) = \left\lfloor \frac{2f(u)f(v)}{f(u) + f(v)} \right\rfloor or \left\lceil \frac{2f(u)f(v)}{f(u) + f(v)} \right\rceil. Then f is called a Super harmonic mean labeling if f(V(G))\cup f(e) / e \in E(G) = \{1, 2, \ldots, p + q\}. A graph which admits Super harmonic mean labeling is called Super harmonic mean graphs. In this paper, we investigate Super harmonic mean labeling of some graphs.

Key words - Graph, Super harmonic mean labeling, Super harmonic mean graphs

I. INTRODUCTION

We begin with simple, finite, connected and undirected graph G(V, E) with p vertices and q edges. For a detailed survey of graph labeling we refer to Gallian[1]. For all other standard terminology and notations we follow Harary[2]. S. Somasundram and R. Ponraj introduced mean labeling of graphs in [3]. R. Ponraj and D. Ramya introduced Super mean labeling of graphs in [4]. S. Somasundram and S.S. Sandhya introduced the concept Harmonic mean labeling in [5] and studied their behavior in [6, 7, 8]. S. Sandhya and C. David Raj introduced Super harmonic labeling in [9]. In this paper, we investigate Super harmonic mean labeling of some graphs. We now give the following definitions which are useful for the present investigation.

Definition 1.1. Let f: V(G) \rightarrow \{1, 2, \ldots, p + q\} be a injective function. For a vertex labeling f, the induced edge labeling f(e = uv) is defined by f(e) = \left\lfloor \frac{2f(u)f(v)}{f(u) + f(v)} \right\rfloor or \left\lceil \frac{2f(u)f(v)}{f(u) + f(v)} \right\rceil. Then f is called a Super harmonic mean labeling if f(V(G))\cup f(e) / e \in E(G) = \{1, 2, \ldots, p + q\}. A graph which admits Super harmonic mean labeling is called Super harmonic mean graphs.

Definition 1.2. The corona G_1 \odot G_2 of two graphs G_1 and G_2 is defined as the graph G obtained by taking one copy of G_1 (which has p_1 vertices) and p_1 copies of G_2 and then joining the i\textsuperscript{th} vertex of G_1 to every vertices in the i\textsuperscript{th} copy of G_2.

Definition 1.3. The graph P_n \odot K_1 is called Comb.

Definition 1.4. The graph C_n \odot K_1 is called crown.

Definition 1.5. The prism D_n, n \geq 3 is a trivalent graph which can be defined as the Cartesian product P_2 \times C_n of a path on two vertices with a cycle on n vertices. We denote a graph obtained by attaching P_2 at each vertex of outer cycle of D_n by (D_n; P_2).

II. SUPER HARMONIC MEAN LABELING FOR CONNECTED GRAPHS

Theorem 2.1 nP_m is a Super harmonic mean graph.
Proof. Let \( v_{i,j}, 1 \leq i \leq n, 1 \leq j \leq m \) be the vertices of \( nP_m \). Then its edge set is \( E = \{u_{i,j}u_{i,j+1}/ 1 \leq i \leq n, 1 \leq j \leq m - 1\} \). Define a
function \( f: V(nP_m) \rightarrow \{1, 2, \ldots, p+q\} \) by
\[
f(v_{i,j}) = (2m - 1)(i - 1) + 2j - 1, 1 \leq i \leq n, 1 \leq j \leq m
\]
Then the induced edge labels are
\[
f^*(v_{i,j}v_{i,j+1}) = (2m - 1)(i - 1) + 2j, 1 \leq i \leq n, 1 \leq j \leq m - 1;
\]
Thus \( f \) provides a Super harmonic mean labeling for \( nP_m \).

**Example 2.2.** A Super harmonic mean labeling of \( 4P_7 \) is shown in figure 2.1.

![Figure 2.1 4P_7](image)

**Theorem 2.3.** \( nK_{1,3} \) is a Super harmonic mean graph.

Proof. Let \( u_i, u_{i,j}, 1 \leq i \leq n, 1 \leq j \leq 3 \) be the vertices of \( nK_{1,3} \) in which \( u_i \) is the central vertex of \( K_{1,3} \). Its edge set is
\( E = \{u_iu_{i,j}/ 1 \leq i \leq n, 1 \leq j \leq 3\} \). Define a function \( f: V(nK_{1,3}) \rightarrow \{1, 2, \ldots, p+q\} \) by
\[
f(u_1) = 7; f(u_i) = 7i - 2; 2 \leq i \leq n;
\]
\[
f(u_{1,1}) = 1; f(u_{1,2}) = 3; f(u_{1,3}) = 5;
\]
\[
f(u_{i,1}) = 7(i - 1) + j, 2 \leq i \leq n, 1 \leq j \leq 2;
\]
\[
f(u_{i,3}) = 7i, 2 \leq i \leq n.
\]
Then the induced edge labels are
\[
f(u_1u_{1,1}) = 2; f(u_1u_{1,2}) = 4; f(u_1u_{1,3}) = 6;
\]
\[
f(u_1u_{i,3}) = 7i - 5 + j; 2 \leq i \leq n, 1 \leq j \leq 2;
\]
Thus both vertices and edges together get distinct labels from \{1, 2, ..., p+q\}. Hence $nK_{1, 3}$ is a Super harmonic mean graph.

