

Mathematical Induction and Graph Coloring

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Abstract— Graph coloring can be used to solve problems in all disciplines. In our work, we have used Mathematical Induction to solve graph coloring problems. In this work, we proved that, a map which is formed by some finite number of line segments joining pairs of points on different sides of a given rectangle is 2-colorable.

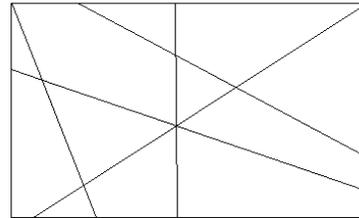


Fig 1- A map

I INTRODUCTION

Mathematical induction is a method of mathematical proof typically used to establish that a given statement is true for all natural numbers. It is done by proving that the first statement in the infinite sequence of statements is true, and then proving that if any one statement in the infinite sequence of statements is true, then so is the next one. Then by mathematical Induction the given statement is true for all natural numbers¹.

A graph is a basically a collection of dots with some pairs of dots being connected by lines. The dots are called vertices and the lines are called edges. More formally, a graph $G = (V, E)$ consists of two sets. The set of vertices denoted by V and the set of edges denoted by E .^{2,3}

The degree of a vertex of a graph is the number of edges adjacent to that vertex.

In Chapter 3, we have proved that a map given in Figure 1 which is formed by some finite number of line segments joining points on different sides of a given rectangle is 2-colorable.

A map which is formed by some finite number of line segments segments joining pairs of points on different sides of a given rectangle in Figure 1.

II. METHODOLOGY

A. Graph Coloring

Perhaps the least obvious application of direct graph theory comes in the form of coloring maps. On any map it is most often the case that any two adjacent regions are colored with a different color so as to help distinguish their geographical features. It turns out that the dual to this problem is to assign a color to each vertex of a simple graph such that no two adjacent vertices share the same color. The chromatic number of a graph is the least number of colors needed to ensure that the vertices can be colored with the above property.^{2,3,4}

There are three types of graph coloring:

- i. Vertex coloring
- ii. Edge coloring
- iii. Face coloring

B. Vertex Coloring

In the most common kind of graph coloring, colors are assigned to the vertices. In, graph theory, graph coloring is a special case of graph labeling; it is an assignment of labels traditionally called "colors" to elements of a graph subject to certain constraints. In its simplest form, it is a way of coloring the vertices of a graph such that no two adjacent vertices share the same color; this is called a vertex coloring.

C. Chromatic Number

The least number of colors needed to color a graph G is called its chromatic number, $\chi(G)$. If G is a graph without loops, then G is k colorable if we assign one of k colors to each vertex so that adjacent vertices have different colors. If G is k - colorable but not $(k - 1)$ - colorable, we say that G is k -chromatic, or that the chromatic number of G is k , and write $\chi(G) = k$ ⁵.

D. Edge Coloring

An edge coloring assigns a color to each edge so that no two adjacent edges share the same color, A graph G is k -colorable (or k -edge colorable) if its edges can be colored with k colors so that no two adjacent edges have the same color.

E. Face Coloring

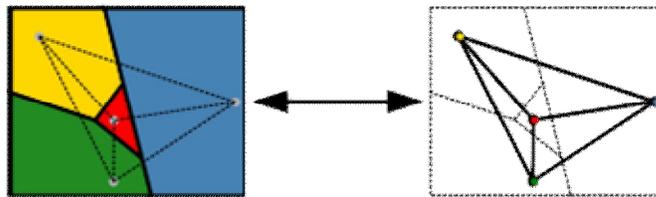


Fig. 2 - Example face coloring

Face coloring of a planar graph assigns a color to each face or region so that *no* two faces that share a common boundary have the same color.

F. Map Coloring

Given a map with various regions in it, we wish to color the regions so that no two regions with a common boundary (consisting of more than one point) having the same color. This map-coloring problem can be restated in terms of graphs by using point-care duality. That is, each region of the map is a vertex of a graph and two vertices are joined by an edge if and only if the corresponding regions have a common boundary. Coloring the map is then equivalent to finding a proper vertex coloring of this graph. The graph that arises in this way is a planar graph.¹

Example for face coloring is depicted in Figure 2.

