On Relative Preclosedness of Strongly Compact (Countably p-Compact) Sets

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Abstract— In this paper, we study the preclosedness of strongly compact (countably p-compact) subsets of subspaces of strongly p-normal spaces. Consequences of the result for unions of specific sets are given. Examples are given to illustrate the results.

Keywords— preclosed, p*closed, pre-R1, strongly compact, countably p-compact, pre-accumulation, p-convergent, pre-sequential, strongly p-normal, net.

I. INTRODUCTION

In [3] Ganster answered the question posed by Katetov as to when preopen sets form a topology. In fact he proved that, for space X having its unique Hewitt representation $X = F \cup G$, where F is closed and resolvable and G is open and hereditarily irresolvable, the preopen sets of X form a topology if and only if closure of G is open and each singleton in the interior of F is preopen in X. In this paper, we will call such spaces in which preopen sets form a topology as spaces having Strong Hewitt Representation. On the other hand, in [4] Garg and Singh took up the question of closedness of a compact (countably compact) set in $S_2$ (sequential, $S_2$) and normal (sequential, normal) spaces. Since normality is not hereditary, in [5] Garg and Singh further generalized the results to closedness of a compact (countably compact) set in subspaces (sequential subspaces) of normal spaces. In [10], Noorie and A. Singh obtained necessary and sufficient conditions for p*closedness of a strongly compact (countably p-compact) set in pre-$R_1$ (pre-sequential, pre- $R_1$), p*normal (pre-sequential, p*normal) spaces and sufficient conditions for p*closedness of a strongly compact (countably p-compact) set in strongly p-normal (pre-sequential, strongly p – normal) and p-normal (pre-sequential, p-normal) spaces. Among others, the following results have been proved in [10]:

Theorem 1.1 [10]:
For a strongly compact (countably p-compact) subset K of a pre- $R_1$(pre-sequential, pre- $R_1$) space X, the following conditions are equivalent:
(i) K is p*closed;
(ii) either K or $K^C$ is a union of p*closed sets;
(iii) both K and $K^C$ are unions of p*closed sets.

Theorem 1.2 [10]:
In a p-normal (pre-sequential, p-normal) space X, strongly compact (countably p-compact) set K is p*closed if K is a union of closed sets and $K^C$ is of the form $G \cup C$, where G and C are arbitrary preopen and closed sets respectively.

Theorem 1.3 [10]:
In a strongly p-normal (pre-sequential, strongly p-normal) space X, strongly compact (countably p-compact) set K is p*closed if K is a union of preclosed sets and $K^C$ is of the form $G \cup F$, where G and F are arbitrary preopen and preclosed sets respectively.

In this paper, results of [10] are generalized to subspaces (pre-sequential subspaces) of spaces having Strong Hewitt Representation, obtaining necessary and sufficient conditions for preclosedness of strongly compact (countably p-compact) subset of the preopen (presequential, preopen) subspace Y in strongly p-normal spaces [Theorem 2.9, Theorem 2.11]. Also sufficient conditions for relative preclosedness of arbitrary union of preclosed sets in strongly p-normal spaces [Corollary 2.10 (a), (b)] are obtained.

A subset A of a space X is preclosed [8] if closure of interior of A is contained in A. The complement of a preclosed set is called a preopen set and preclosure [2] is the intersection of all preclosed sets containing A and is denoted by pcl(A). A point $x \in X$ is a pre-accumulation (p-convergent) [6] of a net in X if the net is frequently (eventually) in every preopen set containing x. A subset A of space X is said to be p*closed [10] if no net in A p-converges to a point of $A^C$. A space X is, (i) strongly compact [10] (countably p-compact) [13] if every preopen (countable preopen) cover of X has finite subcover, (ii) p-normal [11] if for each pair of disjoint closed sets of X, there exist disjoint preopen sets containing them, (iii) strongly p-normal
[10] if for each pair of disjoint preclosed sets of X, there exist disjoint preopen sets containing them. (iv) pre-$R_1$ [1] if for points $x, y$ in X with distinct preclosures there exist disjoint preopen sets containing pcl($x$) and pcl($y$). (v) pre-$T_2$ [7] if for each pair of distinct points $x$ and $y$ of X, there exists a pair of disjoint preopen sets, one containing $x$ and the other containing $y$. (vi) pre-sequential [10] if for every non-preclosed subset $A$ of X there is a sequence $(x_n)$ in A which p-converges to a point of $A^C$. (vii) resolvable [3] if it is the disjoint union of two dense subsets, (viii) hereditarily irresolvable [3] if it does not contain a non-empty resolvable set. Also we will call a set pre-$F_{\sigma}$, if it is countable union of preclosed sets.