Example 2.4. A Super harmonic mean labeling of $4K_{1, 3}$ is given in figure 2.2.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.2.png}
\caption{4K_{1, 3}}
\end{figure}

Theorem 2.5. $(D_n; P_2)$ is a Super harmonic mean graph.

Proof. Let $u_i$ and $v_i$ be the vertices of inner and outer cycle of $(D_n; P_2)$ respectively in which $u_i$ and $v_i$ are adjacent, $1 \leq i \leq n$. Let $w_i$ be a vertex which is joined with $v_i$, $1 \leq i \leq n$. Its edge set is $E = \{u_iu_{i+1}, u_iu_{i+1}, v_iw_i, v_{i+1}w_i / 1 \leq i \leq n - 1\} \cup \{u_{n+1}v_1, v_{n+1}w_1 / 1 \leq i \leq n\}$. Define a function $f: V(D_n; P_2) \to \{1, 2, ..., p+q\}$ by

- $f(u_i) = 7$; $f(u_2) = 14$; $f(u_i) = 7i - 2$, $3 \leq i \leq n$;
- $f(v_i) = 3$; $f(v_2) = 11$; $f(v_i) = 7i$, $3 \leq i \leq n$;
- $f(w_i) = 1$; $f(w_2) = 8$; $f(w_i) = 7i - 5$, $3 \leq i \leq n$.

Then the induced edge labels are

- $f(u_2u_3) = 10$; $f(u_3u_4) = 17$; $f(u_iu_{i+1}) = 7i + 1$, $3 \leq i \leq n - 1$;
- $f(u_iu_1) = \begin{cases} 12 & \text{if } n \leq 6; \\ 13 & \text{if } n > 6; \end{cases}$
- $f(u_1v_1) = 5$;
- $f(u_2v_2) = \begin{cases} 13 & \text{if } n \leq 6; \\ 12 & \text{if } n > 6; \end{cases}$
\[ f(u_iv_i) = 7i - 1; \ 3 \leq i \leq n; \]
\[ f(v_1v_2) = 4; f(v_3v_4) = 15; f(v_{i+1}v_i) = 7i + 3, \ 3 \leq i \leq n - 1; f(v_nv_1) = 6; \]
\[ f(v_1w_1) = 2; f(v_2w_2) = 9; f(v_iw_i) = 7i - 3, \ 3 \leq i \leq n. \]

Thus the vertices and edges together get distinct labels from \{1, 2, ..., p+q\}. Thus \( f \) provides a Super harmonic mean labeling for \((D_n; P_2)\).

Example 2.6. A Super harmonic mean labeling of \((D_7; P_2)\) is shown in figure 2.3.

Fig. 2.3 \((D_7; P_2)\)

III. SUPER HARMONIC MEAN LABELING FOR DISCONNECTED GRAPHS

In this section, we prove \( C_m \cup P_n, (P_m \Theta K_1) \cup C_n, (C_m \Theta K_1) \cup P_n, (C_m \Theta K_1) \cup C_n \) and \((C_m \Theta K_1) \cup (P_n \Theta K_1)\) are Super harmonic mean graphs.

Theorem 3.1. \( C_m \cup P_n \) is a Super harmonic mean graph.

Proof. Let \( u_1u_2...u_mu_1 \) be the cycle \( C_m \) and \( v_1v_2...v_n \) be the path \( P_n \). Then \( C_m \cup P_n \) has edge set \( E = \{u_iu_{i+1}, u塞 1 \leq i \leq m - 1\} \cup \{v_iv_{i+1} / 1 \leq i \leq m - 1\} \). Define a function \( f: V(C_m \cup P_n) \rightarrow \{1, 2, ..., p+q\} \) by
\[ f(u_i) = 3; f(u_i) = 2(i + 1), 2 \leq i \leq m; \]
\[ f(v_1) = 1; f(v_i) = 2m + 2i - 1, 2 \leq i \leq n. \]

Then the induced edge labels are
\[ f^*(u_1u_2) = 4; f^*(u_iu_{i+1}) = 2i + 3, 2 \leq i \leq m - 1; f^*(u_1u_m) = 5; \]
\[ f^*(v_1v_2) = 2; f^*(v_iv_{i+1}) = 2m + 2i, 2 \leq i \leq n - 1. \]

Thus the vertices and edges together get distinct labels from \{1, 2, ..., p+q\}. Thus \( f \) provides a Super harmonic mean labeling for \( C_m \cup P_n \).

