On the Parameters of 2- Class Hadamard Association Schemes

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Abstract- We have all possible parameter of 2-class association schemes having the property that suitable (1, -1)- linear combinations of their association matrices yield the blocks of a Hadamard matrix (H-matrix) of certain classical form of Paley and Williamson. Some 2-class association schemes with the above parameters are identified. The known Hadamard Coherent Configurations or 2-Class Association Schemes (CC’s or 2-AS’s) listed in Table 2 do not yield H-matrices of new order. However we have forwarded new methods of constructing H-matrices of the forms II and III. The developed technique gives several easy constructions of H-matrix from any 2-AS, whose parameters are given.

Keywords- Parameter of Hadamard Matrix, Coherent Configuration, Association scheme.

I. INTRODUCTION

An (n x n) matrix H with entries +1 and -1 is called a Hadamard matrix (or H-matrix), if \( H^T = n I_n \). If \( n = 4t \) and H-matrix of order \( n \) exists, then \( n = 4t \), where \( t \) is an integer. It is conjectured that H-matrix of order \( 4t \) exists for every \( t \geq 1 \). It remains unsettled in spite of various methods of constructions forwarded by different authors. For a brief surveys see Hall (1967), Hedayat et.al. (1978). However the conjecture is supported by the fact that for every order \( 4t \) \( (t > 3) \) investigated there are several inequivalent H-matrices of order \( 4t \) reported by Seberry (2001). We recall following definitions from Alejandro et al. (2003) and Raghavarao (1988).

1.1 Coherent configuration (CC):

Let \( X = \{1, 2, ..., n\} \) and \( P = \{R_0, R_1, ..., R_t\} \) be a set of binary relations on \( X \) satisfying the following four relations:

(a) \( P \) is a partition of \( X^2 \).

(b) there is a subset \( P_0 \) of \( P \) which is a partition of the diagonal, \( D = \{(\alpha, \alpha) : \alpha \in X\} \).

(c) for any relation \( R_i \in P_i \), its converse \( R_i^T \) (or \( R_i^{-1} \)) \( \in P \).

(d) for 0 \( \leq i, j, k \leq t \), there exists an integer \( p_{ij}^k \) such that \( (\alpha, \beta) \in R_k \) implies the order of the set \( \{ \gamma : (\alpha, \gamma) \in R_0 \) and \( (\gamma, \beta) \in R_j \} \) is \( p_{ij}^k \) which is independent of the choice of \( (\alpha, \beta) \in R_k \). \( p_{ij}^k \) are called intersection numbers or parameters of the CC.
Let $A=[a_{jk}]$ be the $(0, 1)$-matrix, is called adjacency matrix of the relation $R_i$, defined as:

$$a_{jk} = \begin{cases} 1, & \text{if } (j, k) \in R_i \\ 0, & \text{otherwise} \end{cases}$$

Clearly $A=\{A_0, A_1, \ldots A_t\}$ satisfies

$(c_1)$ $A_0 + A_1 + \ldots + A_t = \mathbf{I}_n$ (all $1$ matrix)

$(c_2)$ there is a subset of the set $A$, with sum $I_n$

$(c_3)$ $A_iA_j = \sum_{k=0}^{t} p_{ij}^k A_k \ldots$ (1)

A is called basis algebra or coherent algebra of the CC as the matrices belonging to $A$ form a basis of an associative algebra over the field of complex numbers. A CC is faithfully represented by basis matrices $A_i$ of its basis algebra. A CC $P=\{R_0, R_1, \ldots, R_t\}$ is called a $t$-class Association Scheme (t-AS) if it contains an identity relation $R_0$ (or $A_0=I$) and its relations $R_i$ are symmetric (or basis matrices $A_i$ of its coherent algebra are symmetric). Basis matrices of an Association Scheme (AS) are called association matrices. A 2-class association scheme is equivalent to a strongly regular graph.

