On Some Properties of a Generalization of Bessel-Maitland Function

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Abstract—The present paper is the investigation of certain properties of generalized Bessel-Maitland function, written in the form

\[ J_{\nu q}^\mu (z) = \sum_{n=0}^{\infty} \frac{(\nu q)^n}{n! \Gamma(n\mu + \nu + 1)} (z)^n \]

where \( \mu, \nu, \gamma \in \mathbb{C}; \text{Re}(\mu) \geq 0 \).

\( \text{Re}(\nu) \geq -1. \text{Re}(\gamma) \geq 0 \) and \( q \in (0.1) \cup N \). For the function \( J_{\nu q}^\mu (z) \), a number of results including differentiation and integration formulas, Mellin-Barnes integral representation, Laplace transform, Euler transform, \( k \)-transform, Varma transform, Mellin transform. Various relationship with other functions including Fox's \( H \)-function and Wright hypergeometric function were also established. In the end certain relations have been obtained by using the Riemann-Liouville fractional integrals and derivatives.

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I. INTRODUCTION

The special function of the form defined by the series representation

\[ J_{\nu}^\mu (z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n! \Gamma(n\mu + \nu + 1)} \]

\[ \phi(\nu, \nu+1; -z) \]

\[ H_{0.2}^{1.0} \left[ z \right| (0.1), (\nu, \mu) \] (1.1)

is known as Bessel-Maitland function, or the Wright generalized function ([15], (8.3)). It has a wide application in the problem of physics, chemistry, biology, engineering and applied sciences. The theory of Bessel functions is intimately connected with the theory of certain types of differential equations. A detailed account of application of Bessel function is represented in the book of Watson [12].

In this paper, a generalization of Bessel-Maitland function is investigated and is defined as

\[ J_{\nu q}^\mu (z) = \sum_{n=0}^{\infty} \frac{(\gamma q^n (\nu q)^n}{n! \Gamma(n\mu + \nu + 1)} (z)^n \] (1.2)

where, \( \mu, \nu, \gamma \in \mathbb{C}; \text{Re}(\mu) \geq 0, \text{Re}(\nu) \geq -1, \text{Re}(\gamma) \geq 0 \) and \( q \in (0,1) \cup N \) and \( (\gamma) = 1 \), \( (\gamma q^n) = \frac{\Gamma(q+\gamma q^n)}{\Gamma(\gamma)} \), denotes the generalized pochhammer symbol (see Rainville, [8]), which in particular reduces to \( q^n \prod_{j=1}^{n} \left( \frac{\gamma + (j-1)}{q} \right)^n \) if \( q \in N \).

Some important special cases of this function are enumerated below:

(i) \( J_{\nu 0}^\mu (z) = J_{\nu}^\mu (z) \), defined by (1.1).

(ii) \( J_{-1,0}^\mu (z) = \phi(\nu, \nu+1; z) \), known as Wright function ([1], section 18.1) was introduced by Wright [9].

(iii) \( \left( \frac{z}{q} \right)^{\nu+1} \phi(\nu, \nu+1; z) \right) = J_{\nu}^\mu (z) \), is the ordinary Bessel function (Rainville, [8], pp.109).

(iv) If \( \mu = k \in N \) and \( q \in N \),

\[ J_{\nu q}^k \left( \frac{\Delta(q; \gamma) ; -q^k z}{\Delta(k; \nu + 1); -q^k k} \right) \] (1.3)

where, \( q \Delta_k (\cdot) \) is the generalized hypergeometric function and the symbol \( \Delta(q; \gamma) \) is a \( q \)-tuple \( \frac{\nu + 1}{q}, \frac{\nu + q + 1}{q} ; \frac{\nu + 2}{q} ; k \), \( \Delta(k; \nu + 1) \) is a \( k \)-tuple \( \frac{\nu + 1}{k}, \frac{\nu + 2}{k} ; k \).

Convergence criteria for the generalized
hypergeometric function \( qF_k \).

(a) If \( q \leq k \), the function \( qF_k \) converges for \( |z| < \infty \).

(b) If \( q = k + 1 \), the function \( qF_k \) converges for \( |z| < 1 \).

(c) If \( q > k + 1 \), the function \( qF_k \) is divergent for \( z \neq 0 \).