**Example 3.2.** A Super harmonic mean labeling of \( C_9 \cup P_7 \) is shown in figure 3.1.

![Figure 3.1 C_9 \cup P_7](image)

**Theorem 3.3.** \((P_m \bowtie K_1) \cup C_n\) is a Super harmonic mean graph.

Proof. Let \( u_1u_2...u_m \) be the path \( P_m \). Add vertices \( v_i \) such that \( v_i \) is adjacent to \( u_i, \ 1 \leq i \leq m \). The resultant graph is \( P_m \bowtie K_1 \). Let \( w_1w_2...w_nw_1 \) be the cycle \( C_n \). Let \( G = (P_m \bowtie K_1) \cup C_n \) whose edge set is \( E = \{ u_iu_{i+1}/1 \leq i \leq m - 1 \} \cup \{ w_iw_{i+1}, w_nw_1/1 \leq i \leq n - 1 \} \cup \{ u_iu_i/1 \leq i \leq m \} \). Define a function \( f \) \( V(G) \rightarrow \{1, 2, ..., q + 1\} \) by
\[ f(w_i) = 3; f(w_i) = 2(i + 1), 2 \leq i \leq n; \]
\[ f(u_i) = 2(m + 1) + 4i + 3, 1 \leq i \leq m; \]
\[ f(v_1) = 1; f(v_i) = 2(m + 2) + 4i - 2, 2 \leq i \leq m. \]

Then the edges are labeled with
\[ f^*(w_1w_2) = 4; f^*(w_iw_{i+1}) = 2i + 3, 2 \leq i \leq n - 1; f^*(w_nw_1) = 5; \]
\[ f^*(u_iu_{i+1}) = 2(m + 2) + 4i + 3, \ 1 \leq i \leq m - 1; \]
\[ f^*(u_1v_1) = 2; f^*(u_iv_i) = 2(m + 2) + 4i, 2 \leq i \leq m. \]

Therefore, \( f \) is a Super harmonic mean labeling for \( G \). Hence \( G \) is a Super harmonic mean graph.
Example 3.4. A Super harmonic mean labeling of $(P_6 \bowtie K_1) \cup C_9$ is given in figure 3.2.

![Graph of $(P_6 \bowtie K_1) \cup C_9$]

Fig. 3.2 $(P_6 \bowtie K_1) \cup C_9$

Theorem 3.5. $(C_m \bowtie K_1) \cup P_n$ is a Super harmonic mean graph.

Proof. Let $u_1u_2u_3\ldots u_mu_1$ be the cycle $C_m$. Add vertices $v_i$ such that $v_i$ is adjacent to $u_i, 1 \leq i \leq m$. The resultant graph is $C_m \bowtie K_1$. Let $w_1w_2\ldots w_n$ be the path $P_n$. Let $G = (C_m \bowtie K_1) \cup P_n$ whose edge set is $E = \{u_iu_{i+1}/1 \leq i \leq m - 1\} \cup \{w_iw_{i+1}/1 \leq i \leq n - 1\} \cup \{u_iv_i/1 \leq i \leq m - 1\}$. Define a function $f: V(G) \to \{1, 2, \ldots, p + q\}$ by

$$f(u_i) = 3; f(u_i) = 4i, 2 \leq i \leq m;$$
$$f(v_i) = 1; f(v_2) = 6; f(v_i) = 4i - 3, 3 \leq i \leq m;$$
$$f(w_i) = 4m + 2i - 1, 1 \leq i \leq n.$$ 

Then the edges are labeled with

$$f'(u_iu_{i+1}) = 4; f'(u_iu_{i+1}) = 4i + 2, 2 \leq i \leq m - 1; f'(u_iu_{i+1}) = 5;$$
$$f'(u_iv_i) = 2; f'(v_i) = 4i - 1, 2 \leq i \leq m;$$
$$f'(w_iw_{i+1}) = 4m + 2i, 1 \leq i \leq n - 1.$$ 

In the view of the above labeling pattern, $f$ provides a Super harmonic mean labeling for $G$. Hence $G$ is a Super harmonic mean graph.

Example 3.6. A Super harmonic mean labeling of $(C_6 \bowtie K_1) \cup P_9$ is given in figure 3.3.
**Theorem 3.7.** \((C_m \circ K_1) \cup C_n\) is a Super harmonic mean graph.

