Some Common Fixed Point Theorems for Sequence of Mappings in Two Metric Spaces

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ABSTRACT:

In this paper we prove some common fixed point theorems for sequence of mappings in two complete metric spaces.

Key words and Phrases: fixed point, common fixed point, sequence of maps and complete metric space.

AMS Mathematics Subject Classification: 47H10, 54H25

1. INTRODUCTION.

Fixed point theory and common fixed point theory have basic roles in the application of some branches of mathematics. There are many articles about common fixed point theorems in metric spaces([3]-[5]). In [6] and [7], B.Fisher proved some theorems in two complete metric spaces. Later some authors proved some kind of fixed and common fixed point theorems in two metric spaces ([1], [2], [8]-[10]. In this paper we prove some common fixed point theorems for sequence of mappings in two complete metric spaces. The following definitions are necessary for the present study.

Definition 1.1 A sequence \{x_n\} in a metric space (X, d) is said to be convergent to a point \( x \in X \) if given \( \varepsilon > 0 \) there exists a positive integer \( n_0 \) such that \( d(x_n, x) < \varepsilon \) for all \( n \geq n_0 \).

Definition 1.2 A sequence \{x_n\} in a metric space (X, d) is said to be a Cauchy sequence in X if given \( \varepsilon > 0 \) there exists a positive integer \( n_0 \) such that \( d(x_m, x_n) < \varepsilon \) for all \( m, n \geq n_0 \).

Definition 1.3 A metric space (X, d) is said to be complete if every Cauchy sequence in X converges to a point in X.

Definition 1.4 Let X be a non-empty set and \( f : X \to X \) be a map. An element \( x \) in X is called a fixed point of X if \( f(x) = x \).

Definition 1.5 Let X be a non-empty set and \( f, g : X \to X \) be two maps. An element \( x \) in X is called a common fixed point of \( f \) and \( g \) if \( f(x) = g(x) = x \).

Definition 1.6 Let X be a non-empty set and a point \( x \) in X is said to be a common fixed point of sequence of maps \( T_n : X \to X \) if \( T_n(x) = x \) for all \( n \).

2. MAIN RESULTS

Theorem 2.1: Let \( (X, d) \) and \( (Y, e) \) be complete metric spaces. Let \( \{A_n\}, \{B_n\} (n \in \mathbb{N}) \) be sequence of mappings of X into Y and \( \{S_n\}, \{T_n\} (n \in \mathbb{N}) \) be sequence of mappings of Y into X satisfying the inequalities:

\[
\begin{align*}
d(S_pA_nx, T_qB_nx') & \leq c_1 \max\{d(x,x'), d(x, S_pA_nx), d(x', T_qB_nx')\} \quad \text{(2.1.1)} \\
ed(y, A_nT_qy') & \leq c_2 \max\{e(y, y'), e(y, B_nS_qy), e(y', A_nT_qy'), d(S_pA_nx, x')\} \quad \text{(2.1.2)}
\end{align*}
\]

for all \( i \neq j \neq p \neq q, \ x, x' \in X \) and \( y, y' \in Y \) where \( 0 \leq c_1 < 1 \) and \( 0 \leq c_2 < 1 \). If one of the mappings \( \{A_n\}, \{B_n\}, \{S_n\} \) and \( \{T_n\} \) is continuous, then \( \{S_nA_n\} \) and \( \{T_nB_n\} \) have a common fixed point \( z \) in X and \( \{B_nS_n\} \) and \( \{A_nT_n\} \) have a common fixed point \( w \) in Y. Further, \( \{A_n\}z = \{B_n\}z = w \) and \( \{S_n\}w = \{T_n\}w = z \).
Proof: Let $x_n$ be an arbitrary point in $X$ and we define the sequences \( \{x_n\} \) in $X$ and \( \{y_n\} \) in Y by

\[
A_n \cdot x_{2n} = y_{2n-1}, S_n y_{2n-1} = x_{2n}, B_n x_{2n-1} = y_{2n},
\]

\[
T_n y_{2n} = x_{2n}
\]

for $n = 1, 2, 3 \ldots$

Now using inequality (2.1.1) we have

\[
d(x_{2n+1}, x_{2n}) = d(S_n A_n x_{2n}, T_n B_n x_{2n-1})
\]

\[
\leq c_1 \max \{ d(x_{2n}, x_{2n-1}), d(S_n A_n x_{2n}),
\]

\[
d(x_{2n}, T_n B_n x_{2n-1}), e(A_n x_{2n}, B_n x_{2n-1})
\]

\[
= c_1 \max \{ d(x_{2n}, x_{2n-1}), d(y_{2n-1}, x_{2n-1}),
\]

\[
e(y_{2n+1}, y_{2n}), d(x_{2n}, x_{2n}), e(y_{2n+1}, y_{2n})
\]

\[
e(y_{2n+1}, y_{2n},0)
\]

\[
\leq c_1 \max \{ d(x_{2n+1}, x_{2n}), e(y_{2n+1}, y_{2n}) \} \text{ ------ (2.1.3)}
\]

Now using inequality (2.1.2) we have

\[
e(y_{2n+1}, y_{2n}) = e(B_n S_n y_{2n-1}, A_n T_n y_{2n})
\]

\[
\leq c_2 \max \{ e(y_{2n-1}, y_{2n}), e(y_{2n-1}, B_n S_n y_{2n-1}),
\]

\[
e(y_{2n-1}, A_n T_n y_{2n}), d(S_n y_{2n-1}, T_n y_{2n}),
\]

\[
e(y_{2n-1}, A_n T_n y_{2n}), e(B_n S_n y_{2n-1}, y_{2n})
\]

\[
= c_2 \max \{ e(y_{2n-1}, y_{2n}), e(y_{2n-1}, y_{2n}),
\]

\[
d(x_{2n-1}, x_{2n}), e(y_{2n-1}, y_{2n}), e(y_{2n-1}, y_{2n})
\]

\[
= c_2 \max \{ e(y_{2n-1}, y_{2n}), e(y_{2n-1}, y_{2n}),
\]

\[
d(x_{2n-1}, x_{2n}), 0
\]

\[
\leq c_2 \max \{ e(y_{2n-1}, y_{2n}), d(x_{2n-1}, x_{2n}) \} \text{ ------ (2.1.4)}
\]