(d) If \( q = k + 1 \), the function \( qF_k \) is absolutely convergent on the unit circle \( |z| = 1 \), if

\[
\text{Re} \left( \sum_{j=1}^{k} \frac{v+j}{k} - \sum_{i=1}^{q} \frac{y+i-1}{q} \right) > 0.
\]

(v) \( J_{\mu,\nu}^{\alpha,\beta}(z) = \frac{1}{\Gamma(\gamma)} \psi_1 \left[ \left( y, q \right); (v+1,\mu n); -z \right] \)

\[
= \frac{1}{\Gamma(\gamma)} H_{1,2}^{1,1} \left[ \left( 1-y,q \right);(0,1),(-\nu,\mu); \right] (1.4)
\]

where \( \psi_1(\cdot) \) and \( H_{1,2}^{1,1}(\cdot) \) are respectively Wright generalized hypergeometric function [10] and \( H \)-function [6].

(vi) \( J_{\nu-1,q}^{\alpha,\beta}(z) = E_{\nu,q}^\gamma(z) \), where \( \alpha, \beta, \gamma \in C \); \( \text{Re}(\alpha) > 0, \text{Re}(\beta) > 0, \text{Re}(\gamma) > 0 \) and \( q \in (0,1) \cup \mathbb{N} \), was given by Shukla and Prajapati [3].

\( J_{\nu-1,1,q}^{\alpha,\beta}(z) \), was introduced by Prabhakar [17].

\( J_{\nu,1,1}^{\alpha,\beta}(z) = E_{\nu,1}^\gamma(z) \), where \( \nu, \beta \in C; \text{Re}(\alpha) > 0, \text{Re}(\beta) > 0 \), was studied by Wiman [2].

\( J_{0,1}^{\alpha,\beta}(z) = E_{\alpha,1}^\gamma(z) \), where \( z \in C \) and \( \text{Re}(\alpha) > 0 \), was introduced by Gostya Mitag-Leffler [11].

(vii) \( t^{\mu-1}J_{\mu,1}^{\alpha,\beta}(az^\mu) = F_\mu[-\alpha, z] \), \( \mu > 0 \), was studied by Robotov [18], with respect to hereditary integrals for application to solid mechanics.

In the investigation of various properties and relations of the function \( J_{\nu,q}^\gamma(z) \), we need the following well known fact.

**Beta transform** (Sneddon [13]): The Beta (Euler) transform of the function \( f(z) \) is defined by

\[
B\{f(z); a,b\} = \int_0^1 z^{a-1}(1-z)^{b-1}f(z)\,dz \quad (1.5)
\]

where, \( \text{Re}(a) > 0, \text{Re}(b) > 0 \).

**Laplace transform** (Sneddon [13]): The Laplace transform of the function \( f(z) \) is defined as

\[
L\{f(z)\} = \int_0^\infty e^{-sz}f(z)\,dz. \quad (1.6)
\]

**K-Transform** (Meijer [7]): The K-transform is defined by the following integral equation

\[
R_v\{f(x); p\} = \int_0^\infty \frac{1}{(px)^2}K_v(px)f(x)\,dx \quad (1.7)
\]

where \( p \) is a complex parameter and \( K_v(z) \) represent a modified Bessel function of third kind defined by ([7], p.28, eq.1.168).

**Varma Transform** (Meijer [8]): The transform is defined by the integral equation

\[
V(f, k, m; s) = \int_{0}^{\infty} (sx)^{m-1}e^\left(-\frac{1}{2}sx\right)W_{k,m}(sx)f(x)\,dx \quad (1.8)
\]

where \( W_{k,m}(z) \) represents a Wittaker function defined by ([5], p.55, eq.2.39).

**Riemann-Liouville fractional derivative and integral** (see, Samko, Kilbas and Marichev ([16], sect. 2)), for \( \alpha \in C, \text{Re}(\alpha) > 0 \):

The operators are defined by

\[
(I_0^s)^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}\,dt; \quad (1.9)
\]

\[
(I_0^s)^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{f(t)}{(t-x)^{1-\alpha}}\,dt; \quad (1.10)
\]

\[
(D_0^s)^\alpha f(x) = \frac{d}{dx} |x|^{\alpha+1} \left( I_0^1(a) f \right)(x) = \frac{1}{\Gamma(1-\{\alpha\})} \frac{d}{dx} |x|^{\alpha+1} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}\,dt; \quad (1.11)
\]
\[(D^\alpha f)(x) = \left(\frac{d}{dx}\right)^{[\alpha]+1} (I_{1-\alpha} f)(x)\]

\[= \frac{1}{\Gamma(1-\alpha)} \left(\frac{d}{dx}\right)^{[\alpha]+1} \int_x^\infty \frac{f(t)}{(t-x)^{\alpha}} dt, \quad (1.12)\]

where \([\alpha]\) means the integral part of number \(\alpha\) and \(\{\alpha\}\) means the fractional part of number \(\alpha\), \(0 \leq \{\alpha\} < 1\). The number \(\alpha = \{\alpha\} + [\alpha]\).

