Connected Total Dominating Sets and Connected Total Domination Polynomials of Square of Paths

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ABSTRACT

Let G be a simple connected graph of order n. Let $D_{ct}(G, i)$ be the family of connected total dominating sets in G with cardinality i. The polynomial $D_{ct}(G, x) = \sum_{i=\gamma_{ct}(G)}^{n} d_{ct}(G, i)\, x^i$ is called the connected total domination polynomial of G. In this paper, we obtain a recursive formula for $d_{ct}(P_n^2, i)$. Using this recursive formula, we construct the connected total domination polynomial $D_{ct}(P_n^2, x) = \sum_{i=\left\lceil \frac{n-3}{2} \right\rceil}^{n} d_{ct}(P_n^2, i)\, x^i$, where $d_{ct}(P_n^2, i)$ is the number of connected total dominating sets of P_n^2 of cardinality i and some properties of this polynomial have been studied.

Keywords: Square of Path, connected total dominating set, connected total domination number, connected total domination polynomial.

1. Introduction

Let G = (V, E) be a simple graph of order n. For any vertex $v \in V$, the open neighbourhood of v is the set $N(v) = \{u \in V / uv \in E\}$ and the closed neighbourhood of v is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighbourhood of S is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighbourhood of S is $N[S] = N(S) \cup S$. The maximum degree of the graph G is denoted by $\Delta(G)$ and the minimum degree is denoted by $\delta(G)$.

A set S of vertices in a graph G is said to be a total dominating set if every vertex $v \in V$ is adjacent to an element of S.

A total dominating set S of G is called a connected total dominating set if the induced subgraph $\langle S \rangle$ is connected.

The minimum cardinality of a connected total dominating set of G is called the connected total domination number of G and is denoted by $\gamma_{ct}(G)$.

A connected total dominating set with carnality $\gamma_{ct}(G)$ is called γ_{ct} - set. We use the notation $\lceil x \rceil$ for the smallest integer greater than or equal to *x* and $\lfloor x \rfloor$ for the largest integer less than or equal to *x*. Also, we denote the set {1, 2, ..., n} by [n], throughout this paper.

2. CONNECTED TOTAL DOMINATING SETS OF SQUARE OF PATHS

In this section, we state the connected total domination number of the square of path and some of its properties.

Definition 2.1

Let G be a graph of order n with no isolated vertices. Let $D_{ct}(G, i)$ be the family of connected total dominating sets of G with cardinality i and let $d_{ct}(G,i)=|D_{ct}(G,i)|$. Then the connected total domination polynomial $D_{ct}(G,x)$ of G is defined as $D_{ct}(G,x) = \sum_{i=\gamma_{ct}(G)}^{n} d_{ct}(G,i)\, x^i$.

Lemma 2.2

Let P_n^2 be the square of the path P_n with n vertices, then its connected total domination number is $\gamma_{ct}(P_n^2) = \left\lceil \frac{n-3}{2} \right\rceil$.

Lemma 2.3

Let P_n^2, $n \geq 6$ be the square of path with $|V(P_n^2)| = n$. Then $d_{ct}(P_n^2, i) = 0$ if $i < \left\lceil \frac{n-3}{2} \right\rceil$ or $i > n$ and $d_{ct}(P_n^2, i) > 0$ if $\left\lceil \frac{n-3}{2} \right\rceil \leq i \leq n$.

Proof

If $i < \left\lceil \frac{n-3}{2} \right\rceil$ or $i > n$, then there is no connected total dominating set of cardinality i. Therefore, $d_{ct}(P_n^2, i) = \phi$.

By Lemma 2.2, the cardinality of the minimum connected total dominating set is $\left\lceil \frac{n-3}{2} \right\rceil$. Therefore, $d_{ct}(P_n^2, i) > 0$ if $i \geq \left\lceil \frac{n-3}{2} \right\rceil$ and $i \leq n$. Hence, we have $d_{ct}(P_n^2, i) = 0$ if $i < \left\lceil \frac{n-3}{2} \right\rceil$ or $i > n$ and $d_{ct}(P_n^2, i) > 0$ if $\left\lceil \frac{n-3}{2} \right\rceil \leq i \leq n$.

Lemma 2.4

Let P_n^2, $n \geq 6$ be the square of path with $|V(P_n^2)| = n$. Then

1. $D_{ct}(P_n^2, i) = \phi$ if $i < \gamma_{ct}(P_n^2)$ or $i > n$.
2. $D_{ct}(P_n^2, x)$ has no constant term and first degree terms.
3. $D_{ct}(P_n^2, x)$ is a strictly increasing function on $[0,\infty)$.

