On Generalized Concircular $\phi$-Recurrent $N(k)$-Contact Metric Manifold

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Abstract: The object of the present paper is to study generalized concircular $\phi$-recurrent $N(k)$-contact metric manifold and obtained some important results.

Key Words: $N(k)$-contact metric manifold, $\eta$-Einstein manifold, Generalized Concircular $\phi$-recurrent manifold, constant curvature.


1. Introduction

In 1988, S. Tanno [12] introduced the notion of $k$-nullity distribution of a Contact metric manifold as a distribution such that the characteristic vector field $\xi$ of the Contact metric manifold belongs to the distribution. The Contact metric manifold with $\xi$ belonging to the $k$-nullity distribution is called $N(k)$-Contact metric manifold and such a manifold is also studied by various authors. In 2008, De, Gazi [6] studied $\phi$-recurrent $N(k)$-Contact metric manifold.

In this paper we study Generalized Concircular $\phi$-recurrent $N(k)$-Contact metric manifold. Here we show that Generalized Concircular $\phi$-recurrent $N(k)$-Contact metric manifold is an $\eta$-Einstein manifold, and we find a relation between the associated 1-forms A and B. We also prove that the characteristic vector field $\xi$ and the vector field $\rho$ associated to the 1-forms A and B are co-directional. Finally we prove that a generalized Concircular $\phi$-recurrent $N(k)$-Contact metric manifold is of constant curvature.

2. Contact Metric Manifold

A $(2n+1)$-dimensional manifold $M^{2n+1}$ is said to admit an almost Contact structure if it admits a tensor field $\phi$ of type $(1,1)$, a vector field $\xi$ and a 1-form $\eta$ satisfying

$$(a) \quad \phi^2(X) = -X + \eta(X)\xi, \quad (b) \quad \eta(\xi) = 1, \quad (c) \quad \eta \circ \phi = 0, \quad (d) \quad \phi \xi = 0.$$  \tag{2.1}$$

An almost contact metric structure is said to be normal if the induced almost complex structure $J$ on the product manifold $M^{2n+1} \times \mathbb{R}$ defined by

$$J(X, f \frac{d}{dt}) = (\phi X - f\xi, \eta(X) \frac{d}{dt})$$

is integrable, where $X$ is tangent to $M$, $t$ is the coordinate of $\mathbb{R}$ and $f$ is a smooth function on $M \times \mathbb{R}$. Let $g$ be a compatible Riemannian metric with almost contact structure $(\phi, \xi, \eta)$, that is

$$(2.2) \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$
then $M$ becomes an almost contact metric manifold equipped with an almost contact structure $(\phi, \xi, \eta, g)$. From (2.1) it can be easily seen that

\begin{equation}
(a) \quad g(X, \phi Y) = -g(\phi X, Y), \quad (b) \quad g(X, \xi) = \eta(X),
\end{equation}

for all vector fields $X, Y$. An almost contact metric structure becomes a contact metric structure if

\begin{equation}
g(X, \phi Y) = d\eta(X, Y),
\end{equation}

for all vector fields $X, Y$. The 1-form $\eta$ is then a contact form and $\xi$ is its characteristic vector field. The $k$-nullity distribution $N(k)$ of a Riemannian manifold $M$ is defined by [12]

\[N(k): p \to Np(k) = \{Z \in TpM : R(X,Y)Z = g(Y,Z)X - g(X,Z)Y\},\]

$k$ being a constant. If the characteristic vector $\xi \in N(k)$, then we call a Contact metric manifold an $N(k)$-Contact metric manifold.

In $N(k)$-Contact metric manifold the following relations hold [6]:

\begin{equation}
h^2 = (k - 1)\phi^2, \quad k \leq 1,
\end{equation}

\begin{equation}
(\nabla_X \phi)(Y) = g(X + hX, Y) \xi - \eta(Y)(X + hX),
\end{equation}

\begin{equation}
R(\xi, X)Y = k[g(X,Y)\xi - \eta(Y)X],
\end{equation}

\begin{equation}
S(X, \xi) = 2nk\eta(X),
\end{equation}

\begin{equation}
S(X, Y) = 2(n - 1)g(X, Y) + 2(n - 1)g(hX, Y) + 2(1 - n) + 2nk]\eta(X)\eta(Y), \quad n \geq 1,
\end{equation}

\begin{equation}
\eta(\xi) = 2n(2n - 2k),
\end{equation}

\begin{equation}
\nabla_X \xi = \phi X - \phi hX,
\end{equation}

\begin{equation}
S(\phi X, \phi Y) = S(X, Y) - 2nk\eta(X)\eta(Y) - 4(n - 1)g(hX,Y),
\end{equation}

\begin{equation}
(\nabla_X \eta)(Y) = g(X + hX, \phi Y),
\end{equation}

\begin{equation}
R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y],
\end{equation}

\begin{equation}
\eta(R(X,Y)Z) = k[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)].
\end{equation}

