RW-CONTINUOUS MAPS AND RW-IRRESOLUTE MAPS IN TOPOLOGICAL SPACES

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Abstract

In this paper we introduce and study the concept of regular weakly continuity (briefly rw-continuity) and regular weakly irresolute (briefly rw-irresolute) in topological spaces and discuss some of their properties in topological spaces.

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1. Introduction

Topologist studied weaker and stronger forms of continuous functions in topology using the sets stronger and weaker than open and closed sets. Balachandran et.al [4], Levine [14], Mashhour et.al [16], Gnambah et.al [11] have introduced g- continuity, Semi - continuity, pre- continuity, gpr - continuity respectively.


S.S. Benchalli and R.S Wali [5] introduced new class of sets called regular weakly closed (briefly rw-closed) sets in topological spaces which lies between the class of all w - closed sets and the class of all regular g - closed sets.

The aim of this paper is to introduce and study the concepts of new class of maps namely rw-continuous maps and rw-irresolute maps.

Throughout this paper (X, τ) and (Y,σ) (or simply X and Y) represents the non-empty topological spaces on which no separation axiom are assumed, unless otherwise mentioned. For a subset A of X, cl(A) and int(A) represents the closure of A and interior of A respectively.

2. Preliminaries

In this section we recollect the following basic definitions which are used in this paper.

Definition 2.1 [5]: A subset A of a topological space (X, τ) is called rw-closed (briefly rw-closed) if cl(A) ⊆ U, whenever A ⊆ U and U is regular semiopen in X.

Definition 2.2 [18]: A subset A of a topological space (X, τ) is called regular generalized closed (briefly rg-closed) if cl(A) ⊆ U whenever A ⊆ U and U is regular open in X.

Definition 2.3 [19]: A subset A of a topological space (X, τ) is called weakly closed (briefly w-closed) if cl(A)) ⊆ U whenever A ⊆ U and U is semi open in X.
**Definition 2.4** [18]: A map \( f : (X, \tau) \rightarrow (Y, \sigma) \) from a topological space \( X \) into a topological space \( Y \) is called \( rg \) continuous if the inverse image of every closed set in \( Y \) is \( rg \)-closed in \( X \).

**Definition 2.5** [19]: A map \( f : (X, \tau) \rightarrow (Y, \sigma) \) from a topological space \( X \) into a topological space \( Y \) is called \( w \)-continuous if the inverse image of every closed set in \( Y \) is \( w \)-closed in \( X \).

**Definition 2.6** [6]: A map \( f : (X, \tau) \rightarrow (Y, \sigma) \) from a topological space \( X \) into a topological space \( Y \) is called irresolute if the inverse image of every semi-closed set in \( Y \) is semi-closed in \( X \).

**3. RW - continuous mappings**

In this chapter we introduce and study rw-continuous mappings in topological spaces.

**Definition 3.1:** Let \( f : X \rightarrow Y \) from a topological space \( X \) into a topological space \( Y \) is called \( rw \)-continuous if the inverse image of every closed set in \( Y \) is \( rw \) closed in \( X \).

**Theorem 3.2:** If a map \( f : X \rightarrow Y \) from a topological space \( X \) into a topological space \( Y \) is continuous, then it is \( rw \) continuous but not conversely.

**Proof:** Let \( f : X \rightarrow Y \) be continuous and \( F \) be any closed set in \( Y \). Then the inverse image \( f^{-1}(F) \) is closed in \( X \). Since every closed set is \( rw \)-closed, \( f^{-1}(F) \) is \( rw \)-closed in \( X \). Therefore \( f \) is \( rw \)-continuous.

**Remark 3.3:** The converse of the above theorem need not be true as seen from the following example

**Example 3.4:** Let \( X = Y = \{a,b,c\} \) with topologies \( \tau = \{X, \emptyset, \{a\}, \{a,b\}\} \), \( \sigma = \{Y, \emptyset, \{a\}\} \). Let \( f : X \rightarrow Y \) be a map defined by \( f(a) = a, f(b) = b, f(c) = c \). Here \( f \) is \( rw \) continuous but not continuous since for the closed set \( F = \{a\} \) in \( Y \), \( f^{-1}(F) = \{a\} \) is not closed in \( X \).

**Theorem 3.5:** If a map \( f : X \rightarrow Y \) from a topological space \( X \) into a topological space \( Y \) is \( rw \)-continuous, then it is \( rg \) continuous but not conversely.