Proof. Let \(u_1u_2\ldots u_mu_1\) be the cycle \(C_m\). Add vertices \(v_i\) such that \(v_i\) is adjacent to \(u_i\), \(1 \leq i \leq m\). The resultant graph is \(C_m \circ K_1\). Let \(w_1w_2\ldots w_nw_1\) be the cycle \(C_n\). Let \(G = (C_m \circ K_1) \cup C_n\) whose edge set is \(E = \{u_iu_{i+1}, u_mu_1 : 1 \leq i \leq m-1\} \cup \{w_iw_{i+1}, w_nw_1 : 1 \leq i \leq n-1\} \cup \{v_i / 1 \leq i \leq m\}\).

A Super harmonic mean labeling of \((C_m \circ K_1) \cup C_n\) when \(m, n \leq 4\) are given in figures 3.4, 3.5, 3.6 and 3.7 respectively.
Assume that $m, n > 4$. Define a function $f: V(G) \to \{1, 2, \ldots, p + q\}$ by

- $f(u_i) = 3; f(u_k) = 4i + 3, 2 \leq i \leq m$;
- $f(v_1) = 1; f(v_2) = 9; f(v_i) = 4i, 3 \leq i \leq m$;
- $f(w_1) = 4; f(w_i) = 4m + 2i, 2 \leq i \leq n$.

Then the edges are labeled with
\[
\begin{align*}
\ell(u_1u_2) &= 5; \ell(u_iu_{i+1}) = 4i + 5, 2 \leq i \leq m - 1; \ell(u_mu_1) = 6; \\
\ell(u_1v_1) &= 2; \ell(u_iv_i) = 4i + 2, 2 \leq i \leq m; \\
\ell(w_1w_2) &= 7; \ell(w_iw_{i+1}) = 4m + 2i + 1, 2 \leq i \leq n - 1; \ell(w_nw_1) = 8.
\end{align*}
\]

In the view of the above labeling pattern, \( \ell \) provides a Super harmonic mean labeling for \( G \). Hence \( G \) is a Super harmonic mean graph.

**Example 3.8.** A Super harmonic mean labeling of \((C_5 \circ K_1) \cup C_7\) is given in figure 3.8.

![Figure 3.8. \((C_5 \circ K_1) \cup C_7\) Example 3.8](image)

**Theorem 3.9.** \((C_m \circ K_1) \cup (P_n \circ K_1)\) is a Super harmonic mean graph.

**Proof.** Let \( u_1u_2...u_m \) be the cycle \( C_m \) and let \( v_1 \) be the vertex which is joined to the vertex \( u_i \), \( 1 \leq i \leq m \), of the cycle \( C_m \). The resultant graph is \( C_m \circ K_1 \). Let \( s_1s_2...s_n \) be the path \( P_n \) and let \( t_i \) be the vertex which is joined to the vertex \( s_i \), \( 1 \leq i \leq n \), of the path \( P_n \). The resultant graph is \( P_n \circ K_1 \). Let \( G = (C_m \circ K_1) \cup (P_n \circ K_1) \). Define a function \( \ell : V(G) \rightarrow \{1, 2, ..., p + q\} \) by

\[
\begin{align*}
\ell(u_i) &= 3; \ell(u_i) = 4i, 2 \leq i \leq m; \\
\ell(v_1) &= 1; \ell(v_2) = 6; \ell(v_3) = 4i - 3, 3 \leq i \leq m; \\
\ell(s_i) &= 4m + 4i - 1, 1 \leq i \leq n; \\
\ell(t_i) &= 4m + 1; \ell(t_i) = 4(m - 1) + 4i, 2 \leq i \leq n.
\end{align*}
\]

Then the edges are labeled with

\[
\begin{align*}
\ell(u_1u_2) &= 4; \ell(u_iu_{i+1}) = 4i + 2, 2 \leq i \leq m - 1; \ell(u_mu_1) = 5; \\
\ell(u_1v_1) &= 2; \ell(u_iv_i) = 4i - 1, 2 \leq i \leq m; \\
\ell(s_is_{i+1}) &= 4m + 4i + 1, 1 \leq i \leq n - 1; \\
\ell(s_1t_1) &= 4m + 4i - 2, 1 \leq i \leq n.
\end{align*}
\]

Thus \( \ell \) provides a Super harmonic mean labeling for \( G \). Hence \( G \) is a Super harmonic mean graph.
Example 3.10. A Super harmonic mean labeling of \((C_6 \circ K_1) \cup (P_7 \circ K_1)\) is given in figure 3.9.

![Image of a graph depicting \((C_6 \circ K_1) \cup (P_7 \circ K_1)\)]

Fig. 3.9 \((C_6 \circ K_1) \cup (P_7 \circ K_1)\)

REFERENCES