G. A Graph Coloring Algorithm

- Assign color 1 to the vertex with highest degree.
- Also assign color 1 to any vertex that is not connected to this vertex.
- Assign color 2 to the vertex with the next highest degree that is not already colored.
- Also assign color 2 to any vertex not connected to this vertex and that is not already colored.

- If uncolored vertices remain, assign color 3 to the uncolored vertex with next highest degree and other uncolored, unconnected vertices.
- Proceed in this manner until all vertices are colored^{1,6}.

H. Four Color Theorem

Every planar graph is four colorable. An example is shown in figure 3.

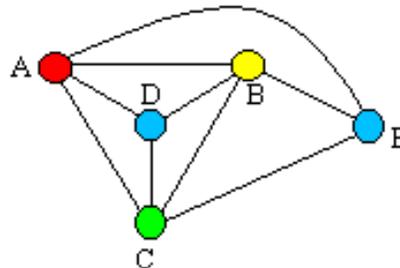
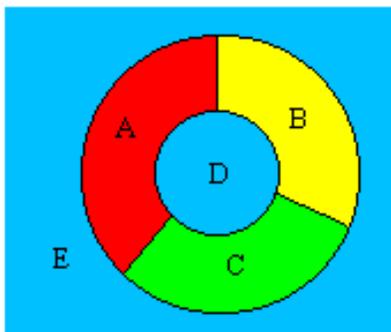


Fig. 3 – Example for four color theorem

A graph is said to be n -colorable if it's possible to assign one of n colors to each vertex in such a way that no two connected vertices have the same color. Obviously the above graph is not 3-colorable, but it is 4-colorable. The Four Color Theorem asserts that every planar graph - and therefore every "map" on the plane or sphere - no matter how large or complex, is 4-colorable.

[1]. *Mathematical Induction*

The simplest and most common form of mathematical induction proves that a statement involving a natural number n holds for all values of n . The proof consists of two steps:

1. The *initial step*: showing that the statement holds when $n = 0$ or $n = 1$.
2. The *inductive step*: assuming that *if* the statement holds for some n , *then* prove that the statement holds when $n + 1$.

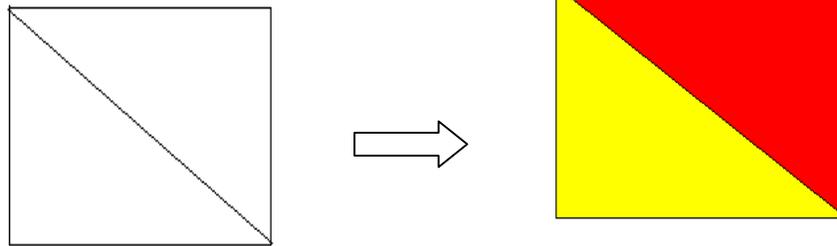


Fig 4 -with one line segments

- with one line segments

When $n = 2$:

Two line segments given in Figure 5 and Figure 6. Here following two cases are possible.

Case 1: Two lines do not intersect each other.

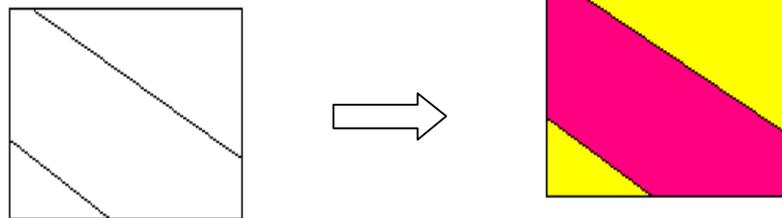


Fig. 5 –with two line segments do not intersect each other

Then the map is 2-colorable.

The description above of the initial step applies when 0 is considered a natural number, as is common in the fields of combinatorics and mathematical logic. If, on the other hand, 1 is taken to be the first natural number, then the base case is given by $n = 1$.¹

2.5 Coloring as a Mathematical Induction Problem.

Consider plane maps, formed by some finite number of line segments joining points on different sides (or opposite corners) of a given rectangle, see Figure 1. Mathematical Induction can be used to show that this is 2-colorable.

III RESULTS

We will use mathematical induction on the number of line segments n .

When $n = 1$:

Only one line segment given by Figure 4, and the result is very clear.

Then the map is 2-colorable.

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Case 2: Two lines intersect each other.