**II. BASIC PROPERTIES**

In this section we derive several interesting properties of the function \(J_{v,q}^\mu y(z)\).

**Theorem (2.1):** If \(\mu, v, y \in C; Re(\mu) \geq 0, Re(y) \geq -1, Re(v) \geq 0\) and \(q \in (0,1) \cup N\) is satisfied, then for \(m \in N\)

\[\left(\frac{d}{dz}\right)^m J_{v,q}^\mu y(z) = (-1)^m (y)_q m J_{v+\mu+q q}^\mu y(z) \quad (2.1.1)\]

\[J_{v,q}^\mu y(z) = (v+1)J_{v+1,q}^\mu y(z) + \mu z \frac{d}{dz} J_{v+1,q}^\mu y(z) \quad (2.1.2)\]

\[J_{v,q}^\mu y(z) - J_{v,q}^{\mu y-1}(z) = -zq \sum_{n=0}^\infty \frac{(y)_{q+n-1}}{\Gamma(\mu m + \mu + v + 1)} \frac{(-z)^n}{n!} \quad (2.1.3)\]

In particular,

\[J_{v,1}^{\mu y-1}(z) - J_{v,1}^{\mu y}(z) = z J_{v+\mu,1}^\mu y(z) \quad (2.1.4)\]

**Proof:** From (1.2),

\[\left(\frac{d}{dz}\right)^m J_{v,q}^\mu y(z) = \sum_{n=m}^\infty \frac{(-1)^n (y)_q m (z)^n}{\Gamma(\mu n + v + 1)} \frac{(z)^{n-m}}{(n-m)!} \]

\[= (-1)^m (y)_q m \sum_{n=0}^\infty \frac{(y+qm)_q}{\Gamma(\mu n + v + 1)} \frac{(z)^n}{n!} \]

\[= (-1)^m (y)_q m J_{v+\mu+q m}^\mu y(z) \]

which is a proof of (2.1.1); \(v+1)J_{v+1,q}^\mu y(z) + \mu z \frac{d}{dz} J_{v+1,q}^\mu y(z) \]

\[= (v+1) \sum_{n=0}^\infty \frac{(y)_{q+n-1}}{\Gamma(\mu m + \mu + v + 1)} \frac{(-z)^n}{n!} + \mu z \sum_{n=0}^\infty \frac{(y)_{q+n-1}}{\Gamma(\mu m + \mu + v + 1)} \frac{(-z)^n}{n!} \]

which proves (2.1.2).

Now,

\[J_{v,q}^\mu y(z) - J_{v,q}^{\mu y-1}(z)\]

\[= \sum_{n=0}^\infty \frac{(y)_q m (z)^n}{n! \Gamma(\mu n + v + 1)} - \sum_{n=0}^\infty \frac{(y-1)_q n (z)^n}{n! \Gamma(\mu n + v + 1)} \]

\[= q \sum_{n=0}^\infty \frac{(y)_q n (z)^n}{\Gamma(\mu n + v + 1)} \frac{(-z)^n}{n!} \]

which proves (2.1.3).

In particular, if \(q = 1\) in (2.1.3), which at once yield (2.1.4).

**Theorem (2.2):** If \(\mu, v, y, \delta \in C; Re(\mu) \geq 0, Re(v) \geq -1, Re(y) \geq 0, Re(\delta) \geq 0\) and \(q \in (0,1) \cup N\) is satisfied, then

\[\frac{1}{\Gamma(\delta)} \int_0^1 \frac{1}{(\omega)^{v+\delta}} (1-\omega)^{-\delta-1} J_{v,q}^\mu y(z \omega^\mu) d\omega \]

\[= J_{v+\delta,q}^\mu y(z) \quad (2.2.1)\]

If \(\mu, v, y, \delta, \alpha \in C; Re(\mu) \geq 0, Re(v) \geq -1, Re(y) \geq 0, Re(\delta) \geq 0\) and \(q \in (0,1) \cup N\) is satisfied, then

\[\frac{1}{\Gamma(\delta)} \int_0^1 (x-s)^{-\delta-1} (s-t)^{\delta} J_{v,q}^\mu y [\sigma(s-t) \mu] d\]