Lemma 2.5

Let P_n^2, $n \geq 6$ be the square of path with $|V(P_n^2)| = n$.

1. If $D_{ct}(P_{n-1}^2, i-1) = \phi$ and $D_{ct}(P_{n-3}^2, i-1) = \phi$, then $D_{ct}(P_{n-2}^2, i-1) = \phi$.
2. If $D_{ct}(P_{n-1}^2, i-1) \neq \phi$ and $D_{ct}(P_{n-3}^2, i-1) \neq \phi$, then $D_{ct}(P_{n-2}^2, i-1) \neq \phi$.
3. If $D_{ct}(P_{n-1}^2, i-1) = \phi$, $D_{ct}(P_{n-2}^2, i-1) = \phi$ and $D_{ct}(P_{n-3}^2, i-1) = \phi$, then $D_{ct}(P_n^2, i) = \phi$.

Proof

1. Since $D_{ct}(P_{n-1}^2, i-1) = \phi$ and $D_{ct}(P_{n-3}^2, i-1) = \phi$, by Lemma 2.3, $i-1 > n-1$ or $i-1 < \left\lceil \frac{n-4}{2} \right\rceil$ and $i-1 > n-3$ or $i-1 < \left\lceil \frac{n-6}{2} \right\rceil$. Therefore, $i-1 > n-1$ or $i-1 < \left\lceil \frac{n-6}{2} \right\rceil$. Hence $i-1 > n-2$ or $i-1 < \left\lceil \frac{n-5}{2} \right\rceil$ holds.

   Therefore, $D_{ct}(P_{n-2}^2, i-1) = \phi$.
2. Suppose that $D_{ct}(P_{n-2}^2, i-1) = \phi$, so by Lemma 2.3, we have $i-1 > n-2$ or $i-1 < \left\lceil \frac{n-5}{2} \right\rceil$. If $i-1 > n-2$, then $i-1 > n-3$. Therefore, $D_{ct}(P_{n-3}^2, i-1) = \phi$, a contradiction. If, $i-1 < \left\lceil \frac{n-5}{2} \right\rceil$, then $i-1 < \left\lceil \frac{n-4}{2} \right\rceil$.

   Therefore, $D_{ct}(P_{n-1}^2, i-1) = \phi$, a contradiction.
   Therefore our assumption is wrong.
   Therefore, $D_{ct}(P_{n-2}^2, i-1) \neq \phi$.
3. By hypothesis, $i-1 < \left\lceil \frac{n-4}{2} \right\rceil$ or $i-1 > n-1$ and $i-1 < \left\lceil \frac{n-5}{2} \right\rceil$ or $i-1 > n-2$ and $i-1 < \left\lceil \frac{n-6}{2} \right\rceil$ or $i-1 > n-3$. Therefore, $i-1 < \left\lceil \frac{n-6}{2} \right\rceil$ or $i-1 > n-1$. Therefore, $i < \left\lceil \frac{n-3}{2} \right\rceil$ or $i > n$.

   Therefore $D_{ct}(P_n^2, i) = \phi$.

Lemma 2.6

Let P_n^2, $n \geq 6$ be the square of path with $|V(P_n^2)| = n$. Suppose that $D_{ct}(P_n^2, i) \neq \phi$, then we have

1. $D_{ct}(P_n^2, i-1) = \phi$ and $D_{ct}(P_{n-1}^2, i-1) = \phi$ if and only if $n = 2k+1$ and $i = k-1$ for some $k \geq 4$.
2. $D_{ct}(P_{n-2}^2, i-1) = \phi$, $D_{ct}(P_{n-3}^2, i-1) = \phi$ and $D_{ct}(P_{n-1}^2, i-1) \neq \phi$ if and only if $i = n$.
3. $D_{ct}(P_{n-1}^2, i-1) \neq \phi$, $D_{ct}(P_{n-2}^2, i-1) \neq \phi$ and $D_{ct}(P_{n-3}^2, i-1) = \phi$ if and only if $i = n-1$.