Again using inequality (2.1.1) we have

\[
d(x_{2n}, x_{2n-1}) = d(x_{2n}, x_{2n})
\]

\[
= d(S_n A_n x_{2n-2}, T_n B_n x_{2n-1})
\]

\[
\leq c_1 \max \{ d(x_{2n-2}, x_{2n-1}), d(S_n A_n x_{2n-2}),
\]

\[
d(x_{2n-2}, T_n B_n x_{2n-1}), e(A_n x_{2n-2}, B_n x_{2n-1})
\]

\[
d(x_{2n-2}, T_n B_n x_{2n-1}), d(S_n A_n x_{2n-2}, x_{2n-1})
\]

\[
= c_1 \max \{ d(x_{2n-2}, x_{2n-1}), d(x_{2n-2}, 1),
\]

\[
ed(x_{2n-1}, x_{2n}), e(y_{2n-1}, y_{2n}),
\]

\[
d(x_{2n-2}, x_{2n}), d(x_{2n-1}, x_{2n-1})
\]

\[
= c_1 \max \{ d(x_{2n-2}, x_{2n-1}), d(x_{2n-2}, x_{2n}),
\]

\[
d(x_{2n-1}, x_{2n}), e(y_{2n-1}, y_{2n}), 0
\]

\[
\leq c_1 \max \{ d(x_{2n-2}, x_{2n-1}), e(y_{2n-1}, y_{2n}) \} \text{ ------ (2.1.5)}
\]

Now using inequality (2.1.2)

\[
e(y_{2n+1}, y_{2n}) = e(B_n S_n y_{2n-1}, A_n T_n y_{2n})
\]

\[
\leq c_2 \max \{ e(y_{2n-1}, y_{2n}), e(y_{2n-1}, B_n S_n y_{2n-1})
\]

\[
e(y_{2n-2}, A_n T_n y_{2n}), d(S_n y_{2n-1}, T_n y_{2n})
\]

\[
e(y_{2n-1}, A_n T_n y_{2n}), e(B_n S_n y_{2n-1}, y_{2n})
\]

\[
= c_2 \max \{ e(y_{2n-1}, y_{2n}), e(y_{2n-1}, y_{2n}),
\]

\[
e(y_{2n-1}, y_{2n}), d(x_{2n-1}, x_{2n-2})
\]

\[
e(y_{2n-1}, y_{2n}), e(y_{2n-1}, y_{2n})
\]

\[
\leq c_2 \max \{ e(y_{2n-1}, y_{2n}), e(y_{2n-1}, y_{2n}),
\]

\[
e(y_{2n-1}, y_{2n}), d(x_{2n-1}, x_{2n-2}), 0
\]

\[
\leq c_2 \max \{ e(y_{2n-1}, y_{2n}), d(x_{2n-1}, x_{2n-2}) \} \text{ ------ (2.1.6)}
\]

from inequalities (2.1.3), (2.1.4), (2.1.5) and (2.1.6), we have

\[
d(x_{2n+1}, x_n) \leq c_1 c_2^{n-1} \max \{ d(x_1, x_0), e(y_1, y_2) \} \to 0 \text{ as } n \to \infty
\]

Thus \( \{x_n\} \) is a Cauchy sequence in \((X,d)\). Since \((X,d)\) is complete, \( \{x_n\} \) converges to a point \(z\) in \(X\). Similarly using inequalities (2.1.3), (2.1.4), (2.1.5) and (2.1.6), we prove \( \{y_n\} \) is a Cauchy sequence in \((Y,e)\) with the limit \(w\) in \(Y\).

Suppose \( \{A_n\} \) is continuous, then

\[
\lim_{n \to \infty} A_n x_n = A_z = \lim_{n \to \infty} y_{2n+1} = w
\]

Now we prove \(S_n A_n z = z\).

Suppose \(S_n A_n z \neq z\).

We have

\[
d(S_n A_n z, z) = \lim_{n \to \infty} d(S_n A_n z, T_n B_n x_{2n-1})
\]

\[
\leq c_1 \max \{ d(z, x_{2n-1}), d(z, S_n A_n z),
\]

\[
d(x_{2n-1}, T_n B_n x_{2n-1}), e(A_n z, b_n x_{2n-1}),
\]

\[
d(z, T_n B_n x_{2n-1}), s_n A_n z, z_{2n-1})
\]

\[
\leq c_1 \max \{ d(z, x_{2n-1}), d(z, S_n A_n z),
\]

\[
d(x_{2n-1}, T_n B_n x_{2n-1}), e(A_n z, b_n x_{2n-1}),
\]

\[
d(z, x_{2n-1}), s_n A_n z, z_{2n-1})
\]

\[
= c_1 \max \{ 0, d(z, S_n A_n z), 0, 0, 0
\]

\[
\leq c_1 \max \{ d(z, S_n A_n z), 0, 0, 0
\]

\[
\leq c_1 \max \{ d(z, S_n A_n z), 0
\]

\[
< d(z, S_n A_n z) \text{ (Since } 0 \leq c_1 < 1\text{)}
\]

Which is a contradiction.

Thus \(S_n A_n z = z\).

Hence \(S_n w = z\). (Since \(A_n z = w\))

Now we prove \(B_n S_n w = w\).

Suppose \(B_n S_n w \neq w\).
We have
\[ e(B_nS_nw, w) = \lim_{n \to \infty} e(B_nS_nw, y_{2n+1}) \]
\[ = \lim_{n \to \infty} e(B_nS_nw, A_nT_ny_{2n}) \]
\[ \leq \lim_{n \to \infty} c_2 \max \{ e(y_{2n}, A_nT_ny_{2n}), d(S_nw, T_ny_{2n}), e(w, A_nT_ny_{2n}), e(B_nS_nw, y_{2n}) \} \]
\[ = c_2 \max \{ e(w, w), e(w, B_nS_nw), e(w, w), e(B_nS_nw, w) \} \]
\[ < e(w, B_nS_nw) \quad \text{(Since } 0 \leq c_2 < 1) \]

Which is a contradiction.

Thus \( B_nS_nw = w \).

The same results hold if one of the mappings \( \{B_n\}, \{S_n\} \) and \( \{T_n\} \) is continuous.

So the point \( z \) is the common fixed point of \( \{S_nA_n\} \) and \( \{T_nB_n\} \). Similarly we prove \( w \) is a common fixed point of \( \{B_nS_n\} \) and \( \{A_nT_n\} \).

