Quasi Conformal Curvature Tensor on a P-Sasakian Einstein Manifold

Gajendra Nath Tripathi, Sudhir Dubey, Dhruwa Narain

Department of Mathematics and Statistics,
D D U Gorakhpur University,
Gorakhpur-273009, INDIA

Abstract—In this paper, we have studied p-Sasakian Einstein manifold which satisfy the condition \( r = n(n - 1), a + 2(n - 1)b \neq 0 \) i.e. the constant scalar curvature \( r \). Also the p-Sasakian Einstein manifold satisfying \( \text{div} \tilde{\mathbf{C}} = 0 \) have studied. Where \( \tilde{\mathbf{C}} \) is quasi-conformal curvature tensor and \( r \) is the scalar curvature.

Keywords—P-Sasakian manifold, Quasi-conformal curvature tensor, Einstein manifold.

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1. PRELIMINARIES

Let \( M^n \) be \( n \)-dimensional \( C^\infty \)-manifold. If there exist a tensor field \( F \) of type \( (1, 1) \), a vector field \( \xi \) and a 1-form \( \eta \) in \( M^n \) satisfying

\[
\begin{align*}
\bar{\mathbf{X}} &= X - \eta(X) \xi, \\
\mathbf{X} &= F(X), \\
\eta(\xi) &= 1
\end{align*}
\]

then \( M^n \) is called an almost para contact manifold.

Let \( g \) be the Riemannian metric satisfying

\[
\begin{align*}
g(X, \xi) &= \eta(X) \\
\eta(F, X) &= 0, \\
F\xi &= 0, \\
\text{rank } F &= (n - 1)
\end{align*}
\]

Then the set \( (F, \xi, \eta, g) \) satisfying (1.1), (1.2), (1.3) and (1.4) is called an almost para-contact Riemannian structure. The manifold with such structure is called an almost \( p \)-contact Riemannian manifold [1].

If we define \( F(X, Y) = g(\bar{\mathbf{X}}, Y) \), then in addition to the above relations the following are satisfied:

\[
\begin{align*}
F(X, Y) &= F(Y, X) \\
F(\bar{\mathbf{X}}, \bar{\mathbf{Y}}) &= F(X, Y)
\end{align*}
\]

Let us consider an \( n \)-dimensional differentiable manifold \( M \) with a positive definite metric \( g \) which admits 1-forms \( \eta \) satisfying

\[
(\nabla_X \eta)(Y) - (\nabla_Y \eta)(X) = 0
\]

And

\[
(\nabla_X \nabla_Y \eta)(Z) = g(X, Z) \eta(Y) - g(X, Y) \eta(Z) + 2 \eta(X) \eta(Y) \eta(Z)
\]

Where, \( \nabla \) denote the covariant differentiation with respect to \( g \). Moreover, if we put,

\[
(\nabla_X \xi) = \bar{\mathbf{X}}
\]

Then it can be easily verified that the manifold in consideration becomes an almost para-contact Riemannian manifold. Such a manifold is called p-Saskian manifolds [2].
For a p-Saskian manifold the following relations hold [4]:

\[(\text{1.10}) \quad R(X, Y)\xi = \eta(X)Y - \eta(Y)X\]

\[(\text{1.11}) \quad R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi\]

\[(\text{1.12}) \quad S(X, \xi) = -(n - 1)\eta(X)\]

\[(\text{1.13}) \quad Q\xi = -(n - 1)\xi\]

\[(\text{1.15}) \quad \eta(R(X, Y)U) = g(X, U)\eta(Y) - g(Y, U)\eta(X)\]

\[(\text{1.16}) \quad \eta(R(X, Y)\xi) = 0\]

\[(\text{1.17}) \quad \eta(R(\xi, X)Y) = \eta(X)\eta(Y) - g(X, Y)\]

For any vector field \(X, Y, Z\) where \(R\) and \(S\) are the curvature tensor and Ricci tensor and \(Q\) is the Ricci operator.

2. A P-SASAKIAN EINSTEIN MANIFOLD SATISFYING \(R = -n(n - 1), a + 2(n - 1)b \neq 0\)

A p-Sasakian manifold \(M^n\) is said to be Einstein manifold, if its Ricci tensor \(S\) is of the form

\[(\text{2.1}) \quad S(X, Y) = kg(X, Y)\]

where \(k\) is constant.