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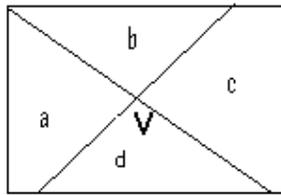


Fig. 6 – with two line segments intersect

Consider the vertex V . Degree of V is equal to 4 which is an even number. There are 4 regions around V . Pick a point from each region and draw the corresponding graph such that each vertex represents a region. Two vertices are adjacent by an edge whenever the corresponding regions have a common boundary. There are four regions adjacent to the vertex V . So the resulting graph is a cycle graph. Therefore it is two colorable.

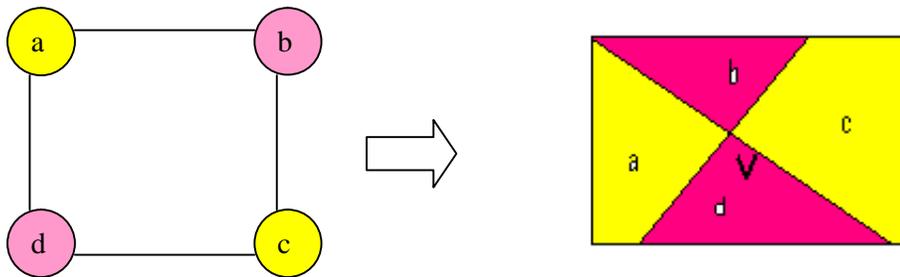


Fig. 7 –coloring of Figure 6

Then the map is 2-colorable.

When $n = 3$:

Three line segments. The following five cases are possible. Given by Figure 8, Figure 9, Figure 10, Figure 11 and Figure 12.

Case 1: Three lines do not intersect each other.

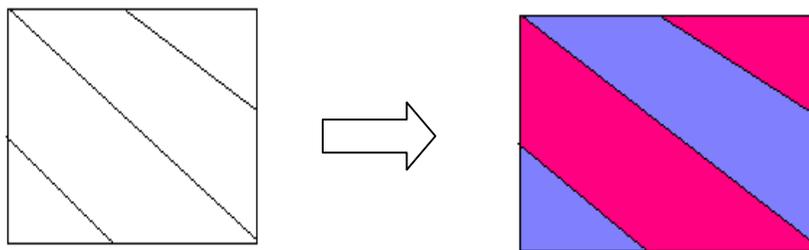
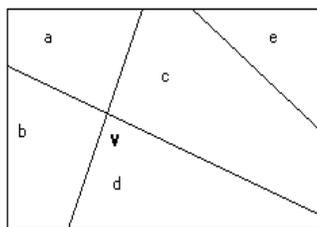


Fig. 8 –with three line segments do not intersect each other

Case 2: Only two lines intersect at a single point.

Fig. 9 – with two line segments intersect



Now consider the vertex V . Degree of V is four. There are four regions adjacent to V and the region c is adjacent to the region e . Then the resulting graph is given below.

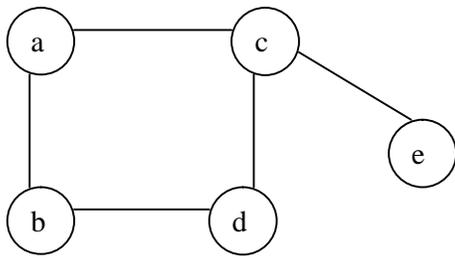


Fig. 10– graph of Figure 9

Consider the sub graph, which is the cycle $a \rightarrow b \rightarrow c \rightarrow d \rightarrow a$. It is an even cycle. So we can assign two colors that alternate around the cycle. It is observed that there are no connection between the vertices a, d , and e . So we can assign the same color to the vertex e as in a and d . Therefore the graph is 2-colorable.

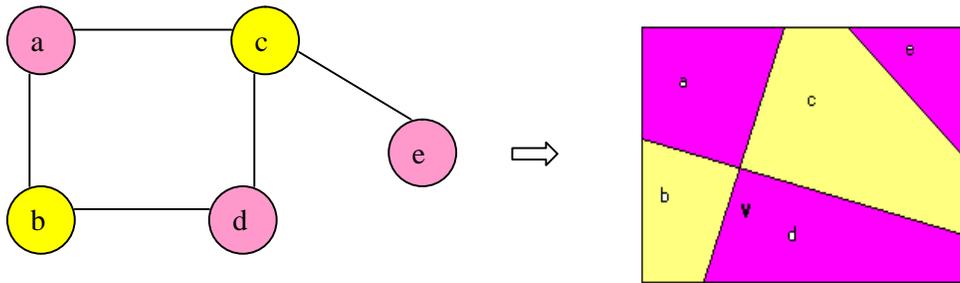


Fig. 11 –coloring of Figure 10

Case 3: Two lines intersect the other line at two distinct points.