**Remark 2.2:** If we put \( A_i = A, B_j = B, S_p = S \) and \( T_q = T \) in the above theorem 2.1, we get the following corollary.

**Corollary 2.3:** Let \( (X, d) \) and \( (Y, e) \) be complete metric spaces. Let \( A, B \) be mappings of \( X \) into \( Y \) and \( S, T \) be mappings of \( Y \) into \( X \) satisfying the inequalities.

\[
d(SAx, T_Bx') \leq c_1, \max \{d(x',x), d(x,Ax), d(x',TBx'), e(Ax,Bx'), d(x,T'Bx), d(SAx,x')\},
\]

\[
e(\text{BSy, AT'y}) \leq c_2, \max \{e(y,y'), e(y,\text{BSy}), e(y',\text{AT'y}), d(\text{SY},\text{Ty'}), e(y, \text{AT'y}), e(\text{BSy}, y')\} -
\]

for all \( x, x' \in X \) and \( y, y' \in Y \) where \( 0 \leq c_1 < 1 \) and \( 0 \leq c_2 < 1 \).

If one of the mappings \( A, B, S \) and \( T \) is continuous, then \( SA \) and \( TB \) have a common fixed point \( z \) in \( X \) and \( BS \) and \( AT \) have a common fixed point \( w \) in \( Y \). Further, \( Az = Bz = w \) and \( Sw = Tw = z \).

**Theorem 2.4:** Let \( (X, d) \) and \( (Y, e) \) be complete metric spaces. Let \( \{A_n\}, \{B_n\} (n \in N) \) be sequence of mappings of \( X \) into \( Y \) and \( \{S_n\}, \{T_n\} (n \in N) \) be sequence of mappings of \( Y \) into \( X \) satisfying the inequalities.

\[
d(S_nA_nx, T_nB_nx') \leq c_1, \max \{d(x',x), d(x,A_nx), d(x',T_nB_nx'), e(A_nx,B_nx'), d(\text{SA}_nx,\text{T}B_nx') \} - \frac{d(S_nA_nx, x')}{2},
\]

\[
e(\text{BS}_ny, \text{AT}_qy') \leq c_2, \max \{e(y, y'), e(y,\text{BS}_ny), e(y',\text{AT}_qy'), d(\text{SA}_ny,\text{T}qy'), e(\text{y, AT}_qy'), e(\text{BS}_ny, y') \} - \frac{d(S_ny,\text{T}qy')}{2}.
\]
for all \( i \neq j \neq p \neq q, x, x' \) in \( X \) and \( y, y' \) in \( Y \) where \( 0 \leq c_1 < 1 \) and \( 0 \leq c_2 < 1 \). If one of the mappings \( \{A_n\}, \{B_n\}, \{S_n\} \) and \( \{T_n\} \) is continuous, then \( \{S_nA_n\} \) and \( \{T_nB_n\} \) have a unique common fixed point \( z \) in \( X \) and \( \{B_nS_n\} \) and \( \{A_nT_n\} \) have a unique common fixed point \( w \) in \( Y \). Further, \( \{A_n\}z = \{B_n\}z = w \) and \( \{S_n\}w = \{T_n\}w = z \).

**Proof:** Let \( x_0 \) be an arbitrary point in \( X \) and we define the sequences \( \{x_n\} \) in \( X \) and \( \{y_n\} \) in \( Y \) by

\[
A_n x_{2n} = y_{2n-1}S_n y_{2n-1} = x_{2n-1}, \quad B_n x_{2n-1} = y_{2n}, \quad T_n y_{2n} = x_{2n}
\]

for \( n = 1, 2, 3 \ldots \).

Now we have

\[
d(x_{2n+1}, x_{2n}) = d(S_nA_n x_{2n}, T_nB_n x_{2n+1}) \\
\leq c_1 \max \{ d(x_{2n-1}, x_{2n}), d(S_nA_n x_{2n}), d(x_{2n-1}, T_nB_n x_{2n+1}), e(A_nB_n x_{2n}, x_{2n+1}) \} \\
= c_1 \max \{ d(x_{2n-1}, x_{2n}), d(x_{2n-1}, x_{2n}), e(y_{2n-1}, y_{2n}), d(x_{2n-1}, x_{2n}) \} \\
\leq c_1 \max \{ d(x_{2n-1}, x_{2n}), e(y_{2n-1}, y_{2n}), 0, d(x_{2n-1}, x_{2n}) \} \\
\leq c_1 \max \{ d(x_{2n-1}, x_{2n}), e(y_{2n-1}, y_{2n}) \}
\]

Now

\[
e(y_{2n}, y_{2n+1}) = e(B_nS_n y_{2n-1}, A_nT_n y_{2n}) \\
\leq c_2 \max \{ e(y_{2n-1}, y_{2n}), e(y_{2n-1}, B_nS_n y_{2n-1}), e(y_{2n-1}, A_nT_n y_{2n}), d(S_n y_{2n-1}, T_n y_{2n}) \} \\
= c_2 \max \{ e(y_{2n-1}, y_{2n}), e(y_{2n-1}, y_{2n}), e(y_{2n-1}, y_{2n+1}), d(x_{2n-1}, x_{2n}) \} \\
\leq c_2 \max \{ e(y_{2n-1}, y_{2n}), e(y_{2n-1}, y_{2n}), e(y_{2n-1}, y_{2n+1}), 0 \} \\
\leq c_2 \max \{ e(y_{2n-1}, y_{2n}), d(x_{2n-1}, x_{2n}) \} \\
= c_2 \max \{ d(x_{2n-1}, x_{2n}) \}
\]

we have

\[
d(x_{2n}, x_{2n+1}) = d(x_{2n-1}, x_{2n})
\]
Thus \( \{x_n\} \) is a Cauchy sequence in \((X,d)\). Since \((X,d)\) is complete, \( \{x_n\} \) converges to a point \( z \) in \( X \). Similarly using inequalities (2.4.4) and (2.4.6), we prove \( \{y_n\} \) is a Cauchy sequence in \((Y,e)\) with the limit \( w \) in \( Y \).

Suppose \( A \) is continuous, then

\[
\lim_{n \to \infty} A_n x_n = A_z = \lim_{n \to \infty} y_{2n+1} = w.
\]

Now we prove \( S_n A_n z = z \).

Suppose \( S_n A_n z \neq z \).

