Observations on the Bi-quadratic Equation

\[ xy + (k^2 + 1)z^2 = 5w^4 \]

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Abstract- Three different patterns of non-zero integral solutions to the bi-quadratic equation with four unknowns given by

\[ xy + (k^2 + 1)z^2 = 5w^4 \]

are obtained. A few interesting relations between the solutions and special polygonal numbers are exhibited.

Keywords- Bi-quadratic equation with four unknowns, Integral solutions. Special polygonal numbers.

I. INTRODUCTION

Diophantine equations have an unlimited field for research by reason of their variety. In particular, the quartic diophantine equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity [1-5]. In this context one may refer [6-18] for various problems on the quartic diophantine equations with 2,3,4 variables. This paper concerns with the problem of determining non-trivial integral solutions of the bi-quadratic equation with four unknowns given by \[ xy + (k^2 + 1)z^2 = 5w^4 \]. Explicit integral solutions of the above equation are presented in different patterns. In each of these patterns, a few interesting relations among the solutions and special polygonal numbers are obtained.

II. NOTATIONS

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III. METHOD OF ANALYSIS

The Diophantine equation representing the bi-quadratic equation under consideration is

\[ xy + (k^2 + 1)z^2 = 5w^4 \]  

......(1)

Different solution patterns to (1) are presented below.

**Pattern 1:**

Introducing the transformations

\[ x = u + kv, \quad y = u - kv, \quad z = v \]  

......(2)

in (1), it simplifies to

\[ u^2 + v^2 = 5w^4 \]  

......(3)

Substituting \( w = a^2 + b^2 \) in (3) and choosing

\[ 5 = (2 + i)(2 - i) \], it becomes

\[ (u + iv)(u - iv) = (2 + i)(2 - i)(a^2 + b^2)^2 \]  

......(4)

Let us define \( u + iv = (2 + i)(a + ib)^4 \)  

......(5)

By equating real and imaginary parts, we get

\[ u = 2a^4 - 4a^3b + 12a^2b^2 + 4ab^3 + 2b^4 \]  

\[ v = a^4 + 8a^3b - 6a^2b^2 - 8ab^3 + b^4 \]  

......(6)

Using these values of \( u \) and \( v \) in (2), the integral solutions to (1) are obtained as,

\[ x(a,b,k) = (2 + k)a^4 - 4(1 - 2k)a^3b - 6(2 + k)a^2b^2 + 4(1 - 2k)ab^3 + (2 + k)b^4 \]  

\[ y(a,b,k) = (2 - k)a^4 - 4(1 + 2k)a^3b - 6(2 - k)a^2b^2 + 4(1 + 2k)ab^3 + (2 - k)b^4 \]  

\[ z(a,b) = a^4 + 8a^3b - 6a^2b^2 - 8ab^3 + b^4 \]  

\[ w(a,b) = a^2 + b^2 \]  

......(7)

**Properties:**

(i) \( 4(2 + k)x(a,1,k) + 16(1 - 2k)(2 + k)[2PP_a - Obl_a] \)  

+ \( 2(2 + k)^2[3CS_a + 12T_a - 5] \)

is a perfect square.

(ii) \( y(a,1,k) = (2 - k)[Star_a^2 - 10Hex_a^2 + 5Oct_a^2] \)  

- \( (1 + 2k)[3OH_a + SO_a - 2Gno_a - 2] \)

(iii) \( (2 - k)x(a,1,k) - (2 + k)y(a,1,k) = -16kT_a(3Dec_a - 8Pen_a + 5) \)

(iv) \( 2[x(a,1,k) + y(a,1,k) + z(a,1,k)] + 20 = 5CS_a^2 - 25Gno_a^2 \)

(v) \( 5[x(a,b,k) + y(a,b,k) + z(a,b,k)] \)

\[ \text{can be written as the difference of two squares.} \]

(vi) \( (2 - k)^3 y(a,1,k) - (2 - k)[Star_a^2 - 4SO_a + 5Gno_a + 3] \)  

+ \( (2 - k)[8PP_a + 16T_a - 24Pen_a + 12Oct_a - 2] \)

is a quartic integer.

**Pattern 2:**

By taking \( 5 = (1 + 2i)(1 - 2i) \) in (3), we get

\[ (u + iv)(u - iv) = (1 + 2i)(1 - 2i)(a^2 + b^2)^4 \]  

......(8)

Now we define

\[ u + iv = (1 + 2i)(a + ib)^4 \]  

......(9)
On equating real and imaginary parts we get,
\[ u = a^4 - 8a^3b - 6a^2b^2 + 8ab^3 + b^4 \]
\[ v = 2a^4 + 4a^3b - 12a^2b^2 - 4ab^3 + 2b^4 \] ………. (10)