**Definition 2.1.** ([6]) A $N(k)$-Contact metric manifold is said to be locally concircular $\phi$-symmetric if

\begin{equation}
\phi^2((\nabla_X \phi)(X, Y)Z = 0,
\end{equation}

for all vector fields $X, Y, Z, W$ orthogonal to $\xi$.

**Definition 2.2.** ([6]) A $N(k)$-Contact metric manifold is said to be concircular $\phi$- recurrent if there exists a non-zero 1-form $A$ such that

\begin{equation}
\phi^2((\nabla_X \phi)(X, Y)Z = A(W)\phi(X, Y)Z,
\end{equation}
for arbitrary vector fields $X, Y, Z$ and $W$, where $\bar{C}$ is a Concircular curvature tensor given by[4]

\begin{equation}
\bar{C}(X,Y)Z = R(X,Y)Z - \frac{r}{2n(2n+1)}[g(Y,Z)X - g(X,Z)Y],
\end{equation}

where $R$ is the curvature tensor, and $r$ is the scalar curvature.

If the 1-form $\alpha$ vanishes, then the manifold reduces to locally concircular $\phi$-symmetric manifold.

**Definition 2.3.** A $N(k)$-Contact metric manifold is said to be generalized concircular $\phi$-recurrent if its curvature tensor $\bar{C}$ satisfies the condition

\begin{equation}
\phi^2((\nabla_w\bar{C})(X,Y)Z) = A(W)\bar{C}(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y],
\end{equation}

where $A$ and $B$ are two 1-forms, $B$ is non-zero and these are defined by

$$A(W) = g(W, \rho_1), \quad B(W) = g(W, \rho_2),$$

and $\rho_1, \rho_2$ are vector fields associated with 1-forms $A$ and $B$, respectively.

**3. Generalized Concircular $\phi$-Recurrent $N(k)$-Contact Metric Manifold**

Let us consider a Generalized Concircular $\phi$-recurrent $N(k)$-Contact metric manifold. Then by virtue of (2.1) and (2.18) we have

\begin{equation}
-(\nabla_w\bar{C})(X,Y)Z + \eta(\nabla_w\bar{C})(X,Y)Z\xi
= A(W)\bar{C}(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y],
\end{equation}

from which it follows that,

\begin{equation}
-g((\nabla_w\bar{C})(X,Y)Z, U) + \eta((\nabla_w\bar{C})(X,Y)Z)\eta(U)
= A(W)g(\bar{C}(X,Y)Z, U) + B(W)[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)].
\end{equation}

Let $\{e_i\}, i = 1, 2, \ldots, 2n + 1$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = \{e_i\}$ in (3.2) and taking summation over $i, 1 \leq i \leq 2n + 1$, we get

\begin{equation}
(\nabla_wS)(Y, Z) = \frac{dr(W)}{2n+1}g(Y,Z) - \frac{dr(W)}{2n(2n+1)}[g(Y,Z)\eta(Y)\eta(Z)]
-A(W)[S(Y,Z) - \frac{r}{2n+1}g(Y,Z)] - 2nB(W)g(Y,Z).
\end{equation}

Replacing $Z$ by $\xi$ in (3.3) and using (2.12), we have

\begin{equation}
(\nabla_wS)(Y, \xi) = \frac{dr(W)}{2n+1}\eta(Y) - A(W)\eta(Y)\left[2nk - \frac{r}{2n+1}\right] - 2nB(W)\eta(Y).
\end{equation}

Now we have,

$$\nabla_wS(Y, \xi) = \nabla_wS(Y, \xi) - S(\nabla_wY, \xi) - S(Y, \nabla_w\xi).$$
Using (2.8) and (2.10) in the above relation, it follows that

\[(\nabla_W S)(Y, \xi) = -2nk\phi g(\phi W + \phi hW, Y) + S(Y, \phi W + \phi hW).\]  

(3.5)

In view of (3.4) and (3.5), we have

\[S(Y, \phi W + \phi hW) = 2nk\phi g(\phi W + \phi hW, Y) + \frac{dr(W)}{2n + 1}\eta(Y)\]

\[- A(W)\eta(Y)2nk2n+1 - 2nB(W)\eta(Y).\]

