On the Zeros of Lacunary Polynomials

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Abstract: In this paper we find the bounds for the zeros of some special lacunary polynomials. Our results generalize some known results in the distribution of zeros of polynomials.

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1. Introduction and Statement of Results

A classic result on the zeros of polynomials is the following theorem due to Cauchy[2]:

**Theorem A:** Let \( P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_{n-1} z^{n-1} + a_n z^n, a_n \neq 0 \) be a complex polynomial. Then all the zeros of \( P(z) \) lie in the closed disk \( |z| \leq r \), where \( r \) is the positive root of the equation

\[
|a_0| + |a_1| |z| + \ldots + |a_{n-1}| |z|^{n-1} - |a_n| z^n = 0.
\]

Another classical result due to Cauchy [2] is the following:

**Theorem B:** Let \( P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_{n-1} z^{n-1} + a_n z^n, a_n \neq 0 \) be a complex polynomial. Then all the zeros of \( P(z) \) lie in the closed disk \( |z| \leq 1 + M \), where

\[
M = \max \left| \frac{a_j}{a_n} \right|, j = 0, 1, 2, \ldots, n - 1.
\]

Similar results due to Dehmer [1] are the following:

**Theorem C:** Let \( P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_{n-1} z^{n-1} + a_n z^n, a_n \neq 0 \) be a complex polynomial. Then all the zeros of \( P(z) \) lie in the closed disk \( |z| \leq \max(1, k) \), where \( k \neq 1 \) is the positive root of the equation

\[
z^{n+1} - (1 + M) z^n + M = 0
\]

and \( M = \max \left| \frac{a_j}{a_n} \right|, j = 0, 1, 2, \ldots, n - 1. \)

**Theorem D:** Let \( P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_{n-1} z^{n-1} + a_n z^n, a_n \neq 0 \) be a complex polynomial. Then all the zeros of \( P(z) \) lie in the closed disk \( |z| \leq \max(1, k) \), where \( k \neq 1 \) is the positive root of the equation
$z^{n+2} - (1 + M')z^{n+1} + M' = 0$

and $M' = \max \left| \frac{a_{n-j} - a_{n-j-1}}{a_n} \right|, j = 0, 1, 2, \ldots, n; a_{-1} = 0$.

The following result is the famous Enestrom-Kakeya Theorem [2]:

**Theorem E:** Let $P(z) = a_0 + a_1z + a_2z^2 + \ldots + a_{n-1}z^{n-1} + a_nz^n, a_n \neq 0$ be a polynomial with real coefficients such that

$a_n \geq a_{n-1} \geq \ldots \geq a_i \geq a_0 > 0$.

Then all the zeros of $P(z)$ lie in the closed disk $|z| \leq 1$.

In this paper, we consider some special lacunary polynomials and see what happens to Theorems A, B, C, D, and E. In fact, we prove the following results:

**Theorem 1:** Let $P(z) = a_0z^n + a_pz^{n+p} + a_{p+1}z^{n+p-1} + \ldots + a_{n-1}z + a_n, a_p \neq 0, p < n$, be a polynomial of degree $n$. Then all the zeros of $P(z)$ lie in $|z| \leq k$, where $k > 1$ is the root of the equation

$k^p - k^{p-1} + M = 0$

and $M = \max \left| \frac{a_j}{a_p} \right|, j = p, p + 1, \ldots, n$.

**Remark 1:** Taking $p = 1$ in Theorem 1, we get $k = 1 + M$ so that Theorem 1 reduces to Theorem A.

**Theorem 2:** Let $P(z) = a_0z^n + a_1z + a_2z^2 + \ldots + a_pz^p + a_nz^n, a_p \neq 0, 1 \leq p \leq n - 1$, be a polynomial of degree $n$. Then all the zeros of $P(z)$ lie in $|z| \leq \max(1, k)$, where $k \neq 1$ is the positive root of the equation

$z^{n+1} - z^n - Mz^{p+1} + M = 0$

and $M = \max \left| \frac{a_j}{a_p} \right|, j = 0, 1, 2, \ldots, p$.