4. $D_{ct}(P^2_{n-1}, i-1) = \phi$, $D_{ct}(P^2_{n-2}, i-1) \neq \phi$ and $D_{ct}(P^2_{n-3}, i-1) \neq \phi$ if and only if $n = 2k+1$ and $i = k-1$ for some $k \geq 4$.
5. $D_{ct}(P^2_{n-1}, i-1) \neq \phi$, $D_{ct}(P^2_{n-2}, i-1) \neq \phi$ and $D_{ct}(P^2_{n-3}, i-1) \neq \phi$ if and only if
$$\left\lceil \frac{n-4}{2} \right\rceil + 1 \leq i \leq n-2.$$

Proof

1. Assume $D_{ct}(P^2_n, i-1) = \phi$ and $D_{ct}(P^2_{n-1}, i-1) = \phi$.

Then by Lemma 2.3, $i-1 > n$ or $i-1 < \left\lceil \frac{n-3}{2} \right\rceil$ and $i-1 > n-1$ or $i-1 < \left\lceil \frac{n-4}{2} \right\rceil$. Suppose $i-1 > n-1$, then $i > n$.

Therefore, $D_{ct}(P^2_n, i) = \phi$, which is a contradiction.

So, $i < \left\lceil \frac{n-4}{2} \right\rceil + 1$ and since $D_{ct}(P^2_n, i) \neq \phi$, we have $\left\lceil \frac{n-3}{2} \right\rceil \leq i < \left\lceil \frac{n-4}{2} \right\rceil + 1$ ......(1)

If $n \neq 2k+1$, then we obtain an inequality of the form $s \leq i < s$, which is not possible. When $n = 2k+1$, (1) holds and in this case $i = k-1$.

Conversely, assume $n = 2k+1$ and $i = k-1$. Therefore $n-1 = 2k$ and $i-1 = k-2$.

$$k-2 < k-1 = \frac{n-3}{2}.$$

Therefore, $k-2 < \left\lceil \frac{n-3}{2} \right\rceil$. That is $i-1 < \left\lceil \frac{n-3}{2} \right\rceil$.

Therefore, $D_{ct}(P^2_n, i-1) = \phi$.

$$\text{Also, } i-1 = k-2 = \frac{n-1}{2} - 2 = \left\lceil \frac{n-5}{2} \right\rceil < \left\lceil \frac{n-4}{2} \right\rceil$$

Therefore, $D_{ct}(P^2_{n-1}, i-1) = \phi$.

2. Assume $D_{ct}(P^2_{n-2}, i-1) = \phi$, and $D_{ct}(P^2_{n-3}, i-1) = \phi$.

Then by Lemma 2.3, we have, $i-1 > n-2$ or $i-1 < \left\lceil \frac{n-5}{2} \right\rceil$ and $i-1 > n-3$ or $i-1 < \left\lceil \frac{n-6}{2} \right\rceil$

If $i-1 < \left\lceil \frac{n-5}{2} \right\rceil$, then $i-1 < \left\lceil \frac{n-4}{2} \right\rceil$.

Therefore by Lemma 2.3, $D_{ct}(P^2_{n-1}, i-1) = \phi$, which is a contradiction.

So we have $i-1 > n-2$ that is $i > n-1$. Therefore $i \geq n$.

Also, since $D_{ct}(P^2_n, i) \neq \phi$, $i \leq n$. Combining these we get $i = n$.

Conversely if $i = n$,

$D_{ct}(P^2_{n-2}, i-1) = D_{ct}(P^2_{n-2}, n-1) = \phi$,

$D_{ct}(P^2_{n-3}, i-1) = D_{ct}(P^2_{n-3}, n-1) = \phi$ and

$D_{ct}(P^2_{n-1}, i-1) = D_{ct}(P^2_{n-1}, n-1) \neq \phi$.

3. Assume $D_{ct}(P^2_{n-1}, i-1) \neq \phi$, $D_{ct}(P^2_{n-2}, i-1) \neq \phi$ and $D_{ct}(P^2_{n-3}, i-1) = \phi$.

Since $D_{ct}(P^2_{n-3}, i-1) = \phi$, then by Lemma 2.3,

$$i-1 > n-3 \text{ or } i-1 < \left\lceil \frac{n-6}{2} \right\rceil \quad \text{.......(1)}$$

Since $D_{ct}(P^2_{n-1}, i-1) \neq \phi$, we have ,

$$\left\lceil \frac{n-4}{2} \right\rceil \leq i-1 \leq n-2. \quad \text{.......(2)}$$

Suppose $i-1 < \left\lceil \frac{n-6}{2} \right\rceil$, then (2) does not hold.

Therefore our assumption is wrong.

Therefore $i-1 > n-3$.