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\begin{equation}
= (x - t)^{\delta + \nu} f_{\nu,\delta, q}^{\mu, \gamma}[\alpha(x - t)^{\mu}], \quad (2.2.2)
\end{equation}
If \( \mu, \nu, \gamma, \delta, \lambda \in \mathbb{C}; \) \( \Re(\mu) \geq 0, \Re(\nu) \geq -1, \)
\( \Re(\gamma) \geq 0, \Re(\delta), \Re(\lambda) \geq 0 \) and \( q = 1 \) is satisfied, then
\begin{equation*}
\int_0^x (t)^{\lambda}(x - t)^{\nu} f_{\nu, \lambda, 1}^{\mu, \gamma}[\omega(x - t)^{\mu}] f_{\lambda, 1}^{\mu, \delta}[\omega t^{\mu}] dt = x^{\lambda + \nu + 1} f_{\nu, 1}^{\mu, \gamma}[\omega t^{\mu}], \quad (2.2.3)
\end{equation*}
If \( \mu, \nu, \gamma, \delta \in \mathbb{C}; \) \( \Re(\mu) \geq 0, \Re(\nu) \geq -1, \Re(\gamma) \geq 0, \)
\( \Re(\delta) \geq 0 \) and \( q \in (0,1) \cup \mathbb{N} \) is satisfied, then
\begin{equation*}
\int_0^x (t)^{\nu} f_{\nu, q}^{\mu, \gamma}(\omega t^{\mu}) dt = z^{\nu + 1} f_{\nu + 1, q}^{\mu, \gamma}(\omega z^{\mu}). \quad (2.2.4)
\end{equation*}

**Proof:** By using the beta function, L.H.S. of (2.2.1) becomes
\[
\frac{1}{\Gamma(\delta)} \sum_{n=0}^{\infty} \frac{(\gamma)_n (z)^n}{\Gamma(\mu n + \nu + 1)} B(\mu n + \nu + 1, \delta)
\]
\[
= \frac{1}{\Gamma(\delta)} \sum_{n=0}^{\infty} \frac{(\gamma)_n (z)^n}{n! \Gamma(\mu n + \nu + 1)} = f_{\nu + \delta, q}^{\mu, \gamma}(z)
\]
which is the proof of (2.2.1).

By changing the variable \( s = t + \omega(x - t) \), the L.H.S. of (2.2.2) becomes
\[
\frac{1}{\Gamma(\delta)} \sum_{n=0}^{\infty} \frac{(\gamma)_n (\omega)^n}{n! \Gamma(\mu n + \nu + 1)}
\]
\[
= \frac{1}{\Gamma(\delta)} \sum_{n=0}^{\infty} \frac{(\gamma)_n (\omega)^n}{n! \Gamma(\mu n + \nu + 1)} B(\mu n + \nu + 1, \delta)
\]
which yield (2.2.2).

Consider,
\[
\int_0^x (t)^{\lambda}(x - t)^{\nu} f_{\nu, \lambda}^{\mu, \gamma}[\omega(x - t)^{\mu}] f_{\lambda, 1}^{\mu, \delta}[\omega t^{\mu}] dt
\]
\[
= \sum_{n,k=0}^{\infty} \frac{(\gamma)_n (\delta)_k}{\Gamma(\mu n + \nu + 1) \Gamma(\mu k + \lambda + 1)} (-\omega)^{n+k} n! k!
\]

**III. INTEGRAL TRANSFORM OF \( f_{\nu, q}^{\mu, \gamma}(z) \)**
In this section, several integral transforms like Beta, Laplace, Varma, Mellin’s and \( K \)-transform are discussed for the function \( f_{\nu, q}^{\mu, \gamma}(z) \) under the following theorem.

**Theorem(3.1)**(Beta transform): By using the definition of Beta function, one obtain
\[
\int_0^1 (z)^{\alpha - 1}(1 - z)^{\beta - 1} f_{\nu, q}^{\mu, \gamma}(xz^{\delta}) dz
\]
where, $\mu, \nu, \gamma \in \mathbb{C}; Re(\mu) \geq 0, Re(\nu) \geq -1, Re(\gamma) \geq 0, Re(\alpha) \geq 0, Re(\beta) \geq 0$ and $q \in (0,1) \cup \mathbb{N}$

**Proof:** From (1.2) and (1.5), we get

$$\int_{0}^{1} (z)^{\alpha-1}(1-z)^{\beta-1}j_{\nu,q}^{\mu,\gamma}(xz^\delta)dz = \sum_{n=0}^{\infty} \frac{(\gamma)_n(-x)^n}{n! \Gamma(\mu n + \nu + 1)} B(\alpha + \delta n, \beta)$$

which is (3.1.1).