If $p_{ij}^k$ are parameters of a 2-class Association Schemes, $p_{ii}^0=n_i$ is called the number of $i$th associates of a point and $n_1, n_2$ and $p_{ij}^k$ satisfy,

$$p_{ij}^k = p_{ji}^{k'}, p_{ij}^0 = \delta_{ij}, p_{ij}^0 = 0; \text{ when } i \neq j,$$

$$n_1 + n_2 = v-1 \text{ and } p_{ij}^1 + p_{j'i}^1 = n^2 - \delta_{ij}, \text{ i,j}=1,2 \ldots (2)$$

### 1.2 H-matrices from AS or CC:

A project of the first author is to obtain all ASs and CCs defined by minimum number of relations leading to the construction of an H-matrix of given form. It was motivated by the fact that the construction of H-matrices of Paley uses a family of CC’s and that of Williamson uses a family of AS’s.

Today we have several ASs and CCs used in Statistics and Coding theory but a few which are suitable for the construction of H-matrices. In view of the significance of such schemes, we forward the following definition:

**Hadamard CC or AS:** A CC or AS $\{A=(A_i)\}$ will be called Hadamard (or H-CC or H-AS related to an H-matrix of a given form) if suitable $(1, -1)$-linear combination (or combinations) of $(0,1)$-basis matrices $A_i$ of its coherent algebra yields a Hadamard matrix (or blocks of the Hadamard matrix).

The present paper is a part of the above project confined to Hadamard 2-AS’s. A result in this direction is due to Singh, et al. (2002) who forwarded a method of constructing H-matrices from underlying 2-ASs of a partial geometry. Here we obtain the parameters of four families of Hadamard 2-ASs. Finally we have identified and tabulated 2-ASs included in the families.

### II. PARAMETERS OF HADAMARD 2-ASS

We consider Hadamard matrices of the following form and order, whose dependence on the association matrices of Hadamard 2-ASs or basis matrices of H-CC’s (which are trivial extension of 2-ASs) are shown below.
### TABLE 1

<table>
<thead>
<tr>
<th>No.</th>
<th>Form of the H-matrix</th>
<th>Order of the H-matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$H = I - A_1 + A_2$, where $A_1, A_2$ are association matrices of a 2-AS on $v = 4t$ points.</td>
<td>$O(H) = v = 4t$</td>
</tr>
<tr>
<td>II</td>
<td>$H = \begin{bmatrix} 1 &amp; e \ e^t &amp; I-A_1+A_2 \end{bmatrix}$, where $A_1, A_2$ are association matrices of a 2-AS on $v = 4t$ points.</td>
<td>$O(H) = v + 1 = 4t$</td>
</tr>
<tr>
<td>III</td>
<td>$H = H \times I_v + K \times (A_1-A_2)$, where $K = V_n$, $V_n = \begin{bmatrix} 0 &amp; 1 \ -1 &amp; 0 \end{bmatrix}$, $h$ = order of an H-matrix $H$ and $A_1, A_2$ are association matrices of a 2-AS on $v = 2n$ points.</td>
<td>$O(H) = 2nh$</td>
</tr>
<tr>
<td>IV</td>
<td>$H = \begin{bmatrix} A &amp; B \ -B &amp; A \end{bmatrix}$, where $A = I - A_1 + A_2$, $B = I + A_1 - A_2$, and $A_1$ and $A_2$ are association matrices of a 2-AS on $2t$ points.</td>
<td>$O(H) = 4t$</td>
</tr>
</tbody>
</table>

1. A 2-AS of AS(1) has parameters

   (i) $v=4n^2$, $p_{11}^1 = p_{11}^2 = n(2n-1)$, $p_{12}^1 = p_{12}^2 = n(n-1)$.

   or (ii) $v=4n^2$, $n_1 = p_{11}^2 = n(2n+1)$, $p_{11}^1 = p_{12}^2 = n(n+1)$.