Also, since $D_{ct}(P^2_{n-2}, i-1) \neq \phi$,

$$\left\lceil \frac{n-5}{2} \right\rceil \leq i-1 \leq n-2. \quad \text{.......(3)}$$

But $i-1 > n-3$.

Therefore $i-1 \geq n-2$. ....(4)

From (3) and (4) we get

$i-1 = n-2$.

Therefore $i = n-1$.

Conversely, suppose $i = n-1$.

Then $D_{ct}(P^2_{n-1}, i-1) = D_{ct}(P^2_{n-1}, n-2) \neq \phi$,

$D_{ct}(P^2_{n-2}, i-1) = D_{ct}(P^2_{n-2}, n-2) \neq \phi$ and

$D_{ct}(P^2_{n-3}, i-1) = D_{ct}(P^2_{n-3}, n-2) = \phi$,

since $n-2 > n-3$.

That is $D_{ct}(P^2_{n-3}, i-1) = \phi$.

4. Assume $D_{ct}(P^2_{n-1}, i-1) = \phi$, $D_{ct}(P^2_{n-2}, i-1) \neq \phi$, and $D_{ct}(P^2_{n-3}, i-1) \neq \phi$.

Since, $D_{ct}(P^2_{n-1}, i-1) = \phi$, by Lemma 2.3, $i-1 > n-1$ or $i-1 < \left\lceil \frac{n-4}{2} \right\rceil$.

If $i-1 > n-1$, then $i-1 > n-2$ and $i-1 > n-3$. Therefore $D_{ct}(P^2_{n-2}, i-1) = \phi$, and $D_{ct}(P^2_{n-3}, i-1) = \phi$, which is a contradiction.

$$\text{Therefore } i-1 < \left\lceil \frac{n-4}{2} \right\rceil \qquad \text{........(1)}$$

Since $D_{ct}(P^2_{n-2}, i-1) \neq \phi$, we have $\left\lceil \frac{n-5}{2} \right\rceil \leq i-1 \leq n-2$ ....(2)

and since $D_{ct}(P^2_{n-3}, i-1) \neq \phi$, we have

$$\left\lceil \frac{n-6}{2} \right\rceil \leq i-1 \leq n-3. \qquad \text{........(3)}$$

Since $D_{ct}(P^2_n, i) \neq \phi$, we have $\left\lceil \frac{n-3}{2} \right\rceil \leq i \leq n-1$.

$$\text{Therefore, } \left\lceil \frac{n-3}{2} \right\rceil -1 \leq i-1 \leq n-2 \qquad \text{.......(4)}$$

By combining all the above inequalities, we have,

$$\left\lceil \frac{n-3}{2} \right\rceil -1 \leq i-1 < \left\lceil \frac{n-4}{2} \right\rceil. \qquad \text{.......(5)}$$

When $n \neq 2k+1$, we get an inequality of the form $s \leq i-1 < s$, which is not possible. When $n = 2k+1$, we have $s \leq i-1 < s+1$. Therefore (5) holds hood. In this case $i = k-1$.

Conversely, assume $n = 2k+1$ and $i = k-1$. Therefore, $n-1 = 2k$ and $i-1 = k-2$. Therefore

$$k = \frac{n-1}{2} \text{ and } k-1 = \frac{n-3}{2} \quad i-1 = k-2$$
$$= \frac{n-1}{2} - 2$$
$$= \frac{n-5}{2} < \left\lceil \frac{n-4}{2} \right\rceil$$

Therefore, $D_{ct}(P^2_{n-1}, i-1) = \phi$.

Also, $D_{ct}(P^2_{n-2}, i-1) = D_{ct}(P^2_{2k-1}, k-2) \neq \phi$, since $\left\lceil \frac{2k-1-3}{2} \right\rceil = \left\lceil \frac{2k-4}{2} \right\rceil = k-2$ and $D_{ct}(P^2_{n-3}, i-1) = D_{ct}(P^2_{2k-2}, k-2) \neq \phi$, since $\left\lceil \frac{2k-2-3}{2} \right\rceil = \left\lceil \frac{2k-5}{2} \right\rceil = k-2$.

5) Assume $D_{ct}(P^2_{n-1}, i-1) \neq \phi$, $D_{ct}(P^2_{n-2}, i-1) \neq \phi$, and $D_{ct}(P^2_{n-3}, i-1) \neq \phi$.

Then by Lemma 2.3, we have $\left\lceil \frac{n-4}{2} \right\rceil \leq i-1 \leq n-1$, $\left\lceil \frac{n-5}{2} \right\rceil \leq i-1 \leq n-2$, and $\left\lceil \frac{n-6}{2} \right\rceil \leq i-1 \leq n-3$.