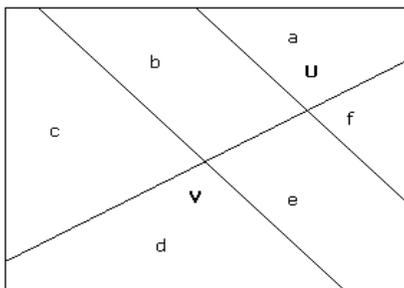


Fig. 12 – two lines intersect the other line at two distinct points

Degree of vertices U and V are equal to four. Considering the regions around U and V , we get

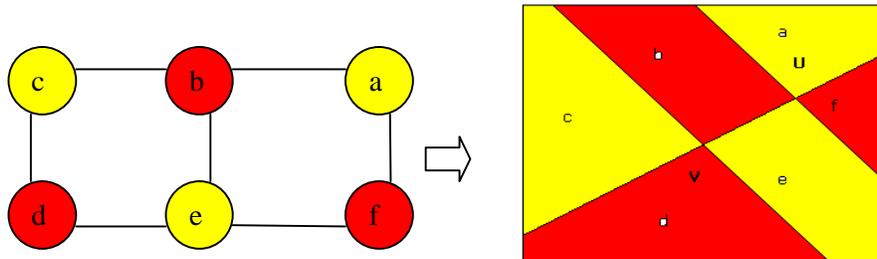


Fig. 13 –coloring of the graph of Figure 12

Case 4: Each distinct pair of lines intersect at a different point.

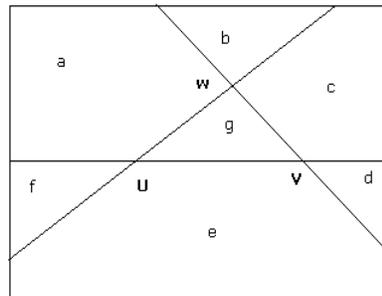


Fig. 14 - each distinct pair of lines intersect at a different point.

Each U, V, W is of degree four. The corresponding graph and its coloring is

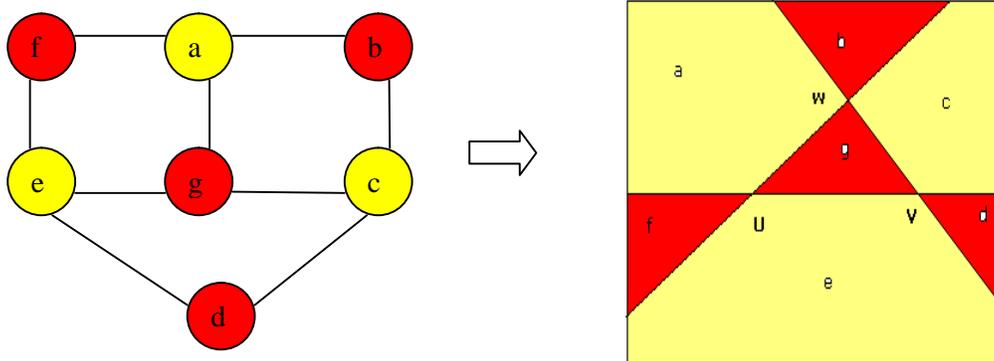


Fig. 15 –coloring of graph of Figure 14

Case 5: Three lines intersect at a single point.