Putting \(Y = \xi\) in (2.1), we get \(S(X, \xi) = kg(X, \xi)\)

Since \(S(X, \xi) = -(n - 1)\eta(X)\) and \(g(X, \xi) = \eta(X)\), we have

\[(\text{2.2}) \quad k = -(n - 1)\]

From (2.1) and (2.2), we get

\[(\text{2.3}) \quad S(X, Y) = -(n - 1)g(X, Y)\]

Contracting (2.3), we get,

\[(\text{2.4}) \quad QY = -(n - 1)Y\]

Where \(S(X, Y) = g(QX, Y)\).

Let \((M^n, g)\) be \(n\)-dimensional Riemannian manifold, the Quasi-conformal curvature tensor \(\tilde{C}\) is defined by [9].

\[(\text{2.5}) \quad \tilde{C}(X, Y)Z = aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] + (\frac{a}{n(n - 1)} + 2b)[g(Y, Z)X - g(X, Z)Y]\]

Using (2.3) and (2.4) in (2.5), we get

\[(\text{2.6}) \quad \tilde{C}(X, Y)Z = aR(X, Y)Z - [2(n - 1)b + \frac{a}{n(n - 1)} + 2b][g(Y, Z)X - g(X, Z)Y]\]

The endomorphism \(X \wedge Y\) and \(X \wedge S\) \(Y\) and the homeomorphism \(R(X, \xi)\tilde{C}\) and \(\tilde{C}(X, \xi)R\) are defined by

\[(\text{2.7}) \quad (X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y\]

\[(\text{2.8}) \quad (X \wedge S)YZ = S(Y, Z)X - S(X, Z)Y\]

\[(\text{2.9}) \quad (R(X, \xi)\tilde{C})(U, Z)W = R(X, \xi)\tilde{C}(U, Z)W - \tilde{C}(R(X, \xi)U, Z)W - \tilde{C}(U, R(X, \xi)Z)W - \tilde{C}(U, Z)R(X, \xi)W\]

\[(\text{2.10}) \quad (\tilde{C}(X, \xi)R)(U, Z)W = \tilde{C}(X, \xi)R(U, Z)W - R(\tilde{C}(X, \xi)U, Z)W - R(U, Z(X, \xi)\tilde{C})W - R(U, Z)\tilde{C}(X, \xi)W\]

respectively, where \(X, Y, Z\) are vector fields of \(M\).
3. Main Results:

**Theorem -1**
An n-dimensional p-Sasakian Einstein manifold M with Quasi-conformal curvature tensor $\tilde{C}$, satisfying $r= -n(n -1)$, $a +2(n -1)b \neq 0$ then we have

$$(R(X, \xi)\tilde{C})= \tilde{C}(X, \xi) R,$$

Proof: Substituting $U$ and $W$ by $\xi$ in (2.9) yields

$$(2.11) \quad (R(X, \xi)\tilde{C})(\xi, Z) \xi = R(X, \xi)\tilde{C}(\xi, Z) \xi - (R(X, \xi)\xi, Z) \xi - \tilde{C}(\xi, R(X, \xi)Z) \xi - \tilde{C}(\xi, Z)R(X, \xi)\xi$$

From (2.6) we get by virtue of (1.2) and (1.12),

$$(2.12) \quad (\xi, Z) \xi = (a +2(n -1)b) \left[1+ \frac{r}{n(n-1)} \right] [Y - \eta(Y) \xi]$$

If $r = -n(n -1)$, provided $a +2(n -1)b \neq 0$ then from (2.12), we have (2.13)

$$(\xi, Z) \xi = 0 \quad \text{and similarly}$$

$$(2.14) \quad \tilde{C}(Z, \xi)\xi = 0, \quad \text{for any vector field Z}.$$

Thus we have,

$$(2.15) \quad (R(X, \xi)\tilde{C})(\xi, Z) \xi = - (R(X, \xi)\xi, Z) \xi - \tilde{C}(\xi, Z)R(X, \xi)\xi$$

Using (1.12), we have

$$\tilde{C}(R(X, \xi)\xi, Z) \xi = - \tilde{C}(X, Z) \xi$$

Thus we have from (2.15)

$$(2.16) \quad (R(X, \xi)\tilde{C})(\xi, Z) \xi = \tilde{C}(X, Z) \xi + \tilde{C}(\xi, Z) X$$