**Particular Cases:** If $\mu = \delta$ and $\alpha = \nu + 1$, then the relation (3.1.1) reduces to

$$\int_{0}^{1} (z)^{\nu}(1-z)^{\beta-1}j_{\nu,q}^{\mu,\gamma}(xz^\delta)dz = \Gamma(\beta)j_{\nu+\beta,q}^{\mu,\gamma}(x)$$

If $\beta = \nu + 1$, $\delta = \mu$ and $z = (1-z)$, then the relation (3.1.1) reduces to

$$\int_{0}^{1} (z)^{\alpha-1}(1-z)^{\beta-1}j_{\nu,q}^{\mu,\gamma}(x(1-z)^\mu)dz = \Gamma(\alpha)j_{\nu+\alpha,q}^{\mu,\gamma}(x)$$

**Theorem (3.2)(Laplace transform):** By using the definition of Laplace transform, one obtain

$$\int_{0}^{\infty} z^{\alpha-1}e^{-sz}j_{\nu,q}^{\mu,\gamma}(xz^\delta)dz = \frac{s^{-\alpha}}{\Gamma(\gamma)} \Psi_{1} \left[ (\gamma, q), (\alpha + \delta, \beta); -x \right]$$

where, $\mu, \nu, \gamma \in \mathbb{C}; Re(\mu) \geq 0, Re(\nu) \geq -1, Re(\gamma) \geq 0, Re(\alpha) \geq 0, Re(\beta) \geq 0$ and $q \in (0,1) \cup \mathbb{N}$.

**Proof:** In virtue of (1.2) and (1.6),

$$L\{z^{\alpha-1}j_{\nu,q}^{\mu,\gamma}(xz^\delta)\} = \int_{0}^{\infty} e^{-sz}z^{\alpha-1} \sum_{n=0}^{\infty} \frac{(\gamma)_n(-x)^n}{n! \Gamma(\mu n + \nu + 1)} dz
\frac{s^{-\alpha}}{\Gamma(\gamma)} \Psi_{1} \left[ (\gamma, q), (\alpha + \delta, \beta); -x \right]$$

which is (3.2.1).

**Particular Cases:** If $q = 1$, $\alpha = \nu + 1$, $\delta = \mu$ in (3.2.1), reduces to

$$\int_{0}^{\infty} z^{\nu}e^{-sz}j_{\nu,1}^{\mu,\gamma}(xz^\mu)dz = \frac{s^{-\nu-1}(1 + xs^{-\mu})^{-\gamma}}{1 - \frac{t}{s\mu}}$$

If $\nu = \delta = \mu, \gamma = q = 1, x = \pm t$, in (3.2.1), reduces to

$$\int_{0}^{\infty} z^{\nu}e^{-sz}j_{\nu,1}^{\mu,\gamma}(zt^\mu)dz = \frac{s^{-\nu-1}}{1 - \frac{t}{s\mu}}$$

where $\left| \frac{t}{s\mu} \right| < 1$ (3.2.3)

**Theorem (3.3)(K-transform):**

$$\int_{0}^{\infty} t^{\alpha-1}K_{\alpha}(st)j_{\nu,q}^{\mu,\gamma}(\omega t^p)dt = \frac{2^{\alpha-2}}{s^{\alpha} \Gamma(\gamma)} \Psi_{1} \left[ (\gamma, q), (\alpha \pm \delta, \beta); -x \omega \left( \frac{2}{s} \right)^p \right]$$

where, $\mu, \nu, \gamma, \alpha, \rho \in \mathbb{C}; Re(\mu) \geq 0, Re(\nu) \geq -1, Re(\gamma) \geq 0, Re(\alpha) \geq 0, Re(\rho) \geq 0$ and $q \in \mathbb{N}$.

**Proof:** By changing the variable $st = z$ in L.H.S. of (3.3.1), we get

$$\int_{0}^{\infty} z^{\alpha-1}K_{\alpha}(z) \sum_{n=0}^{\infty} \frac{(\gamma)_n}{n! \Gamma(\mu n + \nu + 1)} \frac{s^{-\alpha}}{\Gamma(\gamma)} \Psi_{1} \left[ (\gamma, q), (\alpha + \delta, \beta); -x \right]$$

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\[ \times (-\omega)^n \left( \frac{Z}{s} \right)^\rho \frac{1}{s} \int_0^\infty dz = s^{-\alpha} \sum_{n=0}^\infty \frac{(\gamma)^n (-\omega s^\rho)^n}{n! \Gamma(\mu n + \nu + 1)} \int_0^\infty (z)^{\alpha + \rho n - 1} K_\lambda(z)dz \]

Now by using the formula (Mathai and Saxena [4], p. 78):

\[ \int_0^\infty x^{\beta - 1} K_\nu(x) dx = 2^{\rho - 2} \Gamma\left(\frac{\rho + \nu}{2}\right), \]

in the above equation, we obtain the relation (3.3.1).