Also, since $D_{ct}(P^2_n, i) \neq \phi$, we have $\left\lceil \frac{n-3}{2} \right\rceil \leq i \leq n$.

Therefore, $\left\lceil \frac{n-3}{2} \right\rceil -1 \leq i-1 \leq n-1$.

Therefore, $\left\lceil \frac{n-4}{2} \right\rceil +1 \leq i \leq n-2$.

Conversely, suppose $\left\lceil \frac{n-4}{2} \right\rceil +1 \leq i \leq n-2$.

Therefore, $\left\lceil \frac{n-4}{2} \right\rceil \leq i-1 \leq n-3$

and $\left\lceil \frac{n-5}{2} \right\rceil \leq i-1 \leq n-2$, $\left\lceil \frac{n-6}{2} \right\rceil \leq i-1 \leq n-3$ and $\left\lceil \frac{n-4}{2} \right\rceil \leq i-1 \leq n-1$.

From these, we obtain $D_{ct}(P^2_{n-1}, i-1) \neq \phi$, $D_{ct}(P^2_{n-2}, i-1) \neq \phi$ and $D_{ct}(P^2_{n-3}, i-1) \neq \phi$.

Theorem 2.7

For every $n \geq 6$ and $i > \left\lceil \frac{n-3}{2} \right\rceil$,

1. $D_{ct}(P^2_{2n+1}, n-1) = \{\{3, 5, 7, 9, \ldots\ldots\}\}$
2. If $D_{ct}(P^2_{n-2}, i-1) = \phi$, $D_{ct}(P^2_{n-3}, i-1) = \phi$ and $D_{ct}(P^2_{n-1}, i-1) \neq \phi$, then $D_{ct}(P^2_n, i) = D_{ct}(P^2_n, n) = \{\{1,2,3,\ldots\ldots, n\}\}$.
3. If $D_{ct}(P^2_{n-1}, i-1) \neq \phi$, $D_{ct}(P^2_{n-2}, i-1) \neq \phi$ and $D_{ct}(P^2_{n-3}, i-1) = \phi$ then $D_{ct}(P^2_n, n-1) = \{[n] - \{x\} \mid x \in [n]\}$

4. If $D_{ct}(P^2_{n-1}, i-1) \neq \phi$, $D_{ct}(P^2_{n-2}, i-1) \neq \phi$ then

$$D_{ct}(P^2_n, i) = \{\{X \cup \{n\} \text{ if } n-1 \in X\} \cup \{X \cup \{n-1\} \text{ if } n-2 \in X\} \cup \{X \cup \{n-2\} \text{ if } n-3 \in X\} \cup \{Y \cup \{n\} \text{ if } n-2 \in Y\} \cup Y \cup \{n-1) \text{ if } n-3 \in Y\}\},$$

where $X \in D_{ct}(P^2_{n-1}, i-1)$ and $Y \in D_{ct}(P^2_{n-2}, i-1)$.

Proof

1. For every $n \geq 6$,
   $D_{ct}(P^2_{2n+1}, n-1) = \{\{3,5,7,9, \ldots, (2n+1)-6, (2n+1)-4, (2n+1)-2\}\}$
2. Since $D_{ct}(P^2_{n-2}, i-1) = \phi$, $D_{ct}(P^2_{n-3}, i-1) = \phi$ and $D_{ct}(P^2_{n-1}, i-1) \neq \phi$, by Lemma 2.6 (2), $i = n$.
   Therefore, $D_{ct}(P^2_n, i) = D_{ct}(P^2_n, n) = \{[n]\}$.
3. If $D_{ct}(P^2_{n-1}, i-1) \neq \phi$, $D_{ct}(P^2_{n-2}, i-1) \neq \phi$ and $D_{ct}(P^2_{n-3}, i-1) = \phi$, then by Lemma 2.5, $i = n-1$.
   Therefore, $D_{ct}(P^2_n, i) = D_{ct}(P^2_n, n-1) = \{[n] - \{x\} | x \in [n]\}$.
4. The Construction of $D_{ct}(P^2_n, i)$ from $D_{ct}(P^2_{n-1}, i-1)$ and $D_t(P^2_{n-2}, i-1)$ is as follows:

Let X be a connected total dominating set of P^2_{n-1} with cardinality i–1. All the elements of $D_{ct}(P^2_{n-1}, i-1)$ end with n–1 or n–2 or n–3. Therefore, when $n-1 \in X$, adjoin n with X and when $n-2 \in X$ adjoin n–1 with X and when $n-3 \in X$ adjoin n–2 with X. Hence every X of $D_{ct}(P^2_{n-1}, i-1)$ belongs to $D_{ct}(P^2_n, i)$ by adjoining {n} or {n–1} or {n–2} only.