Throughout, by a space X we shall mean a topological space. In a space X, $A^C$ will denote the complement of A for any subset A of X. $Z^+$ will denote the set of all positive integers. G, F and C, respectively, will stand for arbitrary preopen, preclosed and subsets of X. $G_Y$, $F_Y$ and $C_Y$ respectively, will stand for relatively preopen, relatively preclosed and relatively closed subsets of the subspace Y of X. For a set $S \subseteq Y$, $cl_Y(S)$, $pcl_Y(S)$, and $Y - S$, respectively will denote the closure, preclosure and complement of the set S in the subspace Y of X.

The following results will be used in the next section.

**Lemma 1.4** [6]:
A space X is strongly compact if and only if every net in X pre-accumulates to some point of X.

**Lemma 1.5** [7]:
If $A \subseteq Y \subseteq X$ and Y is preopen in X then, A is preopen in Y if and only if A is preopen in X.

**Lemma 1.6** [12]:
If $A \subseteq Y \subseteq X$ and Y is $\alpha$-set in X then $pcl_Y(A) = pcl_X(A) \cap Y$.

**Lemma 1.7** [1]:
For a space X, the following conditions are equivalent:
(i) X is pre-$R_1$;
(ii) X is pre-$T_2$.

## II. RESULTS

Proof of Lemma 2.1 follows from Lemma 1.5 and Lemma 1.6.

**Lemma 2.1:**
If $A \subseteq Y \subseteq X$, A is preclosed in Y and Y is preopen in X then, $A = pcl_Y(A) \cap Y$.

**Remark:**
From now onwards, throughout this section, the space X is assumed to have Strong Hewitt Representation.

With the assumption that preopen sets of space X, form a topology and a directed set, the well known relationship between adherent points of a set and nets has the following analogous form.

**Lemma 2.2:**
For $A \subseteq X$, $x \in pcl(A)$ if and only if there exists a net in A which p-converges to x.

The following Lemma gives another characterization of the preclosed sets.

**Lemma 2.3:**
A subset A of X is p*-closed if and only if it is preclosed.
**Proof:** Since every preclosed set is p*-closed [10]. The converse follows with the help of Lemma 2.2.

We now obtain a necessary and sufficient condition for the preclosedness of a countably p-compact set in a strongly p-normal space in Theorem 2.4 and sufficient condition for the equality of the union of preclosures and the preclosure of the union of countable families of sets in strongly p-normal spaces are also obtained in Corollary 2.5 below.

**Theorem 2.4:**
Let X be a strongly p-normal space and K a countably p-compact subset of X. Then K is preclosed in X if and only if K is a pre-$F_{\sigma}$ and $K^C$ is of the form $G \cup F$. 

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Proof: Let $K = \bigcup_{i=1}^{\infty} F_i$, where each $F_i$ is preclosed in $X$, and $K^c = G \cup F$. We prove $K^c$ is preopen in $X$. For any $x \in K^c$, if $x \in G$, then $x \in G \subset K^c$, where $G$ is preopen in $X$. Otherwise, $x \in F$. Since $X$ is strongly p-normal, for each $n$, there exist disjoint preopen sets $U_n$ and $V_n$ in $X$ containing $F_n$ and $F$ respectively. Then $\{U_n\}^\infty_{n=1}$ is a countable preopen cover of $K$ and therefore, there exists a positive integer $n$, such that, $K \subset \bigcup_{i=1}^{n} U_n$ and $F \subset \bigcap_{i=1}^{n} V_n$. Then $U = \bigcup_{i=1}^{n} U_n$ and $V = \bigcap_{i=1}^{n} V_n$ are disjoint preopen sets such that $x \in V \subset U^c \subset K^c$. Therefore, $K^c$ is preopen and hence $K$ is preclosed.

Corollary 2.5:
In a strongly p-normal space $X$,
(a) a countable union of preclosed sets is preclosed, if it is countably p-compact and is of the form $G \cap F$;
(b) if $c$ is a family of subsets of $X$ such that $\cup\{\text{pcl}(E) : E \in \epsilon\}$, in particular $\cup\{E : E \in \epsilon\}$, is countably p-compact and is of the form $G \cap F$, then $\text{pcl}(\cup\{E : E \in \epsilon\}) = \cup\{\text{pcl}(E) : E \in \epsilon\}$.

Lemma 2.6:
For a preopen subspace $Y$ of $X$, the preopen sets of $Y$ form a topology.