**Theorem (3.4) (Varma transform):**

\[ \frac{\Gamma\left(\frac{1}{2} + \mu + v\right) \Gamma\left(\frac{1}{2} - \mu + v\right)}{\Gamma(1 - \lambda + v)} \]

\[ f_{\alpha+\nu}^{\mu+\nu} = \frac{1}{\Gamma(\alpha)} \int_0^\infty (x - t)^{\alpha - 1} \sum_{n=0}^\infty \frac{(\gamma)^n (-\omega s^\rho)^n}{n! \Gamma(\mu n + \lambda + 1)} \int_0^1 (\delta n + v + 1) \]

\[ \times (1 - z)^{\alpha - 1}dz \]

**Proof:** By changing the variable \(st = z\) in L.H.S. of (3.4.1), we get

\[ \frac{\Gamma\left(\frac{1}{2} + \mu + v\right) \Gamma\left(\frac{1}{2} - \mu + v\right)}{\Gamma(1 - \lambda + v)} \]

in the above equation, we obtain the relation (3.4.1).

**IV. FRACTIONAL INTEGRATION AND DERIVATIVE**

In this section, we establish several interesting properties of the function \(f_{\alpha+\nu}^{\mu+\nu}(z)\) defined by (1.2) associated with the operator of Riemann-Liouville fractional integrals and derivatives.

**Theorem (4.1):** Let \(\alpha, \mu, \nu, \lambda, \gamma, \omega \in C; Re(\alpha) \geq 0, Re(\mu) \geq 0, Re(\lambda) \geq -1, Re(\nu) \geq 0, Re(\gamma) \geq 0, Re(\delta) \geq 0\), and \(q \in N\), then the left sided operator of Riemann-Liouville fractional integral \(I_{0+}^\alpha\) is given for \(x > 0\) by

\[ f_{\alpha+\nu}^{\mu+\nu}(z) = \frac{\Gamma(\lambda + 1, \mu, (\alpha + v + 1, \delta); -\omega x^\delta)}{\Gamma(\gamma)^2 \Psi_2 (\lambda + 1, \mu, (\alpha + v + 1, \delta); -\omega x^\delta)} \]

(4.1.1)

**Proof:** From the relation (1.2) and (1.9), we have

\[ f_{\alpha+\nu}^{\mu+\nu}(z) = \frac{1}{\Gamma(\alpha)} \int_0^\infty (x - t)^{\alpha - 1} \sum_{n=0}^\infty \frac{(\gamma)^n (-\omega)^n (t)^{\delta n + v}}{n! \Gamma(\mu n + \lambda + 1)} \int_0^1 (\delta n + v + 1) \]

\[ \times (1 - z)^{\alpha - 1} \]

\[ \times (1 - z)^{\alpha - 1} \]

evaluating the inner integral by beta-function formula, above relation reduces to (4.1.1).

**Corollary (1.1):** Let \(\alpha, \mu, \nu, \gamma, \omega \in C; Re(\alpha) \geq 0, Re(\mu) \geq 0, Re(\nu) \geq -1, Re(\gamma) \geq 0\) and \(q \in N\), then there holds the relation

\[ f_{\alpha+\nu}^{\mu+\nu}(z) = \frac{1}{\omega} \times f_{\alpha+\nu}^{\mu+\nu}(\omega x^\mu) \]

(4.1.2)

and

\[ f_{\alpha+\nu}^{\mu+\nu}(z) = \frac{1}{\omega} \times f_{\alpha+\nu}^{\mu+\nu}(\omega x^\mu) \]

(4.1.3)
**Theorem(4.2):** Let $\alpha, \mu, v, \gamma, \omega \in C; \text{Re}(\alpha) \geq 0, \text{Re}(\mu) \geq 0, \text{Re}(\lambda) \geq -1, \text{Re}(\nu) \geq 0, \text{Re}(\gamma) \geq 0, \text{Re}(\delta) \geq 0$, and $q \in N$, then the right sided operator of Riemann-Liouville fractional integral $I^{\alpha}_{0+}$ is given by

\[
(I^{\alpha}_{0+}[t^{-\alpha-1}]^{y}_{\lambda,q}(\omega t^{-\delta}))(x) = \frac{x^{-\alpha-1}}{\Gamma(\alpha)} \times \int_{x}^{\infty} (t-x)^{-\alpha-1} \sum_{n=0}^{\infty} \left(\frac{\gamma}{\eta} q_{n+1}(-\omega x^{-\delta})^{n}ight) \frac{\Gamma(\eta n+\lambda+1)}{n!} dt
\]

Evaluating the inner integral by beta-function formula, above relation reduces to (4.2.1).