**Proof:** Let \( f : X \rightarrow Y \) be \( rw \)-continuous and \( F \) be any closed set in \( Y \). Then the inverse image \( f^{-1}(F) \) is \( rw \)-closed in \( X \). Since every \( rw \)-closed set is \( rg \)-closed, \( f^{-1}(F) \) is \( rg \)-closed in \( X \). Therefore \( f \) is \( rg \)-continuous.

**Remark 3.6:** The converse of the above theorem need not be true as seen from the following example

**Example 3.7:** Let \( X = Y = \{a,b,c,d\} \) with topologies \( \tau = \{X, \emptyset, \{a\}, \{a,b\}, \{a,b,c\}\}, \sigma = \{Y, \emptyset, \{a\}\} \). Let \( f : X \rightarrow Y \) be a map defined by \( f(a) = a, f(b) = b, f(c) = c \). Here \( f \) is \( rg \) continuous but not \( rw \)-continuous since for the closed set \( F = \{c\} \) in \( Y \), \( f^{-1}(F) = \{c\} \) is not \( rw \)-closed in \( X \).

**Theorem 3.8:** If a map \( f : X \rightarrow Y \) from a topological space \( X \) into a topological space \( Y \) is \( w \)-continuous, then it is \( rw \) continuous but not conversely.

**Proof:** Let \( f : X \rightarrow Y \) be \( w \)-continuous and \( F \) be any closed set in \( Y \). Then the inverse image \( f^{-1}(F) \) is \( w \)-closed in \( X \). Since every \( w \)-closed set is \( rw \)-closed, \( f^{-1}(F) \) is \( rw \)-closed in \( X \). Therefore \( f \) is \( rw \)-continuous.

**Remark 3.9:** The converse of the above theorem need not be true as seen from the following example

**Example 3.10:** Let \( X = Y = \{a,b,c\} \) with topologies \( \tau = \{X, \emptyset, \{a\}, \{a,b\}\} \), \( \sigma = \{Y, \emptyset, \{a\}\} \). Let \( f : X \rightarrow Y \) be a map defined by \( f(a) = a, f(b) = b, f(c) = c \). Here \( f \) is \( rw \) continuous but not \( w \)-continuous since for the closed set \( F = \{a\} \) in \( Y \), \( f^{-1}(F) = \{a\} \) is not \( w \)-closed in \( X \).

**Theorem 3.11:** A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is \( rw \)-continuous if and only if \( f^{-1}(U) \) is \( rw \)-open in \((X, \tau)\) for every open set \( U \) in \( (Y, \sigma) \).
Proof: Let \( f: (X, \tau) \to (Y, \sigma) \) be \( \text{rw}-\)continuous and \( U \) an open set in \( (Y, \sigma) \). Then \( f^{-1}(U) \) is \( \text{rw}-\)closed in \( (X, \tau) \). But \( f^{-1}(U) = (f^{-1}(U))^c \) and so \( f^{-1}(U) \) is \( \text{rw}-\)open in \( (X, \tau) \).

**Theorem 3.12:** If \( f: X \to Y \) and \( g: Y \to Z \) be any two functions, then \( g \circ f: X \to Z \) is \( \text{rw}-\)continuous if \( g \) is continuous and \( f \) is \( \text{rw}-\)continuous.

**Proof:** Let \( F \) be any closed set in \( Z \). Since \( g \) is continuous, \( g^{-1}(F) \) is closed in \( Y \) and since \( f \) is \( \text{rw}-\)continuous, \( f^{-1}(g^{-1}(F)) \) is \( \text{rw}-\)closed in \( X \). Hence \( (g \circ f)^{-1} \) is \( \text{rw}-\)closed in \( X \). Thus \( g \circ f \) is \( \text{rw}-\)continuous.

**Remark 3.13:** The Composition of two \( \text{rw}-\)continuous maps need not be \( \text{rw}-\)continuous. Let us prove the remark by the following example.