Proof: Preopen sets of $Y$ form a topology if they satisfy finite intersection property. Let $\{U_n\}$ be preopen sets of $Y$. By Lemma 1.5 since $Y$ is preopen in $X$ each $U_n$ is also preopen in $X$. Since, finite intersection of preopen sets is preopen in $X$ and the finite intersection will also be preopen in $Y$ again by Lemma 1.5.

Lemma 2.7:
If $A \subseteq Y \subseteq X$ and $Y$ is preopen in $X$ then, $A$ is strongly compact (countably p-compact) relative to $X$ if and only if $A$ is strongly compact (countably p-compact) relative to $Y$.

Proof: Necessary condition is obvious and the sufficient part follows from Lemma 1.5 and Lemma 2.6.

Proof of Theorem 2.8 follows from Lemma 1.5 and Lemma 1.7.

Theorem 2.8:
Every preopen set $Y$ of a pre-R$_1$ space $X$ is also pre-R$_1$.

We now obtain the necessary and sufficient conditions for the preclosedness of a strongly compact (countably p-compact) subset of the preopen subspace (presequential, preopen subspace) of a strongly p-normal space in Theorem 2.9 and Corollary 2.10 as a generalization of Theorem 1.3 above in subspaces.

Theorem 2.9:
Let $K$ be a strongly compact (countably p-compact) subset of the preopen subspace (presequential, preopen subspace) $Y$ of a strongly p-normal space $X$. Then $K$ is relatively preclosed in $Y$,
(a) if and only if $K$ is a union of relatively preclosed subsets of $Y$ and $Y - K$ is of the form $G \cup F$,
(b) if $K$ is a union of preclosed subsets of $X$ and $Y - K$ is of the form $G \cup F$;
(c) if $K$ is a union of preclosed subsets of $X$ and $Y - K$ is a union of relatively preclosed subsets of $Y$;
(d) if $K$ is a union of relatively preclosed subsets of $Y$ and $Y - K$ is a union of preclosed subsets of $X$.

Proof: (a) Since the necessary condition is obvious we prove the sufficient part. Let $K = \bigcup_{\alpha} V_{\alpha}$, where each $V_{\alpha}$ is a prelosed set in $Y$. Since $Y$ is preopen by Lemma 2.1, $K = \bigcup_{\alpha} (F_{\alpha} \cap Y)$, where each $F_{\alpha}$ is a preclosed set in $X$. By Lemma 2.3, if $K$ is not relatively preclosed in $Y$ then there exists a net $\{x_{\lambda}\}$ (a sequence $\{x_{n}\}$) in $K$ such that $x_{\lambda} (\{x_{\lambda}\})$ p-converges to point $a$ and $a$ is in $Y - K$. Then as $K$ is strongly compact (countably p-compact) in $Y$, Lemma 1.4 implies that the net $\{x_{\lambda}\}$ (the sequence $\{x_{n}\}$) has a pre-accumulation point $b$ in $K$. By Lemma 2.6 the net $\{x_{\lambda}\}$ (the sequence $\{x_{n}\}$) in $K$ pre-accumulates relative to $X$ to the point $b$. Therefore, there exists an $\alpha$ such that $b \in F_{\alpha}$ and $a \not\in F_{\alpha}$. Thus $a$ and $b$ belong to the disjoint preclosed sets $F$ and $F_n$ of $X$ and since $X$ is strongly p-normal it follows that they have disjoint preopen sets $U$ and $V$ of $X$ containing them respectively, and since $Y$ is preopen in $X$ and preopen sets of $X$ form a topology $U \cap Y$ and $V \cap Y$ are disjoint preopen sets of $Y$ (Lemma 1.5) containing $a$ and $b$ respectively, contradicting to the fact that $x \subset p$ converges to $a$ (the sequence $\{x_{n}\}$). Hence $K$ must be preclosed in $Y$ and (a) follows.

The proofs of (b) - (d) are similar to that of part (a).

Corollary 2.10:
Let $K$ be a strongly compact (countably p-compact) subset of the preopen subspace (presequential, preopen subspace) $Y$ of a strongly p-normal space $X$. Then
(a) a union K of relatively preclosed subsets of Y is relatively preclosed in Y, if K is strongly compact (countably p-compact) and Y − K is either of the form G″ V U F or is a union of preclosed subsets of X;
(b) a union K ⊂ Y of preclosed subsets of X is relatively preclosed in Y if K is strongly compact (countably p-compact) and Y − K is either of the form G″ V U F or is a union of relatively preclosed subsets of Y;
(c) if ε is a family of subsets of X such that K = ∪ E E ε is strongly compact (countably p-compact) and Y − K is of the form G″ V U F or is a union of preclosed subsets of X, then pcl Y(∪E E E ε) = ∪ {pcl Y(E) : E E ε};
(d) if ε is a family of subsets of X such that K = ∪ E E ε is strongly compact (countably p-compact) and Y − K is of the form G″ V U F or is a union of relatively preclosed subsets of Y, then pcl Y(∪E E E ε) = ∪ {pcl Y(E) : E E ε}.