**Corollary(1.2):** Let $\alpha, \mu, v, \gamma, \omega \in C; \text{Re}(\alpha) \geq 0, \text{Re}(\mu) \geq 0, \text{Re}(v) \geq -1, \text{Re}(\gamma) \geq 0$ and $q \in N$, then there holds the relation

\[
(I^{\alpha}_{0+}[t^{-\alpha-1}]^{y}_{\nu,q}(\omega t^{-\mu}))(x) = \frac{x^{-\alpha-1}}{\omega^{\alpha-\mu}} I^{\alpha}_{\alpha+v+\mu}(\omega x^{-\mu}) \quad (4.2.2)
\]

and

\[
(I^{\alpha}_{0+}[t^{-\alpha-1}]^{y}_{\nu,1}(\omega t^{-\mu}))(x) = \frac{1}{\omega^{\alpha-\mu-1}} I^{\mu+1}_{\alpha+v+\mu+1}(\omega x^{-\mu}) \quad (4.2.3)
\]

**Theorem(4.3):** Let $\alpha, \mu, v, \gamma, \omega \in C; \text{Re}(\alpha) \geq 0, \text{Re}(\mu) \geq 0, \text{Re}(\lambda) \geq -1, \text{Re}(\nu) \geq 0, \text{Re}(\gamma) \geq 0, \text{Re}(\delta) \geq 0$, and $q \in N$, then the left sided operator of Riemann-Liouville fractional derivative $D^{\alpha}_{0+}$ is given for $x > 0$ by

\[
(D^{\alpha}_{0+}[t^{-\alpha}]^{y}_{\lambda,q}(\omega t^{-\delta}))(x) = \frac{(x)^{\nu-\alpha}}{\Gamma(\nu)} \times \int_{0}^{\infty} t^{\nu-1} \sum_{n=0}^{\infty} \left(\frac{\gamma}{\eta} q_{n+1}(-\omega x^{-\delta})^{n}ight) \frac{\Gamma(\eta n+\lambda+1)}{n!} \times \int_{0}^{x} t^{\delta n+v}(x-t)^{-\alpha} dt \quad (4.3.1)
\]

**Proof:** By virtue of (1.2) and (1.11), we have

\[
(D^{\alpha}_{0+}[t^{-\alpha}]^{y}_{\lambda,q}(\omega t^{-\delta}))(x) = \frac{1}{\omega^{\alpha-\mu}} I^{\mu+1}_{\alpha+v+\mu+1}(\omega x^{-\mu}) \quad (4.3.1)
\]

Evaluating the inner integral by beta-function formula, above relation reduces to (4.2.1).

**Corollary(1.3):** Let $\alpha, \mu, v, \gamma, \omega \in C; \text{Re}(\alpha) \geq 0, \text{Re}(\mu) \geq 0, \text{Re}(v) \geq -1, \text{Re}(\gamma) \geq 0$ and $q \in N$, then there holds the relation,
\[ (D_{0}^{\alpha}[t^{\nu}j_{\nu,q}^{\mu,y} (\omega t^{\mu})])(x) = x^{\nu-\alpha}j_{\nu-\alpha,q}^{\mu,y} (\omega x^{\mu}) \quad (4.3.2) \]

and

\[ (D_{0}^{\alpha}[t^{\nu}j_{\nu,1}^{\mu,y} (\omega t^{\mu})])(x) = \frac{1}{\omega} x^{\nu-\mu-\alpha}[j_{\nu-\mu-\alpha,1}^{\mu,y} (\omega x^{\mu}) - j_{\nu-\mu-\alpha,1}^{\mu,y} (\omega x^{\mu})] \quad (4.3.3) \]