**Remark 2:** Taking $p = n - 1$ in Theorem 1, we get Theorem B of Dehmer.

**Theorem 3:** Let $P(z) = a_0 + a_1z + a_2z^2 + \ldots + a_pz^p + a_nz^n, a_p \neq 0, 1 \leq p \leq n - 1$, be a polynomial of degree $n$. Then all the zeros of $P(z)$ lie in $|z| \leq k$, where $k$ is the greatest positive root of the equation

$z^{n+2} - 2z^{n+1} + z^n - Mz^{p+1} + M' = 0$

and $M' = \max \left| \frac{a_j - a_{j-1}}{a_n} \right|, j = 0, 1, 2, \ldots, p; a_{-1} = 0$. 
Theorem 4: Let \( P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_p z^{n-p} + a_n z^n \), where \( a_p \neq 0 \), \( p < n \), be a polynomial of degree \( n \) such that
\[
a_p \geq a_{p-1} \geq \cdots \geq a_1 \geq a_0.
\]
Then all the zeros of \( P(z) \) lie in the closed disk
\[
|z| \leq \frac{1}{|a_n|} [ |a_n| + a_p - a_0 + |a_0| ]
\]

2. Proofs of Theorems

Proof of Theorem 1: For \( |z| > 1 \), \( \frac{1}{|z|} < 1 \), \( \forall j = p, p+1, \ldots, n \), so that, by using the hypothesis, we have
\[
|P(z)| = \left| a_0 z^n + a_p z^{n-p} + a_{p+1} z^{n-p-1} + \cdots + a_{n-1} z + a_n \right|
\]
\[
\geq |z|^n \left[ 1 - \left( \frac{|a_p|}{|a_0|} \cdot \frac{1}{|z|^p} + \frac{|a_{p+1}|}{|a_0|} \cdot \frac{1}{|z|^{p+1}} + \cdots + \frac{|a_n|}{|a_0|} \cdot \frac{1}{|z|^n} \right) \right]
\]
\[
\geq |z|^n \left[ 1 - M \left\{ \frac{1}{|z|^p} + \frac{1}{|z|^{p+1}} + \cdots + \frac{1}{|z|^n} \right\} \right]
\]
\[
> |z|^n \left[ 1 - M \left\{ \frac{1}{|z|^p} + \frac{1}{|z|^{p+1}} + \cdots + \frac{1}{|z|^n} \right\} \right]
\]
\[
= |z|^n \left[ 1 - \frac{M}{|z|^{n-p}} \right]
\]
\[
= \frac{|z|^n}{|z|^{n-p}} \left( |z|^{p-1} - M \right)
\]
\[
> 0
\]
if
\[
|z|^p - |z|^{p-1} - M > 0.
\]
This shows that the zeros of \( P(z) \) with modulus greater than 1 lie in the closed disk \( |z| \leq k \), where \( k > 1 \) is the root of the equation
\[
k^p - k^{p-1} + M = 0.
\]
Since the zeros of \( P(z) \) of modulus less than or equal to 1 are already in \( |z| \leq k \), the result follows.
**Proof of Theorem 2:** For $|z| > 1$, we have $\frac{1}{|z|^{n-j}} < 1, \forall j = 0,1,2,\ldots,p$, so that

$$|P(z)| = |a_n z^n + a_p z^p + a_{p-1} z^{p-1} + \ldots + a_1 z + a_0|$$

$$\geq |a_n| |z|^n - |a_p| \frac{|z|^p}{|z|} + |a_{p-1}| \frac{|z|^{p-1}}{|z|^2} + \ldots + |a_1| \frac{|z|}{|z|^{p-1}} + |a_0| \frac{|z|^p}{|z|^p}$$

$$= |a_n| |z|^n - |a_p| \left( M + \frac{M}{|z|^1} + \frac{M}{|z|^2} + \ldots + \frac{M}{|z|^{p-1}} \right)$$

$$= |a_n| |z|^n - |z|^p M\left( 1 + \frac{1}{|z|} + \frac{1}{|z|^2} + \ldots + \frac{1}{|z|^{p-1}} + \frac{1}{|z|^p} \right)$$

$$= |a_n| |z|^n - \frac{M(|z|^{p+1} - 1)}{|z| - 1}$$

$$= |a_n| |z|^n - |z|^{n+1} - M|z|^{p+1} + M$$

$$> 0$$

if

$$|z|^{n+1} - |z|^n - M|z|^{p+1} + M > 0.$$ 

Define

$$F(z) = z^{n+1} - z^n - Mz^{p+1} + M.$$ 

By using Descarte’s Rule of Signs, $F(z)$ has exactly two positive zeros $k_1$ and $k_2$ and $F(k_i) = 0$ with $\text{Sign}(F(0)) = 1$. Hence we conclude that

$$|F(z)| > 0 \text{ for } |z| > \max(1,k).$$

Hence, it follows that all the zeros of $P(z)$ lie in $|z| \leq \max(1,k)$, where $k \neq 1$ is the positive root of the equation

$$z^{n+1} - z^n - Mz^{p+1} + M = 0.$$ 

That proves Theorem 2.

