Antiflexible Rings with Weak Novikov Identity

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Abstract - If R is an antiflexible ring of characteristic ≠ 2, 3 with Weak Novikov identity (w, x, y z) = y (w, x, z) then Strong Novikov identity x (y z) = y(x z). Using this results we prove that, if R is a prime not associative antiflexible ring of characteristic ≠ 2, 3 satisfying the Weak Novikov identity (w, x, yz) = y (w, x, z) then R is either an alternative ring (or) strongly (-1, 1) ring.

Key words - Antiflexible rings, Weak Novikov identity, Strong Novikov identity, alternative ring, strongly (-1, 1) ring.

I. INTRODUCTION

E. Kleinfeld in 1994 [1] proved that a prime non-associative Weakly Novikov ring (x, y, z) = (x, z, y) must be Strong Novikov. Again Kleinfeld in 1996 [2] proved that a semi prime ring of characteristic ≠ 2 satisfying the variations of the Novikov identities (x y) z = (x z) y and (x, y, z) = - (x, z, y) is associative. In the another paper of Kleinfeld [3], it is proved that a prime right alternative ring with minimum condition on right ideals which satisfies the identity (w, x, yz) = y (w, x, z) must be associative. Lastly, K. Subhashini in [4] has proved that, if R is a prime (-1,1) ring of characteristic ≠ 2, 3 then R must be commutative and associative.In this paper, first we prove that, a Weak Novikov identity is a Strong Novikov identity. Using this condition of Weak Novikov identity, we prove that an antiflexible ring of characteristic ≠ 2, 3 is either an alternative ring (or) strongly (-1, 1) ring.

II. PRELIMINARIES

A ring is said to be antiflexible ring if it satisfy the identity

A(x, y, z) = (x, y, z) - (z, y, x) ------(1)

The identity (w, x, y z) = y (w, x, z) ------(2)

is known as Weak Novikov identity.

Where as the identity x(yz) = y(xz) ------(3)

is refered as Strong Novikov identity.

A ring is Strong Novikov then it is Weakly Novikov. Moreover, Weakly Novikov rings are a subclass of associative rings where as Strong Novikov rings are not.

The Teichmuller identity which holds in any ring.

B(w, x, y, z) = (wx, y, z) - (w, xy, z) + (w, x, yz) − w(x, y, z) − (w, x, y)z = 0 ------(4)

An antiflexible ring R is a non-associative ring in which the following identities hold.

(w, (x, y, z)) = 0 by [5] ------(5)

The Semi-Jacobi identity is

C(x, y, z) = (x y, z) − x (y, z) − (x, z) y − (x, y, z) − (z, x, y) + (x, z, y) = 0 ------(6)
The nucleus N of any ring is defined as
\[ N = \{ n \in R / (n, R, R) = (R, R, n) = (R, n, R) = 0 \}. \]

An alternative ring R is a ring in which
\[ (x x) y = x (x y), \, y (x x) = (y x) x, \text{ for all } x, y \text{ in } R. \]

These equations are known as the left and right alternative laws respectively.

A right alternative ring R satisfying the identity \( (R, R), R) = 0 \) is called a strongly (-1,1) ring.

**Lemma 2.1**: Let \( n \in N \) then \( (R, N) \subseteq N \).

**Proof**: Let \( w, x, y, z \in R \) and \( n \in N \).

We now take a turn letting one of four elements in Teichmuller identity (4) be in the nucleus N. Thus
\[
\begin{align*}
(n x, y, z) &= n (x, y, z) \\
(w n, y, z) &= (w, n y, z) \\
(w, x n, z) &= (w, x, n z) \\
(w, x, y n) &= (w, x, y) n
\end{align*}
\]

By using equations (5), (1), (10), (1) and (9), we have
\[
W = n \text{ in (5)}
\]
\[
\begin{align*}
n (x, y, z) &= (x, y, z) n \quad \text{(by (5))} \\
&= (z, y, x) n \quad \text{(by (1))} \\
&= (z, y, x n) \quad \text{(by (10))} \\
&= (x n, y, z) \quad \text{(by (8))} \\
&= (x, y, z n) \quad \text{(by (9))} \\
&= (x, n z) \quad \text{(by (10))} \\
&= (x, y, n z) \\
\end{align*}
\]

Hence \((x, y, zn) – (x, y, nz) = 0\)
implies \((x, y, (z, n)) = 0\)

Hence \((R, N) \subseteq N\).