Replacing \(Y\) by \(\phi Y\) in (3.6), and after a brief simplification, we get

\[S(Y, W) = 2[(n + k - 1) + n(k - 1)(nk + n - 1)]g(Y, W)\]

\[+ 2[(n - 1)(k - 1) - n(k - 1)(nk + n - 1)]\eta(Y)\eta(W),\]

or,

\[S(Y, W) = ag(Y, W) + b\eta(Y)\eta(W),\]  

(3.7)

Where \(a = 2[(n + k - 1) + n(k - 1)(nk + n - 1)],\)

\[b = 2[(n - 1)(k - 1) - n(k - 1)(nk + n - 1)]\]

are constants.

Therefore we state the following:

**Theorem 3.1.** A Generalized Concircular \(\phi\)-recurrent \(N(k)\)-Contact metric manifold is an \(\eta\)-Einstein manifold.

Now putting \(Y = Z = ei\) in (3.2) and taking summation over \(i, i = 1, 2, \ldots, 2n + 1,\) we get

\[-(\nabla_W S)(X, U) + \frac{dr(W)}{2n + 1}g(X, U) + (\nabla_W S)(X, \xi)\eta(U)\]

\[- A(W)[S(X, U) - \frac{r}{2n + 1}g(X, U)] + 2nB(W)g(X, U).\]  

(3.8)

Putting \(U = \xi\) in (3.8), we have

\[A(W)\eta(X)2nk2n+1 + 2nB(W)\eta(X) = 0.\]  

(3.9)

Putting \(X = \xi\) in (3.9) we have,

\[B(W) = \left[\frac{r}{2n(2n + 1)} - k\right]A(W).\]  

(3.10)

Hence we state the following theorem:

**Theorem 3.2.** In a generalized Concircularly \(\phi\)-recurrent \(N(k)\)-Contact metric manifold, the 1-forms \(A\) and \(B\) are related as in (3.10).

Now from (3.1) we have

\[(\nabla_W \bar{C})(X, Y)Z = \eta(\nabla_W \bar{C})(X, Y)Z,\]

\[\xi - A(W)\bar{C}(X, Y)Z - B(W)g(Y, Z)X - g(X, Z)g(Y, Z).\]  

(3.11)

This implies
\[(\nabla_W R)(X,Y)Z = \eta(\nabla_W R)(X,Y)Z - A(W)R(X,Y)Z \]
\[+ \frac{r}{2n(2n+1)}[g(Y,Z)X - g(X,Z)Y - g(Y,Z)\eta(X)\xi + g(X,Z)\eta(Y)\xi] \]
\[
(3.12) + \frac{r}{2n(2n+1)}A(W) [g(Y,Z)X - g(X,Z)Y] - B(W) [g(Y,Z)X - g(X,Z)Y].
\]

From (3.12) and the Bianchi identity we get
\[
A(W)\eta(R(X,Y)Z) + A(X)\eta(R(Y,W)Z) + A(Y)\eta(R(W,X)Z)
\]
\[
= \left[ \frac{r}{2n(2n+1)}A(W) - B(W) \right] [g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]
\]
\[
+ \left[ \frac{r}{2n(2n+1)}A(X) - B(X) \right] [g(W,Z)\eta(Y) - g(Y,Z)\eta(W)]
\]
\[
+ \left[ \frac{r}{2n(2n+1)}A(Y) - B(Y) \right] [g(X,Z)\eta(W) - g(W,Z)\eta(X)].
\]

(3.13)

By virtue of (2.14), we obtain from (3.13) that
\[
A(W)k[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] + A(X)k[g(W,Z)\eta(Y) - g(Y,Z)\eta(W)]
\]
\[
+ A(Y)k[g(X,Z)\eta(W) - g(W,Z)\eta(X)]
\]
\[
= \left[ \frac{r}{2n(2n+1)}A(W) - B(W) \right] [g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]
\]
\[
+ \left[ \frac{r}{2n(2n+1)}A(X) - B(X) \right] [g(W,Z)\eta(Y) - g(Y,Z)\eta(W)]
\]
\[
+ \left[ \frac{r}{2n(2n+1)}A(Y) - B(Y) \right] [g(X,Z)\eta(W) - g(W,Z)\eta(X)].
\]

(3.14)

Putting \( Y = Z = e_i \) in (3.14) and taking summation over \( i, 1 \leq i \leq 2n+1 \), we get
\[
(a) A(W)\eta(X) = A(X)\eta(W),
\]
\[
(b) B(W)\eta(X) = B(X)\eta(W)
\]

for all vector fields \( X, W \).