Now let us consider $D_{ct}(P^2_{n-2}, i-1)$. Here all the elements of $D_{ct}(P^2_{n-2}, i-1)$ end with n–2 or n–3. Let Y be the connected total dominating set of P^2_{n-2} with cardinality i–1. Therefore, when $n-2 \in Y$, adjoin n with Y and when $n-3 \in Y$, adjoin n–1 with Y. Hence, every Y of $D_{ct}(P^2_{n-2}, i-1)$ belongs to $D_{ct}(P^2_n, i)$ by adjoining {n} or {n-1} only. Hence, we cover all the elements of $D_{ct}(P^2_n, i)$ by means of the elements of $D_{ct}(P^2_{n-1}, i-1)$ and $D_{ct}(P^2_{n-2}, i-1)$.

Conversely, suppose $Z \in D_{ct}(P^2_n, i)$. Here all the elements of $D_{ct}(P^2_n, i)$ end with n or n – 1 or n – 2.

Suppose $n \in Z$, then $Z = X \cup \{n\}$, where $X \in D_{ct}(P^2_{n-1}, i-1)$ and X ends with n – 1 or $Z = Y \cup \{n\}$, where $Y \in D_{ct}(P^2_{n-2}, i-1)$ and Y ends with n – 2.

Suppose $n-1 \in Z$, then $Z = X \cup \{n-1\}$, where $X \in D_{ct}(P^2_{n-1}, i-1)$ and X ends with n – 2 or $Z = Y \cup \{n-1\}$, where $Y \in D_{ct}(P^2_{n-2}, i-1)$ and Y ends with n – 3.

Suppose $n-2 \in Z$, then $Z = X \cup \{n-2\}$, where $X \in D_{ct}(P^2_{n-1}, i-1)$ and X ends with n – 3.

Theorem 2.8

If $D_{ct}(P^2_n, i)$ is the family of the connected total dominating sets of P^2_n with cardinality i, where $i \geq \left\lceil \frac{n-3}{2} \right\rceil$, then

$$d_{ct}(P^2_n, i) = d_{ct}(P^2_{n-1}, i-1) + d_{ct}(P^2_{n-2}, i-1)$$

Proof

From theorem 2.7, we consider all the three cases as given below, where $i \geq \left\lceil \frac{n-3}{2} \right\rceil$.

i) If $D_{ct}(P^2_{n-1}, i-1) = \phi$ and $D_{ct}(P^2_{n-2}, i-1) = \phi$, then $D_{ct}(P^2_n, i) = \phi$.

ii) If $D_{ct}(P^2_{n-1}, i-1) \neq \phi$ and $D_{ct}(P^2_{n-2}, i-1) = \phi$, then $D_{ct}(P^2_n, i) = \{\{n\} \cup X | X \in D_{ct}(P^2_{n-1}, i-1)\}$

iii) If $D_{ct}(P^2_{n-1}, i-1) \neq \phi$ and $D_{ct}(P^2_{n-2}, i-1) \neq \phi$, then

$$D_{ct}(P^2_n, i) = \begin{cases} \{X \cup \{n\} & \text{if } n-1 \in X\} \cup \\ \{X \cup \{n-1\} & \text{if } n-2 \in X\} \cup \\ \{X \cup \{n-2\} & \text{if } n-3 \in X\} \cup \\ \{Y \cup \{n\} & \text{if } n-2 \in Y\} \cup \\ \{Y \cup \{n-1\} & \text{if } n-3 \in Y\} \cup \end{cases}$$

Where $X \in D_{ct}(P^2_{n-1}, i-1)$ and $Y \in D_{ct}(P^2_{n-2}, i-1)$.