**Lemma 2.2**: The nucleus N of R is an ideal such that \( NA = 0 \). If R is prime and non-associative ring then \( N = 0 \).

**Proof**: For arbitrary elements \( x, y, z \in R \) and \( n \in N \).

From (2), we have
\[
(x, y, z n) = z (x, y, n) = 0
\]
also from (10) \((x, y, n z) = (x, y, z n) = 0\).

Therefore N is both left and right ideal have an ideal of R.

Again using (2) and (5), we have
\[
(x, y, n z) = 0 = n (x, y, z) = (x, y, z) n
\]
i.e., \( N A = A N = 0 \)

Since \( R \) is prime and not associative

and hence \( N = 0 \). ♦

**Lemma 2.3**: If \( R \) is prime and not associative then \( R \) is Strongly Novikov.

**Proof**: Through the repeated use of (2) and (1),

For any \( a, b \in R \), we obtain,

\[
(a, b, xyz) = x (a, b, y z) \quad \text{(by (2))}
\]
\[
= x (y z, b, a) \quad \text{(by (1))}
\]
\[
= (y z, b, x a) \quad \text{(by (2))}
\]
\[
= y (x a, b, z) \quad \text{(by (2))}
\]
\[
= y (z, b, x a) \quad \text{(by (1))}
\]
\[
= y (z, b, x a) \quad \text{(by (1))}
\]
\[
= y (a, b, x z) \quad \text{(by (2))}
\]

Therefore \( (a, b, x.yz) = (a, b, y.xz) \)

\[
\Rightarrow (a, b, x.yz) - (a, b, y.xz) = 0
\]

\[
\Rightarrow (a, b, x.yz - y.xz) = 0
\]

Therefore \( x.yz - y.xz \in N \).

From lemma 2.2, \( N = 0 \),

Hence we have Strong Novikov identity \( x.yz = y.xz \) holds in \( R \). ♦

**Lemma 2.4**: If \( R \) is a prime and not associative ring then \( U \) is an ideal.

**Proof**: Note that

\[
(xy, y) = xy.y - y.xy
\]
\[
= xy.y - x.yy \quad \text{(by (3))}
\]
\[
= xy^2 - xy^2
\]
\[
= 0.
\]

Linearization results in \( (xy, z) = -(xz, y) \)

If \( u \in U \) and \( y = u \) then \( (xu, z) = 0 \)

Thus \( U \) is a left ideal.

Since \( xu = ux \), it follows that

\( U \) is an ideal of \( R \). ♦

Consider the equation \( (y, (x, x, y)) = 0 \)

Replacing \( y \) by \( y + (a, b) \) in the equation then we obtain
\[(a, b), (x, x, y)) = - (y, (x, x, (a, b))) \quad \text{(12)}\]

In \(D(x, y, z) = (x, (yz)) + (y, (zx)) + (z, (x, y)) = 0\)

Put \(y = (R, R, R)\) an arbitrary associator and apply (5), then we have

\[((R, R, R), (z, x)) = 0 \quad \text{(13)}\]

Let \(I\) be the linear span of the alternators in \(R\).

Obviously \(I\) is an ideal of \(R\).

**Lemma 2.5**: Let \(I\) be an ideal of an antiflexible ring with characteristic \(\neq 2, 3\) then

(a) \(\text{ann}(I) = \{x \in R/ xI = Ix = 0\}\) is an ideal.

(b) \(\text{ANN}(I) = \{x \in \text{ann}(I) / (I, R, x) = 0\}\) is the largest ideal of \(R\) containing in \(\text{ann}(I)\).

**Proof**:

By virtue of \(B(x, y, z) = (x, y, z) + (y, z, x) + (z, x, y) = 0\)

Then we claim that \(\text{ann}(I)\) is an ideal.