2. A 2-AS of AS(2) has the parameters $v = 4n^2 - 1$, $n_1 = 2n^2$, $p_{12}^1 = n^2 - 1$, $p_{12}^2 = n^2$, and contains the 2-AS of pg $(n,2n+1,n)$.

3. A 2-AS of AS(3) has parameters $v = (2n-1)^2 + 1$, $n_1 = p_{11}^2 = m(2n-1)$, $p_{12}^1 = p_{12}^2 = n^2 - n$.

   where $n \geq 2$, $\left\lfloor \frac{n}{2} \right\rfloor \leq m \leq n-1$ and contains the AS of pg$(n,2n,n)$.

4. AS(4) = AS(3).

A common property shared by all the Hadamard 2-ASs belonging to the above four families is that the parameters $p_{12}^1$ and $p_{12}^2$ are as equal as possible i.e. $|p_{12}^1 - p_{12}^2| = 0$ or 1.

### III. TABLE OF KNOWN HADAMARD 2-ASs

<table>
<thead>
<tr>
<th>Hadamard 2-AS</th>
<th>Family AS (i) and form of the H-matrix</th>
<th>Source of the Hadamard 2-AS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(v, p_{11}^0, p_{11}^1, p_{12}^2)$</td>
<td>AS $(4n^2,n)$</td>
<td>Infinitely many 2-ASs obtained from Bush type</td>
</tr>
</tbody>
</table>

**Theorem:** Let AS(i), $i = 1, 2, 3, 4$ be four families of the association schemes required by the forms I, II, III, IV respectively in Table 1 as per Singh et al. (2009). Then for some values of $n$ and $m$,
Where (a) \( n = 2r, 4r \) being the order of an \( H \)-matrix.

(b) \( n = r^2 \), where \( r \) is odd.

(c) \( n = 3 \) \( \lambda = 1 \) for \( n = 3, \ldots , 9 \) vide Hall no. 14, 32, 51, 77, 111, 145, 174 respectively by Hall (1967). Also see Mathon et al. (1992).

AS(1) & H-matrices of order \( 16r^2 \), see Bonato, et al. (2001).
(a) for \( n = 2 \) & (2) CC derived from AS of \( pg(n, 2n + 1, n) \) \( n = 3, 4, \ldots , 9 \).
(b) for \( n = 3 \) & AS(2) \( n = 3, 4, \ldots , 9 \).
(c) \( n = 4 \) & AS(3) Dual of BIBD \( (2n^2-2n+1, k=n, \lambda = 1) \) with Hall (1967) No.9, 22, 42 for \( n = 3, 4, 5 \) respectively. Also see Mathon et al. (1992).

(2n-1), n(n-1) & 2-ASs of Clatworthy’s (1952) Nos. pg9(n=3) and pg13(n=4),

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IV. CONCLUSIONS

Hall (1967); Clatworthy (1973); Spence (1995); Brouwer (1996); Spence; Hanaki et al. reported that \( H \)-matrices of new order will be obtained when 2-ASs of corresponding parameters are known. The identifications of the following 2-ASs as Hadamard ones appear to be new:

(i) Clatworthy’s AS of Misc. PBIBD \#M_1, M_{11a}, 2-ASs of Clatworthy’s Nos. pg9(n=3) and pg13(n=4),

(ii) Hanaki’s AS16 #4, 5 and 6,
(iii) 180 2-ASs with parameters (36, 14, 4, 6) reported by Spence and AS (64, 36, 22, 22) by Brouwer.

(iv) dual of BIBDs with \( v = 2n^2 - n, k = n, \lambda = 1 \) for \( n = 3 \), . \( . , 9 \) vide Hall no.14, 32, 51, 77, 111, 145, 174 respectively,

(v) dual of BIBD \( (2n^2 - 2n + 1, k = n, \lambda = 1) \) with Hall No.9, 22, 42 for \( n = 3, 4, 5 \) respectively.

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