We now obtain the necessary and sufficient conditions for the preclosedness of a countably p-compact subset of the preopen subspace of a strongly p-normal space as a generalization of Theorem 2.4 above in subspaces.

Theorem 2.11: Let K be a countably p-compact subset of the preopen subspace Y of a strongly p-normal space X. Then K is relatively preclosed in Y
(a) if and only if K is a relatively pre-Fσ set in Y and Y − K is of the form G″ V U F,
(b) if any one of the following conditions holds:
(i) K is an pre-Fσ in X and Y − K is of the form G″ V U F;
(ii) K is a union of preclosed subsets of Y;
(iii) K is a relatively pre-Fσ set in Y and Y − K is a union of closed subsets of X.

Proof: (a) Since necessity is obvious for any set K, we need prove only the sufficient part. Let K = ∪ F∞ i=1 Fi, where each Fi is preclosed in X. Since Y is preopen set by Lemma 2.1, we have K = ∪ Fi Y ∩ Y Fi of the form G″ V U F by Theorem 2.5. Therefore, Y − K is relatively preclosed in Y. For any x ∈ Y − K, if x ∈ Gx, then Y − K = Gx V U F implies x ∈ Gx ⊂ Y − K. If x ∈ F, then since X is strongly p-normal and each Fσ is disjoint from F, there exist, for each positive integer n, disjoint preopen sets Un and Vn in X containing Fσ and respectively. Then \{Un \}n=1∞ is a countable preopen cover of K and there exists a positive integer n such that K ⊂ ∪[n=1∞ Un] and F ⊂ ∩[n=1∞ Vn]. Then U = ∪[n=1∞ Un] and V = ∩[n=1∞ Vn] are disjoint preopen sets such that x ∈ V ⊂ U ⊂ Y − K. It follows that x ∈ V ∩ Y ⊂ Y − K, where V ∩ Y is relatively precopen in Y (Lemma 1.5). Therefore, Y − K is relatively preopen and hence K is preclosed in Y.
(b) The proof is similar to that of part (a) above.

III. Examples

Example 3.1 below shows that in a space which does not have strong Hewitt representation, (i) a preopen subspace of strongly p-normal need not be strongly p-normal and (ii) a preopen subspace of pre-R1 need not be pre-R1.

Example 3.1:
Let X = Z∗, together with the topology, T = {G ⊂ X | G = ∅ or {1, 2} ⊂ G}. Then (X, T) strongly p-normal space which is not normal and a pre-R1 space which is not R1 where preopen sets of X do not form a topology. Let Y be any set containing {1} but not containing {2}. Y is a preopen subspace of X which is neither strongly p-normal space nor a pre-R1 space.

Example 3.2 below shows, (i) a strongly p-normal space which is not pre-R1, (ii) a preopen subspace of strongly p-normal space need not be strongly p-normal even if the space has strong Hewitt representation, (iii) “G″ V U F” in cannot be replaced by “G″ V U F” or “G″ V U F” in Theorem 2.9(a) and Corollary 2.10(a).

Example 3.2 ([c.f. 15; 8.1 Problem 1 and 13; Example 27]):
Let Y = N ⊂ X ⊂ X, x1, x2 are two distinct points and N is any infinite set neither containing x1 nor x2. We topologize Y with topology T by calling any subset of N open and calling any set containing x1 or x2 open if and only if it contains all but finite number of points in N. Let p ∈ X and Xp = Y ⊂ X, with the topology Tp = {G ⊂ Xp | G ∈ T or G = Xp}. Then (Xp, Tp) is a strongly p-normal space but (Y, T) is not strongly p-normal space. Let K = G ⊂ Xp, where G contains all but finite number of points in N, K is a strongly compact (countably p-compact) set which is union of relatively closed and preclosed sets of Y but K is not of the of the form G″ V U F or G″ V U F and K is not preclosed.

Example 3.3 below shows that if a space X has Strong Hewitt representation then subspace Y of X need not have Strong Hewitt representation.
Example 3.3: Let $X = \{a,b,c,d\}$, together with the topology $T = \{\emptyset, \{a\}, \{a, b, c\}, X\}$. Then preopen sets of $(X, T)$ form a topology. Further, $Y = \{b,c,d\}$ is a preclosed subspace of $X$ the preopen sets of which do not form a topology.

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