We have

\[
d(S_n A_n z, z) = \lim_{n \to \infty} d(S_n A_n z, T_n B_n x_{2n-1})
\]

\[
\leq \lim_{n \to \infty} c_1 \max \{ d(z, x_{2n-1}), d(z, S_n A_n z), e(A_n z, y_{2n}),
\frac{d(z, T_n B_n x_{2n-1})}{2}, \frac{d(S_n A_n z, x_{2n-1})}{2} \}
\]

\[
= c_1 \max \{ d(z, x_{2n-1}), d(z, S_n A_n z), e(A_n z, y_{2n}),
\frac{d(z, T_n B_n x_{2n-1})}{2}, \frac{d(S_n A_n z, x_{2n-1})}{2} \}
\]

\[
\leq c_1 d(z, S_n A_n z)
\]

\[
< d(z, S_n A_n z) \quad (\text{Since } 0 \leq c_1 < 1)
\]

Which is a contradiction.

Thus \( S_n A_n z = z \).

Hence \( S_n w = z \). (Since \( A_n z = w \))

Now we prove \( B_n S_n w = w \).

Suppose \( B_n S_n w \neq w \).

We have

\[
e(B_n S_n w, w) = \lim_{n \to \infty} e(B_n S_n w, y_{2n+1})
\]

\[
= \lim_{n \to \infty} e(B_n S_n y_{2n}, A_n T_n y_{2n})
\]

\[
\leq \lim_{n \to \infty} c_2 \max \{ e(w, y_{2n}), e(w, B_n S_n w),
\frac{e(y_{2n}, A_n T_n y_{2n})}{2}, \frac{d(S_n w, T_n y_{2n})}{2},
\frac{e(w, A T_n y_{2n})}{2}, \frac{e(B S_n y_{2n})}{2} \}
\]

\[
= c_2 \max \{ e(w, y_{2n}), (w, B_n S_n w), e(w, w),
\frac{d(z, z)}{2}, \frac{e(w, w)}{2}, \frac{e(B S_n w, w)}{2} \}
\]

\[
< e(w, B_n S_n w) \quad (\text{Since } 0 \leq c_2 < 1)
\]

Which is a contradiction.

Thus \( B_n S_n w = w \).

Hence \( B_n z = w \). (Since \( S_n w = z \))

Now we prove \( T_n B_n z = z \).

Suppose \( T_n B_n z \neq z \).

We have

\[
d(z, T_n B_n z) = \lim_{n \to \infty} d(x_{2n+1}, T_n B_n z)
\]

\[
= \lim_{n \to \infty} d(S_n A_n x_{2n}, T_n B_n z)
\]

\[
\leq \lim_{n \to \infty} c_1 \max \{ d(x_{2n+1}, z), d(x_{2n+1}, S_n A_n x_{2n}),
\frac{d(x_{2n+1}, T_n B_n z)}{2}, \frac{d(S_n A_n x_{2n}, z)}{2} \}
\]

\[
= c_1 \max \{ d(x_{2n+1}, z), d(x_{2n+1}, S_n A_n x_{2n}),
\frac{d(x_{2n+1}, T_n B_n z)}{2}, \frac{d(S_n A_n x_{2n}, z)}{2} \}
\]

\[
\leq c_1 d(x_{2n+1}, S_n A_n x_{2n})
\]

\[
< d(x_{2n+1}, S_n A_n x_{2n}) \quad (\text{Since } 0 \leq c_1 < 1)
\]

Which is a contradiction.

Thus \( T_n B_n z = z \).

Hence \( T_n w = z \). (Since \( B_n z = w \))

Now we prove \( A_n T_n w = w \).

Suppose \( A_n T_n w \neq w \).

We have

\[
e(w, A_n T_n w) = \lim_{n \to \infty} e(y_{2n}, A_n T_n w)
\]

\[
= \lim_{n \to \infty} e(B_n S_n y_{2n-1}, A_n T_n w)
\]
We have $d(z, z') = e(Bz, y) = d(Sw, y)$.

$\frac{e(y, z')}{2} = c_2. \max\{ e(w, w'), e(w, w), e(w', w'), d(z', z') \}
\frac{d(z', z)}{2}
\frac{e(w, w')}{2}
\frac{e(w, w')}{2}$

$d(z', z') < e(w, w') < d(z, z')$

Which is a contradiction.

Thus $z = z'$.

So the point $z$ is the unique common fixed point of $(S_2A_2)$ and $(T_2B_2)$. Similarly we prove $w$ is a unique common fixed point of $(B_2S_2)$ and $(A_2T_2)$.

**Remark 2.5**: If we put $A = A, B = B, S = S$ and $T = T$ in the above theorem 2.4, we get the following corollary.

**Corollary 2.6**: Let $(X, d)$ and $(Y, e)$ be complete metric spaces. Let $A, B$ be mappings of $X$ into $Y$ and $S, T$ be mappings of $Y$ into $X$ satisfying the inequalities.

$d(SAx, TBx') \leq c_2. \max\{ d(x, x'), d(x, SAx), d(x', TBx'), e(Ax, Bx'), d(x, TBx') - d(SAx, x') \}$

$e(Ax, Bx') - e(BSy, ATy') \leq c_2. \max\{ e(y, x'), e(y, BSy), e(y', ATy'), e(y, ATy') - e(BSy, y') \}$

for all $x, x' \in X$ and $y, y' \in Y$ where $0 \leq c_1 < 1$ and $0 \leq c_2 < 1$.

If one of the mappings $A, B, S$ and $T$ is continuous, then $SA$ and $TB$ have a unique common fixed point $z$ in $X$ and $BS$ and $AT$ have a unique common fixed point $w$ in $Y$. Further, $Az = w$ and $Sw = Tw = w$.

**Theorem 2.7**: Let $(X, d)$ and $(Y, e)$ be complete metric spaces. Let $(A_n), (B_n) (n \in N)$ be sequence of mappings of $X$ into $Y$ and $(S_n), (T_n) (n \in N)$ be sequence of mappings of $Y$ into $X$ satisfying the inequalities.