Let \(t \in I, h \in \text{ann}(I), k \in \text{ANN}(I)\) and \(x, y \in R\).

Since \(\text{ANN}(I) \subseteq \text{ann}(I)\)

We know that all six associators

\[(k, t, x) = (k, x, t) = (x, k, t) = (t, x, k) = (t, k, x) = 0\]

Thus \(kx.t = k.xt = 0\)

And \(t.kx = tk.x = 0\)

i.e., \(kx \in \text{ann}(I)\).

Also from \(D(x, w, y, z) = (xw, y, z) - (x, w, yz) + (x, y, wz) - (x, w, z)y - (x, y, z)w = 0\)

We have \(0 = (t, y, kx) + (t, k, yx) - (t, y, x)k - (t, k, x)y = (t, y, kx)\)

Since \((t, y, x) \in I\) Therefore \(\text{ANN}(I)\) is a right ideal.

Now \((xk)t = x(kt) = 0\) and

\[t(xk) = (tx)k = 0\]

so \(xk \in \text{ann}(I)\)

\(\Rightarrow \text{ann}(I)\) is an ideal.

To show \((t, y, xk) = 0\)

We consider \(B(t, x, k, y) = (tx, k, y) - (t, xk, y) + (t, x, ky) - t(x, k, y) - (t, x, ky) = 0\)

Since \(I\) is an ideal, \(\text{ANN}(I)\) is a right ideal contained in \(\text{ann}(I)\) and any associator with elements from \(R, I\) and \(\text{ANN}(I)\) is zero, then these two identities reduce to

\[-(t, xk, y) - t(x, k, y) = 0\] and \((x, k, y)t = 0\)

Adding these two identities and applying (5) we have

\[\text{ANN}(I) \subseteq \text{ann}(I)\]

Thus \((t, y, xk) = 0\).
Which establishes ANN(I) is an ideal of R. ◆

**Theorem 2.1** : Let R be a prime not associative antiflexible ring of characteristic ≠ 2, 3 satisfying the weak Novikov identity \((w, x, yz) = y(w, x, z)\), then R is either an alternative ring or a strongly (-1, 1) ring.

**Proof** : By semi-Jacobi identity, we have
\[ C(x, y, z) = (x y, z) - (x, z) y - (x, y, z) - (z, x, y) + (x, z, y) = 0 \]
Interchanging x and y in this equation, we have
\[ C(y, x, z) = (y x, z) - (y, z) x - (y, x, z) - (z, y, x) + (y, z, x) = 0 \]
Subtracting these two equations, we have
\[ (x y, z) - (x, z) y - (x, y, z) - (z, x, y) + (x, z, y) - (y x, z) + y (x, z) + (y, z) x + (y, x, z) + (z, y, x) - (y, z, x) = 0. \]
\[ \Rightarrow (xy - yx, z) - (x(y, z) - (y, z)x) + (y(x, z) - (x, z)y) = 0 \]
\[ ------(14) \]
Since I is an ideal and also from (14), \((I, Z) = 0\), we obtain
\[ (x(y, z) - (y, z)x) = 0 \]
\[ \Rightarrow x(y, z) = (y, z)x \]
Let \( x = (x, x, z) \) be an alternator and \( z = (R, R) \), we have
\[ (x, x, z) (y, (R, R)) = 0 = (y, (R, R)) (x, x, z) \]
Thus we have established \((y, (R, R)) ∈ Ann(I)\)
Next using linearized (14) and the fact that I is an ideal, we have
\[ (I, R, (y, (R, R))) = -(R, I, (y, (R, R))) = 0 \]
Thus \((y, (R, R)) ∈ ANN(I)\)
But ANN(I) is an ideal of R from Lemma 6
Since I. \((ANN(I)) = 0\) and R is prime.
Then either \(Z = 0\) or \((R, (R, R)) = 0\).
If \(1 = 0\) then R is alternative ring.
If \((R, (R, R)) = 0\) then R is strongly (-1, 1) ring. ◆

**III. REFERENCES**