**Theorem (4.4):** Let \( \alpha, \mu, \nu, \lambda, \gamma, \omega \in C; Re(\alpha) \geq 0, Re(\mu) \geq 0, Re(\lambda) \geq -1, Re(\nu) \geq 0, Re(\gamma) \geq 0, \) and \( q \in N, \) then the right sided operator of Riemann-Liouville fractional derivative \( D_{x}^{\alpha} \) is given for \( x > 0 \) by

\[ (D_{x}^{\alpha}[t^{\nu-\lambda}j_{\lambda,q}^{\mu,y} (\omega t^{-\delta})])(x) = \frac{1}{\Gamma(x^{-\nu})} \left[ \frac{\gamma_{q} \nu}{\omega x^{-\delta}} \right] (\lambda + 1, \mu) (1 + \nu - \alpha + \delta) \]

By virtue of (1.2) and (1.12), we have

\[ \begin{aligned} (D_{x}^{\alpha}[t^{\nu-\lambda}j_{\lambda,q}^{\mu,y} (\omega t^{-\delta})])(x) &= \left( \frac{d^{\alpha}}{dx^{\alpha}} \right) (\lambda + 1, \mu) (1 + \nu - \alpha + \delta) \\
&= \sum_{n=0}^{\infty} \frac{(\gamma_{q} \nu)(-\omega)^{n}}{n! \Gamma(\mu n + \lambda + 1) \Gamma(1 - (\alpha))} (\lambda + 1, \mu) (1 + \nu - \alpha + \delta) \\
&\times \left( \frac{d^{\alpha}}{dx^{\alpha}} \right) (\nu + 1, \delta) \\
&\times \left( \frac{-d}{dx} \right)^{\alpha+1} \int_{0}^{x} t^{-\delta n + \alpha - \nu - 1}(x - t)^{-\alpha} dt \end{aligned} \]

By changing the variable \( t = \frac{x}{z}, \) the above expression reduces into the form

\[ \begin{aligned} (D_{x}^{\alpha}[t^{\nu-\lambda}j_{\lambda,q}^{\mu,y} (\omega t^{-\delta})])(x) &= \sum_{n=0}^{\infty} \frac{(\gamma_{q} \nu)(-\omega)^{n}}{n! \Gamma(\mu n + \lambda + 1) \Gamma(1 - (\alpha))} (\lambda + 1, \mu) (1 + \nu - \alpha + \delta) \\
&\times \left( \frac{d^{\alpha}}{dx^{\alpha}} \right) (\nu + 1, \delta) \\
&\times \left( \frac{-d}{dx} \right)^{\alpha+1} \int_{0}^{x} t^{-\delta n + \alpha - \nu - 1}(x - t)^{-\alpha} dt \end{aligned} \]

By the reflection formula for the gamma-function (see, [16], (1.60)),

\[ \frac{1}{\Gamma(-\nu - \delta n)} = \frac{\Gamma(1 + \nu + \delta n) \Gamma(-\nu - \delta n)}{\Gamma(1 + \nu + \delta n) \Gamma(-\nu - \delta n)} = \frac{\Gamma(1 + \nu + \delta n) \Gamma(-\nu - \delta n)}{\Gamma(1 + \nu + \delta n) \Gamma(-\nu - \delta n)} \]

and

\[ \Gamma(\nu - \alpha + (\alpha) + \delta n) \Gamma(1 - \nu - \alpha - (\alpha) - \delta n) = \pi \]

substituting these relations into (4.4.2), we obtain (4.4.1).

**Corollary (1.4):** Let \( \alpha, \mu, \nu, \gamma, \omega \in C; Re(\alpha) \geq 0, \]

\[ \begin{aligned} Re(\mu) &\geq 0, Re(\nu) \geq -1, Re(\gamma) \geq 0 \quad \text{and} \quad q \in N, \end{aligned} \]

then there holds the relation

\[ \begin{aligned} (D_{x}^{\alpha}[t^{\nu-\lambda}j_{\lambda,q}^{\mu,y} (\omega t^{-\delta})])(x) &= x^{\nu-\lambda}j_{\nu-\lambda,q}^{\mu,y} (\omega x^{\mu}) \quad (4.4.3) \]

and

\[ \begin{aligned} (D_{x}^{\alpha}[t^{\nu-\lambda}j_{\lambda,q}^{\mu,y} (\omega t^{-\delta})])(x) &= \frac{1}{\omega} x^{\nu-\lambda} \left[ j_{\nu-\lambda,q}^{\mu,y} (\omega x^{\mu}) - j_{\nu-\lambda,q}^{\mu,y} (\omega x^{\mu}) \right] \quad (4.4.4) \end{aligned} \]

**REFERENCES**