$d(S_nAx, T_nB_nx') \leq c_2. \max\{ d(x, x'), d(x, S_nAx), e(A_nx, B_nx'), d(x, TB_nx') - d(S_nAx, x') \}$

$e(B_ny, AT_ny') \leq c_2. \max\{ e(y, y'), e(y, BS_ny), e(y', AT_ny'), e(y, AT_ny') - e(B_nSy, y') \}$

for all $x, x' \in X$ and $y, y' \in Y$ where $0 \leq c_1 < 1$ and $0 \leq c_2 < 1$.
for all \( i \neq j \neq \emptyset \neq q, x, x' \in X \) and \( y, y' \) in \( Y \) where \( 0 \leq c_1 < 1 \) and \( 0 \leq c_2 < 1 \). If one of the mappings \( \{A_n\}, \{B_n\}, \{S_n\} \) and \( \{T_n\} \) is continuous, then \( \{S_nA_n\} \) and \( \{T_nB_n\} \) have a common fixed point \( z \) in \( X \) and \( \{B_nS_n\} \) and \( \{A_nT_n\} \) have a common fixed point \( w \) in \( Y \). Further, \( \{A_n\}z = \{B_n\}z = w \) and \( \{S_n\}w = \{T_n\}w = z \).

Proof: Let \( x_0 \) be an arbitrary point in \( X \) and we define the sequences \( \{x_n\} \) in \( X \) by \( A_nx_{n+1} = y_{2n+1}, S_ny_{2n+1} = x_{2n+1}, B_nx_{2n+1} = y_{2n}, T_ny_{2n} = x_{2n} \) for \( n = 1, 2, 3 \ldots \).

Now we have

\[
\begin{align*}
\text{d}(x_{2n+1}, x_n) & = \text{d}(A_nx_{2n}, T_nB_nx_{2n+1}) \\
& \leq c_1 \max \{ \text{d}(x_{2n}, x_{2n+1}), \text{d}(x_{2n}, S_nA_nx_{2n+1}), \\
& e(A_nx_{2n}, B_nx_{2n+1}), \text{d}(x_{2n}, T_nB_nx_{2n+1}) / 2, \\
& \text{d}(S_nA_nx_{2n}, x_{2n+1}) / 2, \\
& \text{d}(x_{2n}, T_nB_nx_{2n+1}) / \text{d}(x_{2n}, x_{2n+1}) \} \\
& = c_1 \max \{ \text{d}(x_{2n}, x_{2n+1}), \text{d}(x_{2n}, x_{2n+1}), \text{d}(y_{2n+1}, y_{2n}), \\
& \text{d}(x_{2n}, y_{2n}) / 2, \text{d}(x_{2n}, x_{2n+1}) / 2, \\
& \text{d}(x_{2n}, x_{2n+1}) / \text{d}(x_{2n}, x_{2n+1}) \} \\
& \leq c_1 \max \{ \text{d}(x_{2n}, x_{2n+1}), \text{d}(y_{2n+1}, y_{2n}) \} \quad (2.7.3)
\end{align*}
\]

Now

\[
\begin{align*}
\text{e}(y_{2n}, y_{2n+1}) & = \text{d}(B_nS_ny_{2n+1}, A_nT_ny_{2n}) \\
& \leq c_2 \max \{ \text{e}(y_{2n}, y_{2n+1}), \text{d}(S_ny_{2n+1}, T_ny_{2n}), \text{e}(y_{2n}, A_nT_ny_{2n}) / 2, \\
& \text{d}(B_nS_ny_{2n+1}, y_{2n}), \text{e}(y_{2n}, A_nT_ny_{2n}) / 2, \\
& \text{d}(e(y_{2n}, y_{2n+1}), \text{e}(y_{2n}, y_{2n+1}), \text{d}(x_{2n}, x_{2n+1}), \\
& \text{e}(y_{2n}, y_{2n+1}) / 2, \text{d}(y_{2n}, y_{2n+1}) / 2, \\
& \text{e}(y_{2n}, y_{2n+1}) / \text{e}(y_{2n}, y_{2n+1}) \} \\
& \leq c_2 \max \{ \text{e}(y_{2n}, y_{2n+1}), \text{d}(x_{2n}, x_{2n+1}) \} \quad (2.7.4)
\end{align*}
\]

Similarly,

\[
\begin{align*}
\text{d}(x_{2n}, x_{2n+1}) & \leq c_1 \max \{ \text{d}(x_{2n}, x_{2n+1}), \text{e}(y_{2n}, y_{2n+1}) \} \quad (2.7.5) \\
\text{e}(y_{2n}, y_{2n+1}) & \leq c_2 \max \{ \text{e}(y_{2n}, y_{2n+1}), \text{d}(x_{2n}, x_{2n+1}) \} \quad (2.7.6)
\end{align*}
\]

from inequalities (2.7.3), (2.7.4), (2.7.5) and (2.7.6), we have

\[
\begin{align*}
\text{d}(x_{n+1}, x_n) & \leq c_1^{n} c_2^{n-1} \max \{ \text{d}(x_1, x_0), \text{e}(y_1, y_2) \} \to 0 \quad \text{as} \ n \to \infty
\end{align*}
\]

Thus \( \{x_n\} \) is a Cauchy sequence in \((X,d)\). Since \((X,d)\) is complete, \( \{x_n\} \) converges to a point \( z \) in \( X \). Similarly using inequalities (2.7.3), (2.7.4), (2.7.5) and (2.7.6), we prove \( \{y_n\} \) is a Cauchy sequence in \((Y,e)\) with the limit \( w \) in \( Y \).

Suppose \( \{A_n\} \) is continuous, then

\[
\lim_{n \to \infty} A_nx_{2n} = A_z = \lim_{n \to \infty} y_{2n+1} = w.
\]

Now we prove \( S_nA_nz = z \).

Suppose \( S_nA_nz \neq z \).

We have

\[
\begin{align*}
\text{d}(S_nA_nz, z) & = \lim_{n \to \infty} \text{d}(S_nA_nz, T_nB_nx_{2n+1}) \\
& \leq \lim_{n \to \infty} c_1 \max \{ \text{d}(z, x_{2n+1}), \text{d}(z, S_nA_nz), \\
& e(A_nz, B_nx_{2n+1}), \text{d}(z, T_nB_nx_{2n+1}) / 2, \\
& \text{d}(S_nA_nz, x_{2n+1}) / 2, \\
& \text{d}(z, S_nA_nz) \cdot \text{d}(x_{2n+1}, T_nB_nx_{2n+1}) / \text{d}(z, x_{2n+1}) \} \\
& \leq \lim_{n \to \infty} c_1 \max \{ \text{d}(z, x_{2n+1}), \text{d}(z, S_nA_nz), \\
& e(A_nz, B_nx_{2n+1}), \text{d}(z, x_{2n+1}) / 2, \\
& \text{d}(z, S_nA_nz) \cdot \text{d}(x_{2n+1}, x_{2n+1}) / \text{d}(z, x_{2n+1}) \} \\
& \leq c_1 \max \{ \text{d}(z, S_nA_nz) \} \\
& < \text{d}(z, S_nA_nz) \quad (\text{Since} \ 0 \leq c_1 < 1)
\end{align*}
\]

Which is a contradiction.