Then the integral solutions to (1) are given by
\[ x(a,b,k) = (1 + 2k)a^4 - 4(2 - k)a^3b - 6(1 + 2k)a^2b^2 + 4(2 - k)ab^3 + (1 + 2k)b^4 \]
\[ y(a,b,k) = (1 - 2k)a^4 - 4(2 + k)a^3b - 6(1 - 2k)a^2b^2 + 4(2 + k)ab^3 + (1 - 2k)b^4 \]
\[ z(a,b) = 2a^4 + 4a^3b - 12a^2b^2 - 4ab^3 + 2b^4 \]
\[ w(a,b) = a^2 + b^2 \] ………. (11)

Observations:
1. \[ x(a,1,1) + y(a,1,1) = z(a,1,1) - 120\text{Tetra}_a - 1 \]
2. \[ z(a,1,1) - w^2(a,1) + 16T_a + 5\text{Gno}_a = 4 \] is a quartic integer.
3. \[ x(a,b,k) - y(a,b,k) = 2kz(a,b,k) \]
4. \[ 10\{x(a,b,k) + y(a,b,k)\} + 40z(a,b) \] can be written as the difference of two squares.
5. \[ (2 + k)x(a,1,k) - (2 - k)y(a,1,k) = 50[3\text{Hex}_{a}^2 - \text{Dec}_{a}^3 + \text{OH}_{a} - 2\text{PP}_{a} + 1] \]
6. \[ (1 - 2k)x(a,1,k) - (1 + 2k)y(a,1,k) = 10\{3\text{OH}_{a} + \text{SO}_{a} - 2\text{Gno}_{a} - 2\} \]
7. \[ 6(1 + 2k)x(a,1,k) + 48(1 + 2k)^2 - 2(k - 2)(1 + 2k)[9\text{OH}_{a} + 3\text{SO}_{a} + \text{Star}_{a} - 12T_a - 1] \]
\[ \text{is a Nasty number.} \]

Pattern 3:
Let us choose 5 as \[ 5 = \frac{(11 + 2i)(11 - 2i)}{25} \]

Then (3) can be written as
\[ (u + iv)(u - iv) = \frac{(11 + 2i)(11 - 2i)}{25}(a^2 + b^2)^4 \] ………. (12)

Now we define
\[ u + iv = \frac{(11 + 2i)}{5}(a + ib)^4 \] ………. (13)

By equating real and imaginary parts on both sides, we obtain the values of \( u \) and \( v \) as
\[ u = \frac{1}{5}[11a^4 - 8a^3b - 66a^2b^2 + 8ab^3 + 11b^4] \]
\[ v = \frac{1}{5}[2a^4 + 44a^3b - 12a^2b^2 - 44ab^3 + 2b^4] \] ………. (14)

With these values of \( u \) and \( v \), the solutions \( x, y, z \) and \( w \) of (1) are
\[ x(a,b,k) = \frac{1}{5}[(11 + 2k)a^4 + 4(11k - 2)a^3b - 6(11k + 2)a^2b^2 - 4(2 + k)ab^3 + (1 + 2k)b^4] \]
\[ y(a,b,k) = \frac{1}{5}[(11 - 2k)a^4 - 4(11k + 2)a^3b - 6(11k - 2)a^2b^2 + 4(2 - k)ab^3 + (1 - 2k)b^4] \]
\[ z(a,b) = \frac{1}{5}[2a^4 + 44a^3b - 12a^2b^2 - 44ab^3 + 2b^4] \]
\[ w(a,b) = a^2 + b^2 \] ………. (15)

Since our aim is to find integral solutions, Let us take \( a = 5A \) and \( b = 5B \)

Then the solutions are given by,
\[ x(A,B,k) = 125(11 + 2k)A^4 + 500(11k + 2)A^3B - 750(11 + 2k)A^2B^2 - 500(11k - 2)AB^3 + 125(11 + 2k)B^4 \]
\[ y(A,B,k) = 125(11 - 2k)A^4 - 500(11k + 2)A^3B - 750(11 - 2k)A^2B^2 + 500(11k + 2)AB^3 + 125(11 - 2k)B^4 \]
\[ z(A,B,k) = 250A^4 + 5500A^3B - 1500A^2B^2 - 5500AB^3 + 250B^4 \]
\[ w(A,B) = 25A^2 + 25B^2 \] ………. (16)
Observations:

1. \(x(A, B, k) - y(A, B, k) = 2kz(A, B)\)

2. \(2w(A, l) - 25GnoA^2 - 75 = 0\)

3. \(11(A, l, k) - x(A, l, k) - y(A, l, k) = 3125[SOA + 20ctA - 3HexA]\)

4. \(y(A, l, k) + y(A, l, k) - 250[ICTA^2 - 11TEDA - 4SOA + 20T_1 + 11] = 0(\text{mod } 15250)\)

5. \(20[x(A, l, k) - y(A, l, k)]k^3 - 5000[22SOA - 12PrOA - 5GnoA - 3]k^4\)

is a quartic integer.

6. \((11 - 2k)x(A, l, k) - (11 + 2k)y(A, l, k) = 62500k[SOA + DecA - 2HexA]\)

CONCLUSION:

One may search for other patterns of solutions and the corresponding observations.

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