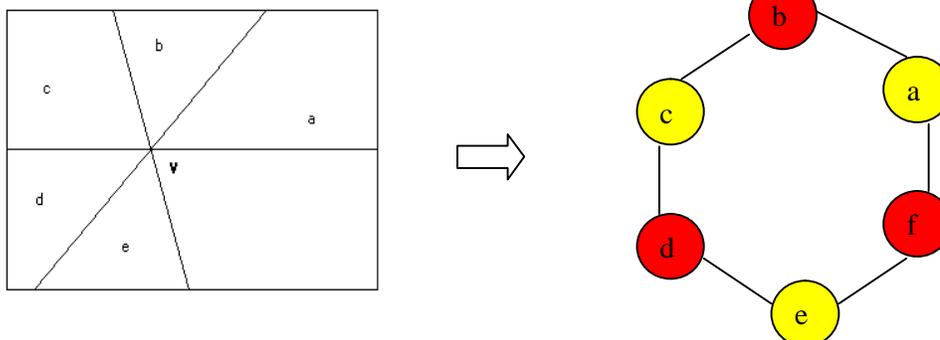


Fig. 16- coloring og graph of case 5.

Degree of V is six. The resulting graph is a cyclic graph of even number of vertices. So we can assign two vertex colors that alternate around the cycle. Therefore the corresponding graph is 2-colorable.

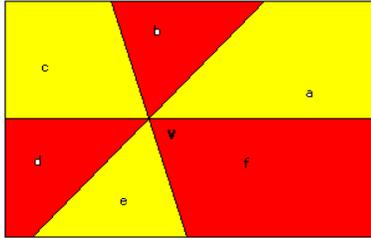


Fig. 17 –coloring of Figure 16

According to the above five situations it can be shown that two colors are sufficient to color the rectangle when it has three lines. When three lines are intersect the degree of vertices inside the rectangle are always of even degree. Therefore the graph is always 2- colorable.

Assume that the assertion is true for $n = p$

Once p lines are drawn by joining different points on the rectangle then the corresponding graph is 2-colorable. Note that degree of each new vertex (if exists) is even.

Now draw the $p + 1^{th}$ line.

Consider the following two cases.

Case 1: Assume that the new line intersects some of the vertices already in the rectangle.

Let m be the degree of such vertex. After drawing the new line segment, degree of that vertex will be increased by 2, i.e. degree is $m + 2$. Since m is an even number $m + 2$ is also an even number. So by considering that vertex we can draw an even cycle graph. It is always 2-colorable. So as previous by considering all the vertices inside the rectangle we can draw the corresponding graph and it is 2-colorable.

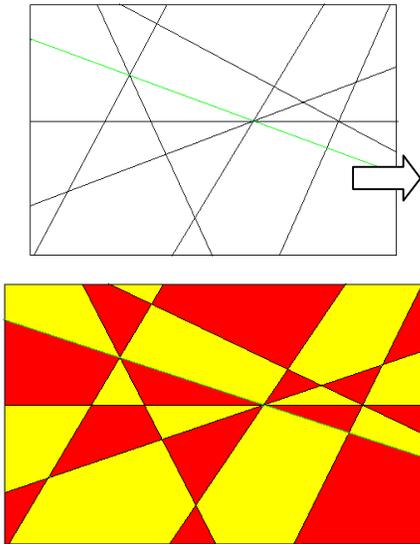


Fig. 18 –coloring when the new line intersects some of the vertices already in the rectangle

Case 2: Assume that the new line does not intersect anyone of the lines which are already in the rectangle. Then as we have shown earlier it is always 2-colorable.

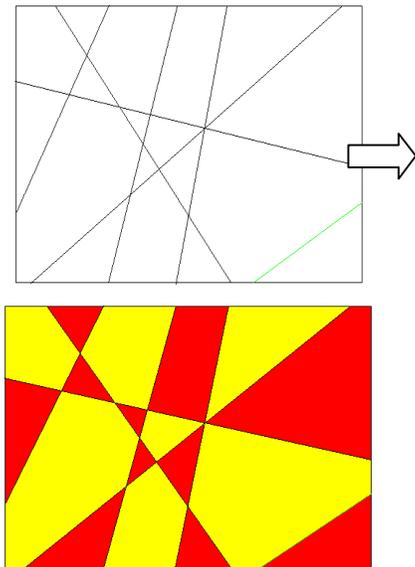


Fig. 19 –coloring when the new line does not intersect anyone of the lines

So, if the graph is two colorable for p lines then it is also two colorable for $p + 1$ lines. So by mathematical induction the given map is 2- colorable.

V CONCLUSION

In this work, we have used Mathematical Induction to prove that a map formed by intersection of finite number of lines joining different points on a rectangle is 2-colorable. Also, degree of each new vertex (if exists) is even.

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