**Proof of Theorem 3:** Consider the polynomial

$$F(z) = (1-z)P(z) = (1-z)(a_n z^n + a_p z^p + \ldots + a_1 z + a_0)$$

$$= -a_n z^{n+1} + a_n z^n + (a_p - a_{p-1}) z^p + (a_{p-1} - a_{p-2}) z^{p-1} + \ldots + (a_1 - a_0) z + a_0.$$ 

For $|z| > 1$, we have, by using the hypothesis,
\[ |F(z)| \geq \left| a_n \right| |z|^{n+1} - |a_n z^n + (a_p - a_{p-1})z^p + (a_{p-1} - a_{p-2})z^{p-1} + \ldotsb + (a_1 - a_0)z + a_0 | \\
\geq \left| a_n \right| |z|^{n+1} - |z|^n - |z|^p \left( \frac{a_p - a_{p-1}}{|a_n|} + \frac{a_{p-1} - a_{p-2}}{|a_n|} + \ldotsb + \frac{|a_1 - a_0| + |a_0 - a_{-1}|}{|a_n||z|^{p-1}} \right) \\
\geq \left| a_n \right| |z|^{n+1} - |z|^n - |z|^p \left( M' + \frac{M'}{|z|} + \ldotsb + \frac{M'}{|z|^{p-1}} + \frac{M'}{|z|^p} \right) \\
= \left| a_n \right| |z|^{n+1} - |z|^n - |z|^p M'(1 + \frac{1}{|z|} + \ldotsb + \frac{1}{|z|^{p-1}} + \frac{1}{|z|^p}) \\
= \left| a_n \right| |z|^{n+1} - |z|^n - \frac{|z|^{p+1} - 1}{|z| - 1} M' \left| z \right|^{p+1} + M' \\
> 0 \\
\]

if 
\[ |z|^{n+2} - 2|z|^{n+1} + |z|^n - M |z|^{n+1} + M' > 0 . \]

This shows that the zeros of F(z) with modulus greater than 1 lie the closed disk |z| \leq k, where k is the greatest positive root of the equation
\[ z^{n+2} - 2z^{n+1} + z^n - M z^{n+1} + M' = 0 \]

Since z=1 is a zero of the above equation, it follows that the zeros of F(z) with modulus less than 1 also lie in |z| \leq k.

Since the zeros of P(z) are also the zeros of F(z), it follows that all the zeros of P(z) lie in the closed disk |z| \leq k, where k is the greatest positive root of the equation
\[ z^{n+2} - 2z^{n+1} + z^n - M z^{n+1} + M' = 0 , \]

thereby proving Theorem 3.

**Proof of Theorem 4:** Consider the polynomial
\[ F(z) = (1 - z)P(z) = (1 - z)(a_n z^n + a_p z^p + \ldotsb + a_1 z + a_0) \\
= -a_n z^{n+1} + a_n z^n + (a_p - a_{p-1})z^p + (a_{p-1} - a_{p-2})z^{p-1} + \ldotsb + (a_1 - a_0)z + a_0 . \]

For |z| > 1, we have, by using the hypothesis,
\[ |F(z)| \geq |a_n|z|^{n+1} - |z|^n \left( |a_n| + \frac{|a_p - a_{p-1}|}{|z|^{n-p}} + \frac{|a_{p-1} - a_{p-2}|}{|z|^{n-p+1}} + \ldots + \frac{|a_0|}{|z|^n} \right) \]

\[ > |z|^n \left( |a_n| - |a_n| + a_p - a_{p-1} + a_{p-1} - a_{p-2} + \ldots + a_0 - a_0 + |a_0| \right) \]

\[ = |z|^n \left( |a_n| - |a_n| + a_p - a_0 + |a_0| \right) \]

\[ > 0 \]

if
\[ |a_n|z - (|a_n| + a_p - a_0 + |a_0|) > 0 \]
i.e. \[ |z| > \frac{1}{|a_n|} (|a_n| + a_p - a_0 + |a_0|) \].

This shows that the zeros of \( F(z) \) with modulus greater than 1 lie in the closed disk
\[ |z| \leq \frac{1}{|a_n|} (|a_n| + a_p - a_0 + |a_0|) \].

Since the zeros of \( F(z) \) with modulus less than 1 already lie in the above disk, it follows that all the zeros of \( F(z) \) lie in the closed disk
\[ |z| \leq \frac{1}{|a_n|} (|a_n| + a_p - a_0 + |a_0|) \].

Since the zeros of \( P(z) \) are also the zeros of \( F(z) \), the result follows.

References