Thus \( S_nA_nz = z \).

Hence \( S_nw = z \). (Since \( A_nz = w \))

Now we prove \( B_nS_nw = w \).

Suppose \( B_nS_nw \neq w \).

We have

\[
\begin{align*}
\text{e}(B_nS_nw, w) & = \lim_{n \to \infty} \text{e}(B_nS_nw, y_{2n+1}) \\
& = \lim_{n \to \infty} \text{e}(B_nS_nw, A_nT_ny_{2n}) \\
& \leq \lim_{n \to \infty} c_2 \max \{ \text{e}(w, y_{2n}), \text{e}(w, B_nS_nw), \\
& \text{d}(S_nw, T_ny_{2n}), \text{e}(w, A_nT_ny_{2n}) / 2, \text{e}(B_nS_nw, y_{2n+1}) / 2, \\
& \text{e}(w, B_nS_nw), \text{e}(y_{2n}, A_nT_ny_{2n}) / \text{e}(w, y_{2n}) \} \\
& < \text{e}(w, B_nS_nw) \quad (\text{Since} \ 0 \leq c_2 < 1)
\end{align*}
\]

Which is a contradiction.
Thus $B_nS_nw = w$.

Hence $B_nz = w$. (Since $S_nw = z$)

Now we prove $T_nB_nz = z$.

Suppose $T_nB_nz \neq z$.

$$d(z, T_nB_nz) = \lim_{n \to \infty} d(x_{2n+1}, T_nB_nz)$$

$$= \lim_{n \to \infty} d(S_nA_nx_{2n}, T_nB_nz)$$

$$\leq \lim_{n \to \infty} c_1 \max\{d(x_{2n}, z), d(x_{2n}, S_nA_nx_{2n}), e(A_nx_{2n}, B_nz), d(x_{2n}, T_nB_nz) / 2\},$$

$$d(SA_{2n}x_{2n} / 2, d(x_{2n}, S_nA_nx_{2n}).d(z, T_nB_nz) / d(x_{2n}z) \}< d(z, T_nB_nz) \quad \text{(Since } 0 \leq c_1 < 1 \text{)}$$

Which is a contradiction.

Thus $T_nB_nz = z$.

Hence $T_nw = z$. (Since $B_nz = w$)

Now we prove $A_nT_nw = w$.

Suppose $A_nT_nw \neq w$.

$$e(w, A_nT_nw) = \lim_{n \to \infty} e(y_{2n}, A_nT_nw)$$

$$= \lim_{n \to \infty} e(B_nS_ny_{2n-1}, A_nT_nw)$$

$$\leq \lim_{n \to \infty} c_2 \max\{e(y_{2n-1}, w), e(y_{2n-1}, B_nS_ny_{2n-1}), d(S_ny_{2n-1}, T_nw), e(y_{2n-1}, A_nT_nw) / 2, e(B_nS_ny_{2n-1}, w) / 2, e(y_{2n-1}, A_nT_nw) / e(y_{2n-1}, w)\}$$

$$< e(w, A_nT_nw) \quad \text{(Since } 0 \leq c_2 < 1 \text{)}$$

Which is a contradiction.

Thus $A_nT_nw = w$.

The same results hold if one of the mappings $\{B_n\}$, $\{S_n\}$ and $\{T_n\}$ is continuous.

So the point $z$ is the common fixed point of $\{S_nA_n\}$ and $\{T_nB_n\}$. Similarly we prove $w$ is a common fixed point of $\{B_nS_n\}$ and $\{A_nT_n\}$.

**Uniqueness**: Let $z'$ be another common fixed point of $\{S_nA_n\}$ and $\{T_nB_n\}$ in $X$, $w'$ be another common fixed point of $\{B_nS_n\}$ and $\{A_nT_n\}$ in $Y$.

We have $d(z, z') = d(S_nA_nz', T_nB_nz')$

$$\leq c_1 \max\{d(z, z'), d(z, S_nA_nz), e(A_nz, B_nz'), d(z, T_nB_nz') / 2, d(S_nA_nz, z') / 2\},$$

$$d(z, z') \leq c_1 \max\{d(z, z'), d(z, S_nA_nz), e(A_nz, B_nz'), d(z, T_nB_nz') / 2, d(S_nA_nz, z') / 2\},$$

$$d(z, z') \leq c_1 \max\{d(z, z'), d(z, S_nA_nz), e(A_nz, B_nz'), d(z, T_nB_nz') / 2, d(S_nA_nz, z') / 2\},$$

$$< e(w, w')$$

$$e(w, w') = e(B_nS_nw, A_nT_nw)$$

$$\leq c_2 \max\{e(w, w'), e(w, B_nS_nw), e(S_nw, T_nw), e(w, A_nT_nw) / 2, e(B_nS_nw, w') / 2, e(w, B_nS_nw).e(w', A_nT_nw) / e(w, w')\}$$

$$< d(z, z')$$

Hence $d(z, z') < e(w, w') < d(z, z')$

Which is a contradiction.

Thus $z = z'$.

So the point $z$ is the unique common fixed point of $\{S_nA_n\}$ and $\{T_nB_n\}$. Similarly we prove $w$ is a unique common fixed point of $\{B_nS_n\}$ and $\{A_nT_n\}$.

**Remark 2.8**: If we put $A_i = A$, $B_j = B$, $S_p = S$ and $T_q = T$ in the above theorem 2.7, we get the following corollary.

**Corollary 2.9**: Let $(X, d)$ and $(Y, e)$ be complete metric spaces. Let $A$, $B$ be mappings of $X$ into $Y$ and $S$, $T$ be mappings of $Y$ into $X$ satisfying the inequalities:

$$d(SAx, TBx') \leq c_1 \max\{d(x, x'), d(x, SAx), e(Ax, Bx'), d(x, TBx') / 2, d(SAx, x') / 2, d(x, SAx).d(x', TBx') / d(x, x')\}$$

$$e(BSy, ATy') \leq c_2 \max\{e(y, y'), e(y, BSy), d(Sy, Ty') / e(y, Ty') / e(y, y')\}$$

for all $x, y \in X$ and $y, y' \in Y$ where $0 \leq c_1 < 1$ and $0 \leq c_2 < 1$.