**Example 3.14:** Let \( X = Y = Z = \{a, b, c\} \) with topologies \( \tau = \{X, \emptyset, \{a, b\}, \{b\}\} = \sigma = \{Y, \emptyset, \{b\}, \{a, b\}\}, \eta = \{Z, \emptyset, \{a, c\}, \{c\}\} \). Let \( g: (X, \tau) \to (Y, \sigma) \) and \( f: (Z, \eta) \to (X, \tau) \) be a map defined by \( g(a) = b, g(b) = b, g(c) = c \). Let \( f = (Z, \eta) \to (X, \tau) \) be any open set of \( (Z, \eta) \). Here \( \{c\} \) is closed set of \( (Y, \sigma) \). Therefore \( g \circ f \) is not \( \text{rw}-\)continuous.

**Theorem 3.15:** Let \( f: X \to Y \) be a \( \text{rw}-\)continuous map from a topological space \( X \) into a topological space \( Y \) and let \( H \) be a closed subset of \( X \). Then the restriction \( f|H: H \to Y \) is \( \text{rw} \)– continuous where \( H \) is endowed with the relative topology.

**Proof:** Let \( F \) be any closed subset in \( Z \). Since \( f \) is \( \text{rw}-\)continuous, \( f^{-1}(F) \) is \( \text{rw}-\)closed in \( X \). If \( f^{-1}(F) \cap H = H_1 \) then \( H_1 \) is a \( \text{rw}-\)closed set in \( X \), since the intersection of two \( \text{rw}-\)closed sets is \( \text{rw}-\)closed. Since \( f|H \) is \( \text{rw}-\)continuous, \( (f|H)^{-1}(F) = H_1 \) it is sufficient to show that \( H_1 \) is \( \text{rw}-\)closed in \( H \). Let \( G_1 \) be any open set of \( H \) such that \( G_1 \) contains \( H_1 \). Let \( G_1 = G \cap H \) where \( G \) is open in \( X \). Now \( H_1 \subseteq G \cap H \subseteq G \cap H \subseteq G \cap H = G_1 \) since \( H_1 \) is \( \text{rw}-\)closed in \( X \). Hence \( f\mid H \) is \( \text{rw}-\)continuous.

**Theorem 3.16:** Let \( f: X \to Y \) be a map from a topological space \( X \) into a topological space \( Y \)

i) The following statements are equivalent
   a) \( f \) is \( \text{rw}-\)continuous
   b) The inverse image of each open set in \( Y \) is \( \text{rw}-\)open in \( X \).
   ii) If \( f: X \to Y \) is \( \text{rw}-\)continuous then \( f(\text{rw cl}(A)) \subseteq \text{cl}(f(A)) \) for every subset \( A \) of \( X \).
   iii) The following statements are equivalent
   a) For each point \( x \) in \( X \) and each open set \( V \) in \( Y \) with \( f(x) \in V \), there is a \( \text{rw}-\)open set \( U \) in \( X \) such that \( x \in U \setminus V \), \( f(U) \subseteq V \).
   b) For every subset \( A \) of \( X \), \( f(\text{rw cl}(A)) \subseteq \text{cl}(f(A)) \).
   c) For each subset \( B \) of \( Y \), \( f^{-1}(\text{rw cl}(B)) \subseteq \text{cl}(f^{-1}(B)) \).

**Proof:** i) Assume that \( f: X \to Y \) be \( \text{rw}-\)continuous. Let \( G \) be open in \( Y \). Then \( G^c \) is closed in \( Y \). Since \( f \) is \( \text{rw}-\)continuous, \( f^{-1}(G^c) \) is \( \text{rw}-\)closed in \( X \). But \( f^{-1}(G^c) = X - f^{-1}(G) \). Thus \( X - f^{-1}(G) \) is \( \text{rw}-\)closed in \( X \) and so \( f^{-1}(G) \) is \( \text{rw}-\)open in \( X \). Therefore (a) implies (b).

Conversely assume that the inverse image of each open set in \( Y \) is \( \text{rw}-\)open in \( X \). Let \( F \) be any closed set in \( Y \). Then \( F^c \) is open in \( Y \). By assumption, \( f^{-1}(F^c) \) is \( \text{rw}-\)open in \( X \). But
f^{-1}(F^c) = X - f^{-1}(F). Thus X - f^{-1}(F) is rw-open in X and so f^{-1}(F) is rw-closed in X. Therefore f is rw-continuous. Hence (b) implies (a). Thus (a) and (b) are equivalent.

ii) Since A \subseteq f^{-1}(f(A)), we have A \subseteq f^{-1}(cl(f(A))). Now cl(f(A)) is a closed set in Y and hence f^{-1}(cl(f(A))) is a rw-closed set containing A. Consequently

\( \text{rw cl}(A) \subseteq f^{-1}(\text{cl}(f(A))). \) Therefore \( f(\text{rw cl}(A)) \subseteq f^{-1}(\text{cl}(f(A))) \).