From the above construction in each case, we obtain that, $d_{ct}(P^2_n, i) = d_{ct}(P^2_{n-1}, i-1) + d_{ct}(P^2_{n-2}, i-1)$

3. Connected Total Domination Polynomial of square of paths

Definition 3.1

Let $D_{ct}(P^2_n, i)$ be the family of connected total dominating sets of P^2_n with cardinality i and let $d_{ct}(P^2_n, i) = |D_{ct}(P^2_n, i)|$. Then the connected total domination Ploynomial $D_{ct}(P^2_n, x)$ of P^2_n is defined as,

$$D_{ct}(P^2_n, x) = \sum_{i=\gamma_{ct}(P^2_n)}^{n} d_{ct}(P^2_n, i)\, x^i.$$

Theorem 3.2

For every $n \geq 8$,

$D_{ct}(P^2_n, x) = x\,[D_{ct}(P^2_{n-1}, x) + D_{ct}(P^2_{n-2}, x)]$

with initial values

$D_{ct}(P^2_2, x) = x^2$,

$D_{ct}(P^2_3, x) = x^3 + 3x^2$,

$D_{ct}(P^2_4, x) = x^4 + 4x^3 + 5x^2$

$D_{ct}(P^2_5, x) = x^5 + 5x^4 + 8x^3 + 5x^2$

$D_{ct}(P^2_6, x) = x^6 + 6x^5 + 12x^4 + 10x^3 + 3x^2$

$D_{ct}(P^2_7, x) = x^7 + 7x^6 + 17x^5 + 18x^4 + 8x^3 + x^2$.

Proof

We have,

$d_{ct}(P^2_n, i) = d_{ct}(P^2_{n-1}, i-1) + d_{ct}(P^2_{n-2}, i-1)$.

Therefore,

$d_{ct}(P^2_n, i)\, x^i = d_{ct}(P^2_{n-1}, i-1)\, x^i + d_{ct}(P^2_{n-2}, i-1)\, x^i$

$\Sigma d_{ct}(P^2_n, i)\, x^i = \Sigma d_{ct}(P^2_{n-1}, i-1)\, x^i + \Sigma d_{ct}(P^2_{n-2}, i-1)\, x^i$

$\Sigma d_{ct}(P^2_n, i)\, x^i = x\Sigma d_{ct}(P^2_{n-1}, i-1)\, x^{i-1} + x\Sigma d_{ct}(P^2_{n-2}, i-1) x^{i-1}$.

$D_{ct}(P^2_n, x) = x\, D_{ct}(P^2_{n-1}, x) + x\, D_{ct}(P^2_{n-2}, x)$

$D_{ct}(P^2_n, x) = x\,[D_{ct}(P^2_{n-1}, x) + D_{ct}(P^2_{n-2}, x)]$.

with the initial values,

$D_{ct}(P^2_2, x) = x^2$,

$D_{ct}(P^2_3, x) = x^3 + 3x^2$,

$D_{ct}(P^2_4, x) = x^4 + 4x^3 + 5x^2$

$D_{ct}(P^2_5, x) = x^5 + 5x^4 + 8x^3 + 5x^2$

$D_{ct}(P^2_6, x) = x^6 + 6x^5 + 12x^4 + 10x^3 + 3x^2$

$D_{ct}(P^2_7, x) = x^7 + 7x^6 + 17x^5 + 18x^4 + 8x^3 + x^2$.

We obtain $d_{ct}(P^2_n, i)$, for $2 \leq n \leq 15$ as shown in Table 1.

Table 1

$d_{ct}(P^2_n, i)$, the number of connected total dominating sets of P^2_n with cardinality i.

n \ i	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	1													
3	3	1												
4	5	4	1											
5	5	8	5	1										
6	3	10	12	6	1									
7	1	8	18	17	7	1								
8	0	4	18	30	23	8	1							
9	0	1	12	36	47	30	9	1						
10	0	0	5	30	66	70	38	10	1					
11	0	0	1	17	66	113	100	47	11	1				
12	0	0	0	6	47	132	183	138	57	12	1			
13	0	0	0	1	23	113	116	283	185	68	13	1		
14	0	0	0	0	7	70	245	299	421	242	80	14	1	
15	0	0	0	0	1	30	183	256	582	606	310	93	15	1

In the following theorem, we obtain some properties of $d_{ct}(P^2_n, i)$.

Theorem 3.3

The following Properties hold for the coefficients of $D_{ct}(P_n^2, x)$.

1. $d_{ct}(P_n^2, n) = 1$, for every $n \geq 2$.
2. $d_{ct}(P_n^2, n-1) = n$, for every $n \geq 3$.
3. $d_{ct}(P_n^2, n-2) = \frac{1}{2}[n^2 - 3n + 6]$, for every $n \geq 4$.
4. $d_{ct}(P_n^2, n-3) = \frac{1}{6}[n^3 - 9n^2 + 38n - 60]$, for every $n \geq 5$.
5. $d_{ct}(P_{2n+1}^2, n-1) = 1$, for every $n \geq 3$.
6. $d_{ct}(P_{2n}^2, n-1) = n$, for every $n \geq 3$.