On the other hand

$$(2.17) \quad (\tilde{C}(X, \xi) R)(\xi, Z) \xi = \tilde{C}(X, \xi) R(\xi, Z) \xi - R(\tilde{C}(X, \xi)\xi, Z) \xi - R(\xi, Z(\tilde{C}(X, \xi)\xi) - R(\xi, Z)\tilde{C}(X, \xi)\xi$$

Using (1.12, (1.15) and (2.14), we obtain the following equations

$$\tilde{C}(X, \xi) R(\xi, Z) \xi = \tilde{C}(X, \xi) Z$$

$$(2.18) \quad (\tilde{C}(X, \xi) R)(\xi, Z) \xi = 0$$

Thus our condition satisfies the following equation

$$(R(X, \xi)\tilde{C})(\xi, Z) \xi = 0$$

Therefore from (2.16), we have

$$\tilde{C}(X, Z) \xi + \tilde{C}(\xi, Z) X = 0$$
Using (1.2), (1.11), (1.12) and (2.6), we have
\[(a + 2(n - 1)b) [1 + \frac{r}{n(n - 1)}] \|2\eta(X)Z - \eta(Z)X - g(X, Z)\xi] = 0\]

Which true for \( r = -n(n - 1), a + 2(n - 1)b \neq 0. \)

Hence the theorem is proved.

**Theorem 2**: An n-dimensional p-Sasakian Einstein manifold M with Quasi-conformal curvature tensor \( \tilde{C} \), satisfying \( r = -n(n - 1), a + 2(n - 1)b \neq 0 \) then we have
\[ R(X, \xi). \tilde{C} = L((X \cdot \xi).C), L \neq -1, \text{ where } L \text{ is some function on } M. \]

**Proof**: We denote the expression in the bracket on the right hand side of (2.9) by A, and we calculate it. Thus
\[(2.19) \quad A = L((X \cdot \xi)\tilde{C})(\xi, Z)\xi = L\{((X \cdot \xi)\tilde{C})(\xi, Z)\xi - \tilde{C}((X \cdot \xi)\tilde{C})(\xi, Z)\xi - \tilde{C}(\xi, (X \cdot \xi)Z)\xi - \tilde{C}(\xi, Z)(X \cdot \xi)\xi\}
\]

Using (2.13), we have
\[(X \cdot \xi)\tilde{C}(\xi, Z)\xi = 0\]
\[\tilde{C}((X \cdot \xi)\tilde{C})(\xi, Z)\xi = \tilde{C}(X - \eta(X)\xi, Z)\xi = \tilde{C}(X, Z)\xi\]
\[\tilde{C}(\xi, (X \cdot \xi)Z)\xi = 0\]
\[\tilde{C}(\xi, Z)(X \cdot \xi)\xi = \tilde{C}(\xi, Z)X\]

From the above and using (2.16), we have
\[\tilde{C}(X, Z)\xi + \tilde{C}(\xi, Z)X = L\{-\tilde{C}(X, Z)\xi - \tilde{C}(\xi, Z)X\}\]
\[(2.20) \quad (1 + L)[\tilde{C}(X, Z)\xi + \tilde{C}(\xi, Z)X] = 0\]

Using (1.2), (1.11), (1.12) and (2.6), we have
\[(2.21) \quad (1 + L)(a + 2(n - 1)b)[1 + \frac{r}{n(n - 1)}]\|2\eta(X)Z - \eta(Z)X - g(X, Z)\xi] = 0\]

Since \( L \neq -1. \)

Thus which true for \( r = -n(n - 1), a + 2(n - 1)b \neq 0. \)

Hence the theorem is proved.

**Theorem 3**: An n-dimensional p-Sasakian Einstein manifold M with Quasi-conformal curvature tensor \( \tilde{C} \), Satisfying \( r = -n(n - 1), a + 2(n - 1)b \neq 0 \) then we have
\[ R(X, \xi). \tilde{C} = f((X \cdot \xi).C), f = -1 \quad (1 - n)^2, \text{ where } f \text{ is some function on } M. \]

**Proof**: We denote the expression in the bracket on the right hand side of (2.9) by A, and we calculate it. Thus
\[(2.22) \quad (R(X, \xi). \tilde{C})(\xi, Z)\xi = f([((X \cdot \xi).C)(\xi, Z)\xi]\]