Conversely if (b) holds and let x \in X a n d \text{let } V be any open set containing f(x).

A. Since f(f^{-1}(V)) \subseteq V \subseteq \text{cl}(\text{f}(A)) \subseteq \text{cl}(A) = A \text{ and hence f(U) holds and hence f}(A) \nsubseteq V \because \text{Therefore we have } y = f(x) \in \text{c l}(f(A)).

4. RW - irresolute mappings

Definition 4.1: Let f: X \rightarrow Y from a topological space X into a topological space Y is called rw-irresolute if the inverse image of every rw-closed set in Y is rw-closed in X.

Theorem 4.2: A map f: X \rightarrow Y is rw-irresolute if and only if the inverse image of every rw-open set in Y is rw-open in X.

Proof: Assume that f is rw-irresolute. Let A be any rw-open set in Y. Then A^c is rw-closed set in Y. Since f is rw-irresolute, f^{-1}(A^c) is rw-closed in X. But f^{-1}(A^c) = X - f^{-1}(A) and so f^{-1}(A) is rw-open in X. Hence the inverse image of every rw-open set in Y is rw-open in X.

Conversely assume that the inverse image of every rw-open set in Y is rw-open in X. Let A be any rw-closed set in Y. Then A^c is rw-open in Y. By assumption, f^{-1}(A^c) is rw-open in X. But f^{-1}(A^c) = X - f^{-1}(A) and so f^{-1}(A) is rw-closed in X. Therefore f is rw-irresolute.

Theorem 4.3: If a map f: X \rightarrow Y is rw-irresolute, then it is rw-continuous but not conversely.

Proof: Assume that f is rw-irresolute. Let F be any closed set in Y. Since every closed set is rw-closed, F is rw-closed in Y. Since f is rw-irresolute, f^{-1}(F) is rw-closed in X. Therefore f is rw-continuous.

Remark 4.4: The converse of the above theorem need not be true as seen from the following example.

Example 4.5: Let X = \{a,b,c\} with topologies \( \tau = \{X,\emptyset,\{a\},\{a,c\} \} \) and \( \sigma = \{Y,\emptyset,\{b\},\{b,c\},\{c\} \} \). Let f: X \rightarrow Y be a map defined by f(a) = a, f(b) = b, f(c) = c. Here f is rw-continuous. However \{a\} is rw-closed in Y but f^{-1}(a) = \{a\} is not rw-closed in X. Therefore f is not rw-irresolute.
Theorem 4.6: Let X, Y and Z be any topological spaces. For any rw-irresolute map f: X → Y and any rw-continuous map g: Y → Z, the composition g ∘ f: (X, τ) → (Z, η) is rw-continuous.

Proof: Let F be any closed set in Z. Since g is rw-continuous, g^−1(F) is rw-closed in Y. Since f is rw-irresolute, f^−1(g^−1(F)) is rw-closed in X. But f^−1(g^−1(F)) = (g ∘ f)^−1. Therefore g ∘ f: X → Z is rw-continuous.

Remark 4.7: The irresolute maps and rw-irresolute maps are independent of each other. Let us prove the remark by the following examples.

Example 4.8: Let X = Y = {a,b,c} with topologies τ = {X, {a,b,c}, {a,c}, {a}} and τ = {Y, {a,b,c}, {a,c}, {a}}. Let f: X → Y be a map defined by f(a) = a, f(b) = b, f(c) = c. Then f is irresolute but it is not rw-irresolute since F = {a} is rw-closed in (Y, τ) but F^−1(F) = {a} is not rw-closed in (X, τ).

Example 4.9: Let X = Y = {a,b,c} with topologies τ = {X, {a,b,c}, {a,c}, {a}} and τ = {Y, {a,b,c}, {a,c}, {a}}. Let f: X → Y be a map defined by f(a) = a, f(b) = b, f(c) = c. Then f is rw-irresolute but it is not irresolute since F = {a,c} is semi-closed in (Y, σ) where F^−1(F) = {a,c} is not semi-closed in (X, τ).

REFERENCES

[16] Mashhour.A.S., Ab

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