Proof

1. Since $D_{ct}(P_n^2, n) = \{ [n] \}$, we have the result.

2. Since $D_{ct}(P_n^2, n-1) = \{ [n] - \{x\} / x \in [n]\}$,

we have $d_{ct}(P_n^2, n-1) = n$.

3. To prove, $d_{ct}(P_n^2, n-2) = \frac{1}{2}[n^2 - 3n + 6]$, for every $n \geq 4$, we apply induction on n.

When n = 4,

L. H. S = $d_{ct}(P_4^2, 4-2) = d_{ct}(P_4^2, 2) = 5$ (from the table) and

R. H. S = $\frac{1}{2}[n^2 - 3n + 6] = \frac{1}{2}[4^2 - 3 \times 4 + 6] = 5$

Therefore, the results is true for n = 4.

Now, suppose that the result is true for all numbers less than 'n' and we prove it for n.

By theorem 3.2, we have,

$$d_{ct}(P_n^2, n-2) = d_{ct}(P_{n-1}^2, n-3) + d_{ct}(P_{n-2}^2, n-3).$$

$$= \frac{1}{2}[(n-1)^2 - 3(n-1) + 6] + n-2.$$

$$= \frac{1}{2}[n^2 - 2n + 1 - 3n + 3 + 6 + 2n - 4].$$

$$= \frac{1}{2}[n^2 - 3n + 6].$$

Hence the result is true for all n.

4. To prove, $d_{ct}(P_n^2, n-3) = \frac{1}{6}[n^3 - 9n^2 + 38n - 60]$, for every $n \geq 5$, we apply induction on n.

When n = 5, L.H.S = $d_{ct}(P_5^2, 2) = 5$ (from the table) and

R.H.S = $\frac{1}{6}[5^3 - 9 \times 5^2 + 38 \times 5 - 60] = 5$.

Therefore the result is true for n = 5.

Now, suppose that the result is true for all numbers less than 'n' and we prove it for n.

By Theorem 3.2, we have,

$$d_{ct}(P_n^2, n-3) = d_{ct}(P_{n-1}^2, n-4) + d_{ct}(P_{n-2}^2, n-4).$$

$$= \frac{1}{6}[(n-1)^3 - 9(n-1)^2 + 38(n-1) - 60] + \frac{1}{2}[(n-2)^2 - 3(n-2) + 6].$$

$$= \frac{1}{6}[(n^3 - 3n^2 + 3n - 1) - 9(n^2 - 2n + 1) + 38n - 38 - 60] + \frac{1}{2}[(n^2 - 4n + 4 - 3n + 6 + 6].$$

$$= \frac{1}{6}[(n^3 - 3n^2 + 3n - 1 - 9n^2 + 18n - 9 + 38n - 38 - 60 + 3n^2 - 21n + 48].$$

$$= \frac{1}{6}[n^3 - 9n^2 + 38n - 60].$$

Hence, the result is true for all n.

5) Since $D_{ct}(P_{2n+1}^2, n-1) = \{3,5,7,9, \ldots, (2n+1) - 6, (2n+1) - 4, (2n+1) - 2\}$, we have, $d_{ct}(P_{2n+1}^2, n-1) = 1$.

6) To prove, $d_{ct}(P_{2n}^2, n-1) = n$, for every $n \geq 3$, we apply induction on n.

When n = 3,

L.H.S = $d_{ct}(P_6^2, 2) = 3$ (from the table) and

R.H.S = n = 3.

Therefore, the result is true for n = 3.

Now, suppose that the result is true for all numbers less than 'n' and we prove it for n. By Theorem 3.2, we have,

$$d_{ct}(P_{2n}^2, n-1) = d_{ct}(P_{2n-1}^2, n-2) + d_{ct}(P_{2n-2}^2, n-2)$$
$$= 1 + n - 1 = n$$

Therefore, the result is true for all $n \geq 3$.

5. Conclusion

In this paper, the connected total domination polynomials of square of paths has been derived by identifying its connected total dominating sets. It also helps us to characterize the connected total dominating sets and to find the number of connected total dominating sets of cardinality i. We can generalize this study to any power of the path and some interesting properties can be obtained via the roots of the connected total domination polynomial of P_n^k.

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