Replacing \( X \) by \( \xi \) in (3.15) we get
\[
(a) A(W) = \eta(W)\eta(\rho_1)
\]
\[
(b) B(W) = \eta(W)\eta(\rho_2)
\]

(3.16)

From (3.15) and (3.16), we can state the following theorem:

**Theorem 3.3.** In a generalized concircular \( \phi \)-recurrent \( N(k) \)-contact metric manifold, the characteristic field \( \xi \) and the vector fields \( \rho_1 \) and \( \rho_2 \) associated to the 1-forms \( A \) and \( B \) respectively are co-directional and the 1-forms \( A \) and \( B \) are given by (3.16).

4. 3-dimensional Generalized Concircular \( \phi \)-Recurrent \( N(k) \) – Contact Metric Manifold

In a 3-dimensional \( N(k) \) – Contact metric Manifold \( (M^3, g) \), we have

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\[ R(X,Y)Z = \left( \frac{r}{2} - 2k \right) [g(Y,Z)X - g(X,Z)Y] + \left( 3k - \frac{r}{2} \right) [g(Y,Z)\eta(X)\xi] - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y, \]

(4.1)

and

\[ S(X,Y) = \left( \frac{r}{2} - k \right) g(X,Y) + \left( 3k - \frac{r}{2} \right) \eta(X)\eta(Y). \]

(4.2)

Using (4.1) in (2.17), we get

\[ \bar{C}(X,Y)Z = \left( \frac{r}{2} - 2k - \frac{r}{2n(2n+1)} \right) [g(Y,Z)X - g(X,Z)Y] + \left[ 3k - \frac{r}{2} \right] [g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \]

(4.3)

Differentiating the equation (4.3) covariantly, we get

\[
\begin{align*}
(\nabla_W \bar{C})(X,Y)Z &= \left[ \frac{10d\tau(W)}{21} \right] [g(Y,Z)X - g(X,Z)Y] \\
&\quad - \left[ \frac{d\tau(W)}{2} \right] [g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y] \\
&\quad + \left[ 3k - \frac{r}{2} \right] [g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi] (\nabla_W \xi) + \left[ 3k - \frac{r}{2} \right] [g(Y,Z)\xi - \eta(Z)Y] (\nabla_W \eta) (X) \\
&\quad - \left[ 3k - \frac{r}{2} \right] [g(X,Z)\xi - \eta(Z)X] (\nabla_W \eta) (Y).
\end{align*}
\]

(4.4)

Noting that we may assume that all vector fields \(X, Y, Z, W\) are orthogonal to \(\xi\) and using (2.1), we get

\[ (\nabla_W \bar{C})(X,Y)Z = \left[ \frac{10d\tau(W)}{21} \right] [g(Y,Z)X - g(X,Z)Y] + \left[ 3k - \frac{r}{2} \right] [g(Y,Z)(\nabla_W \eta)(X) - g(X,Z)(\nabla_W \eta)(Y)] \xi. \]

(4.5)

Applying \(\phi^2\) on both sides of (4.5) and using (2.1), we get

\[ \phi^2(\nabla_W \bar{C})(X,Y)Z = \left[ \frac{10d\tau(W)}{21} \right] [g(X,Z)Y - g(Y,Z)X]. \]

(4.6)

Using (2.18), the equation (4.6) reduces to,

\[ A(W) \bar{C}(X,Y)Z + B(W) [g(Y,Z)X - g(X,Z)Y] = \left[ \frac{10d\tau(W)}{21} \right] [g(Y,Z)X - g(X,Z)Y]. \]

(4.7)

Putting \(W = \{e_i\}\), where \(\{e_i\}, i = 1, 2, 3\), is an orthonormal basis of the tangent space at any point of the manifold and taking summation over \(i, 1 \leq i \leq 3\), we obtain

\[ \bar{C}(X,Y)Z = \lambda [g(Y,Z)X - g(X,Z)Y], \]

(4.8)
where $\lambda = \left[ \frac{104r(e_i)}{21A(e_i)} + \frac{B(e_i)}{A(e_i)} \right]$ is a scalar, since $A$ and $B$ are non-zero 1-forms. Then by Schur’s theorem $\lambda$ will be a constant on the manifold. Therefore, we state the following theorem:

**Theorem 4.4.** A 3-dimensional Generalized Concircular $\phi$-recurrent $N(k)$-Contact metric manifold is of constant curvature.

**References**