If one of the mappings $A$, $B$, $S$ and $T$ is continuous, then $SA$ and $TB$ have a unique common fixed point $z$ in $X$ and $BS$ and $AT$ have a unique common fixed point $w$ in $Y$. Further, $Az = Bz = w$ and $Sw = Tw = z$.

**Theorem 2.10**: Let $(X, d)$ and $(Y, e)$ be complete metric spaces. Let $\{A_n\}$, $\{B_n\}$ (n $\in \mathbb{N}$) be sequence of mappings of $X$ into $Y$ and $\{S_n\}$, $\{T_n\}$, (n $\in \mathbb{N}$) be sequence of mappings of $Y$ into $X$ satisfying the inequalities.

$$d(S_pAx, T_nB_nx') \leq c_1 \max\{d(x, x'), d(x, S_pAx), e(Ax, Bx'), d(x, TBx') / 2, d(SAx, x') / 2, d(x, S_pAx).d(x', TBx') / d(x, x')\}$$

$$e(B_jSy, AT_qy') \leq c_2 \max\{e(y, y'), e(y, B_jSy)\}$$. (2.10.1)
Now we have
\[
d(x_{2n+1}, x_{n}) = d(S_{n}x_{n}, T_{n}B_{n}x_{2n+1})
\]
\[
\leq c_{1}\max\left\{ d(x_{2n}, x_{2n+1}), d(x_{2n}, S_{n}x_{n}), e(A_{n}x_{n}, B_{n}x_{2n+1}), d(y_{2n+1}, y_{2n})\right\}/2,
\]
\[
d(x_{2n}, x_{2n+1}) = c_{1}\max\left\{ d(x_{2n}, x_{2n+1}), d(x_{2n}, S_{n}x_{n}), e(y_{2n+1}, y_{2n})\right\}/2,
\]
\[
d(x_{2n}, x_{2n+1})/2, d(x_{2n}, x_{2n+1})\right\}/2.
\]
\[
\leq c_{1}\max\left\{ d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), e(x_{2n}, y_{2n})\right\} (2.10.3)
\]
\[
e(y_{2n}, y_{2n+1}) = e(B_{n}S_{n}y_{2n}, A_{n}T_{n}y_{2n})
\]
\[
\leq c_{2}\max\left\{ e(y_{2n}, y_{2n}), e(y_{2n}, B_{n}S_{n}y_{2n}), d(S_{n}y_{2n}, T_{n}y_{2n}), e(y_{2n}, A_{n}T_{n}y_{2n})/e(y_{2n}, y_{2n})\right\}/2,
\]
\[
e(y_{2n}, y_{2n+1}), e(y_{2n}, y_{2n+1}), d(x_{2n}, x_{2n+1}), e(y_{2n}, y_{2n+1})\right\}/2, e(y_{2n}, y_{2n+1})\right\}/2.
\]
\[
\leq c_{2}\max\left\{ e(y_{2n}, y_{2n}), d(x_{2n}, x_{2n+1})\right\} (2.10.4)
\]
Similarly,
\[
d(x_{2n}, x_{2n+1}) \leq c_{1}\max\left\{ d(x_{2n}, x_{2n+1}), e(y_{2n}, y_{2n})\right\} (2.10.5)
\]
\[
e(y_{2n}, y_{2n+1}) \leq c_{2}\max\left\{ e(y_{2n}, y_{2n}), d(x_{2n}, x_{2n+1})\right\} (2.10.6)
\]
from inequalities (2.10.3), (2.10.4), (2.10.5) and (2.10.6), we have
\[
d(x_{n+1}, x_{n}) \leq c_{1}^{n}c_{2}^{-n-1}, \max\left\{ d(x_{1}, x_{0}), e(y_{1}, y_{2})\right\} \rightarrow 0 \text{ as } n \rightarrow \infty
\]
Thus \{x_{n}\} is a Cauchy sequence in \((X,d)\). Since \((X,d)\) is complete, \{x_{n}\} converges to a point \(z\) in \(X\). Similarly using inequalities (2.10.3), (2.10.4), (2.10.5) and (2.10.6), we prove \{y_{n}\} is a Cauchy sequence in \((Y,e)\) with the limit \(w\) in \(Y\).

Suppose \{A_{n}\} is continuous, then
\[
\lim_{n \rightarrow \infty} A_{n}x_{2n} = A_{n}z = \lim_{n \rightarrow \infty} y_{2n+1} = w.
\]
Now we prove \(S_{n}A_{n}z = z\).

Suppose \(S_{n}A_{n}z \neq z\).

We have
\[
d(S_{n}A_{n}z, z) = \lim_{n \rightarrow \infty} d(S_{n}A_{n}z, T_{n}B_{n}x_{2n+1})
\]
\[
\leq \lim_{n \rightarrow \infty} c_{1}\max\{ d(z, x_{2n+1}), d(z, S_{n}A_{n}z), e(A_{n}z, B_{n}x_{2n+1}),
\]
\[
d(z, T_{n}B_{n}x_{2n+1}) + d(S_{n}A_{n}z, x_{2n+1})\}/2,
\]
\[
d(z, S_{n}A_{n}z), d(x_{2n}, T_{n}B_{n}x_{2n+1}) / d(z, x_{2n+1})\}
\]
\[
\leq \lim_{n \rightarrow \infty} c_{1}\max\{ d(z, x_{2n+1}), d(z, S_{n}A_{n}z), e(A_{n}z, y_{2n},
\]
\[
d(z, x_{2n}), d(z, S_{n}A_{n}z), e(A_{n}z, y_{2n})\}
\]
\[
\leq \lim_{n \rightarrow \infty} c_{1}\max\{ d(z, x_{2n+1}), d(z, S_{n}A_{n}z), e(w, w),
\]
\[
d(z, x_{2n}), d(z, S_{n}A_{n}z), e(z, z)\}/2, d(z, x_{2n}), d(z, x_{2n+1}) / d(z, x_{2n+1})\}
\]
\[
\leq c_{1}\max\{ d(z, x_{2n}), d(z, S_{n}A_{n}z), e(z, z)\}/2, c_{1}\max\{ d(z, x_{2n}), d(z, x_{2n+1}) / d(z, x_{2n+1})\}
\]
\[
< c_{1}\max\{ d(z, x_{2n}), d(z, S_{n}A_{n}z)\} (Since 0 \leq c_{1} < 1)
\]
Which is a contradiction.