Where \( (X \cdot \xi)\xi = S^n(Y, Z)X - S^n(X, Z)Y, \text{ And } S^n(Y, Z) = g(Q^nX, Y) \)
Then
\[ A = f (((X \wedge^n \xi) \hat{C})(\xi, Z) \xi) = f (((X \wedge^n \xi) \widetilde{C}(\xi, Z) \xi - \hat{C}(X \wedge^n \xi) \xi, Z) \xi \]
\[ - \widetilde{C}(\xi, (X \wedge^n \xi) Z) \xi - \widetilde{C}(\xi, Z)((X \wedge^n \xi) \xi) \]

Using (2.13), we have
\[ (X \wedge^n \xi) \hat{C}(\xi, Z) \xi = 0 \]
\[ \hat{C}( (X \wedge^n \xi) \xi, Z) \xi = [(1-n \gamma) \hat{C}(X, Z) \xi \]
\[ \hat{C}(\xi, (X \wedge^n \xi) Z) \xi = 0 \]
\[ \hat{C}(\xi, Z)((X \wedge^n \xi) \xi) \xi = [(1-n \gamma) \hat{C}(\xi, Z) X \]

From the above and using (2.16), we have
\[ \hat{C}(X, Z)\xi + \hat{C}(\xi, Z) X = f[-(1-n \gamma) \hat{C}(X, Z) \xi - [(1-n \gamma) \hat{C}(\xi, Z) X] \]
\[ = - f(1-n \gamma) \hat{C}(X, Z) \xi + \hat{C}(\xi, Z) X \]
\[ (1 + f[+(1-n \gamma)] \hat{C}(X, Z) \xi + \hat{C}(\xi, Z) X) = 0 \]

Using (2.21), we have
\[ (1 + f[+(1-n \gamma)](a + 2(n-1)b)[1 + \frac{\eta}{n(n-1)} ] [2\eta(X)Z - \eta(Z)X - g(X, Z) \xi] = 0 \]

Since \( f = \frac{-1}{(1 - n)^n} \).

Thus which true for \( r = -n(n - 1) \), \( a + 2(n - 1)b \neq 0 \).
Hence the theorem is proved.

4. A p-Sasakian Einstein manifold satisfying (div \( \hat{C} \)) (X, Y) Z = 0

We assume that
\[ (4.1) \quad \text{div} \hat{C} = 0 \]

Where ‘div’ denotes the divergence.

Now differentiating (2.5) covariantly with respect to \( U \), we get

\[ (4.2) \quad (D_u \hat{C})(X, Y) Z = a(D_u R)(X, Y) Z + b((D_u S)(Y, Z) X - \]
\[ (D_u S)(X, Z) Y - (n - 1) D_u [g(Y, Z) X] + \]
\[ (n - 1) D_u [g(X, Z) Y] \]
\[ - \frac{1}{n} (\frac{a}{n-1}) + 2b(D_u \rho)[g(Y, Z) X - g(X, Z) Y]. \]

contraction of (4.2) with respect to \( X \), we get
\[ (4.3) \quad (\text{div} \hat{C})(X, Y) Z = b(n - 1)(D_u S)(Y, Z) - \frac{n-1}{n} (\frac{a}{n-1}) + 2b(g(Y, Z))(U) \]

From (2.3), We have
\[ (4.4) \quad (D_u S)(Y, Z) = 0 \]

Using (4.1) and (4.4) in (4.3), we obtain
\[
\frac{n-1}{n} \left( \frac{a}{n-1} + 2b \right) g(Y, Z) (U_r) = 0
\]

Since \( g(Y, Z) \neq 0 \), then we have \( U_r = 0 \), \( a + 2(n - 1)b \neq 0 \).

Which gives \( r \) is covariant constant.

Again if \( r \) is covariant constant i.e. \( U_r = 0 \), then from (4.3) and (4.4), we obtain

\[
(\text{div} \, \tilde{\mathfrak{C}})(X, Y)Z = 0.
\]

Hence we can state the following theorem.

**Definition:** A manifold \( M_n \) is said to be Quasi-Conformally conservative if \( \text{div} \, \tilde{\mathfrak{C}} = 0 \) [8].

**Theorem 4:** A \( p \)-Sasaki Einstein manifold is Quasi-Conformally conservative if and only if the scalar \( r \) is covariant constant, \( a + 2(n - 1)b \neq 0 \).

**REFERENCES**