Thus \(S_{n}A_{n}z = z\).

Hence \(S_{n}w = z\). (Since \(A_{n}z = w\))

Now we prove \(B_{n}S_{n}w = w\).

Suppose \(B_{n}S_{n}w \neq w\).

We have
\[
e(B_{n}S_{n}w, w) = \lim_{n \rightarrow \infty} e(B_{n}S_{n}w, y_{2n+1})
\]
\[
= \lim_{n \rightarrow \infty} e(B_{n}S_{n}w, A_{n}T_{n}y_{2n})
\]
\[
\leq \lim_{n \rightarrow \infty} c_{2}\max\{ e(w, y_{2n}), e(w, B_{n}S_{n}w), d(S_{n}w, T_{n}y_{2n}),
\]
\[
e(w, A_{n}T_{n}y_{2n}) + e(B_{n}S_{n}w, y_{2n})\}/2, e(w, B_{n}S_{n}w), e(y_{2n}, A_{n}T_{n}y_{2n}) / e(w, y_{2n})\}
\]
\[
e(w, B_{n}S_{n}w) (Since 0 \leq c_{2} < 1)
\]
Which is a contradiction.
Thus $B_nS_nw = w$.

Hence $B_nz = w$. (Since $S_nw = z$)

Now we prove $T_aw = z$.

Suppose $T_aw \neq z$.

$$d(z, T_aw) = \lim_{n \to \infty} d(x_{2n+1}, T_aw)$$
$$= \lim_{n \to \infty} d(S_nA_nx_{2n}, T_aw)$$
$$\leq \lim_{n \to \infty} c_1 \max\{d(x_{2n}, z), d(x_{2n}, S_nA_nx_{2n})\},$$
$$e(A_nx_{2n}, B_nz), [d(x_{2n}, T_aw)+d(SAx_{2n}, z)] / 2,$$
$$d(x_{2n}, S_nA_nx_{2n}), d(z, T_aw) / d(x_{2n}, z)$$
$$< d(z, T_aw) \quad \text{(Since } 0 \leq c_1 < 1\)$$

Which is a contradiction.

Thus $T_aw = z$.

Hence $T_bw = z$. (Since $B_nz = w$)

Now we prove $A_nT_aw = w$.

Suppose $A_nT_aw \neq w$.

$$e(w, A_nT_aw) = \lim_{n \to \infty} e(y_{2n}, A_nT_aw)$$
$$= \lim_{n \to \infty} e(B_nS_nA_ny_{2n-1}, A_nT_aw)$$
$$\leq \lim_{n \to \infty} c_2 \max\{e(y_{2n-1}, w), e(y_{2n-1}, B_nS_ny_{2n-1})\},$$
$$d(S_ny_{2n-1}, T_aw), [e(y_{2n-1}, A_nT_aw)+e(B_nS_ny_{2n-1}, w)] / 2,$$
$$e(y_{2n-1}, B_nS_ny_{2n-1}), e(w, A_nT_aw) / e(y_{2n-1}, w)$$
$$< e(w, A_nT_aw) \quad \text{(Since } 0 \leq c_2 < 1\)$$

Which is a contradiction.

Thus $A_nT_aw = w$.

The same results hold if one of the mappings $\{A_n\}, \{S_n\}$ and $\{T_a\}$ is continuous.

So the point $z$ is the common fixed point of $\{S_nA_n\}$ and $\{T_Bn\}$. Similarly we prove $w$ is a common fixed point of $\{B_nS_n\}$ and $\{A_nT_a\}$.

**Uniqueness:** Let $z'$ be another common fixed point of $\{S_nA_n\}$ and $\{T_Bn\}$ in $X$, $w'$ be another common fixed point of $\{B_nS_n\}$ and $\{A_nT_a\}$ in $Y$.

We have $d(z, z') = d(S_nA_nz, T_Bnz')$

$$\leq c_1 \max\{d(z, z'), d(z, S_nA_nz), e(A_nz, B_nz'),$$
$$[d(z, T_Bnz')+d(S_nA_nz, z')] / 2,$$
$$d(z, S_nA_nz), d(z', T_Bnz') / d(z, z') \}$$

Thus $A_nT_aw = z'$.

Hence $d(z, z') < e(w, w') < d(z, z')$

Which is a contradiction.

Thus $z = z'$.

So the point $z$ is the unique common fixed point of $\{S_nA_n\}$ and $\{T_Bn\}$. Similarly we prove $w$ is a unique common fixed point of $\{B_nS_n\}$ and $\{A_nT_a\}$.

**Remark 2.11:** If we put $A_i = A$, $B_j = B$, $S_i = S$ and $T_j = T$ in the above theorem 2.10, we get the following corollary.

**Corollary 2.12:** Let $(X, d)$ and $(Y, e)$ be complete metric spaces. Let $A$, $B$ be mappings of $X$ into $Y$, $S$, $T$ be mappings of $Y$ into $X$ satisfying the inequalities.

$$d(SAx, TBx') \leq c_1 \max\{d(x, x'), d(x, SAX), e(Ax, Bx'),$$
$$[d(x, TBx')+d(SAx, x')]/2,$$
$$d(x, SAX), d(x', TBx') / d(x, x')\}$$
$$e(BSy, ATy') \leq c_2 \max\{e(y, y'), e(y, BSy), d(Sy, Ty'),$$
$$[e(y, ATy')+e(BSy, y')]/2,$$
$$e(y, BSy), e(y', ATy') / e(y, y')\}\}

for all $x, x'$ in $X$ and $y, y'$ in $Y$ where $0 \leq c_1 < 1$ and $0 \leq c_2 < 1$.

If one of the mappings $A$, $B$, $S$ and $T$ is continuous, then $SA$ and $TB$ have a unique common fixed point $z$ in $X$ and $BS$ and $AT$ have a unique common fixed point $w$ in $Y$. Further, $Az = Bz = w$ and $Sw = Tw = z$.

**REFERENCES**


