Geraghty Type Contraction and Common Coupled Fixed Point Theorems in Bipolar Metric Spaces with Applications to Homotopy

B.Srinuvasa Rao\textsuperscript{1} G.N.V.Kishore\textsuperscript{2} G.Kiran Kumar\textsuperscript{3}

\textsuperscript{1}Research Scholar, Dept.of Mathematics, KL University, Vaddeswaram, Guntur-522 502, Andhra Pradesh, India.
\textsuperscript{2}Associate Professor, Dept. of Mathematics, SRKR Engineering College, Bhimavaram 534204, Andhra Pradesh, India.
\textsuperscript{3}Assistant Professor, Dept.of Mathematics, Dr.B.R.Ambedkar University, Srikakulam-532410, Andhra Pradesh, India.

Abstract

In this paper, we establish the existence of common coupled fixed point results of two covariant mappings in a complete bipolar metric spaces under Geraghty type contraction by using weakly compatible mappings with an example. We have also provided an applications to Homotopy theory.

Keywords - Bipolar metric space, weakly-compatible mappings, common coupled fixed point.

I. INTRODUCTION

This work is motivated by the recent work on extension of Banach contraction principle on Bipolar metric spaces, which has been done by Mutlu and Gürdal [1]. Also they investigated some fixed point and coupled fixed point results on this spaces (see [1], [2]). Later, we proved some fixed point theorems in our earlier papers (see [3], [4]). Subsequently, many authors established coupled fixed point theorems in different spaces (see [5]-[8]).

In 1982, Sessa [9] initiated to studied common fixed point theorems for weakly commuting pair of mappings. Afterward, in 1986, Jungck [10] introduced concept of weakened weakly commuting mappings to compatible mappings in metric spaces and established compatible pair of mappings commute on the sets of coincidence point of the involved mappings. In 1998, the weak compatibility notion initiated by Jungck and Rhoades [11], if they commute at their coincidence points and proved that compatible mappings are weakly compatible but the converse does not hold. In 1973, Geraghty ([12]) introduced a generalization of Banach contraction principle ([13]) in which the contraction constant was replaced by a function having some specified properties. Later, many authors refereed it to Geraghty type fixed point results and extended different types of distance spaces (see [14],[15]).

The aim of this paper is to initiate the study of a common coupled fixed point results for two covariant mappings under Geraghty type contractive conditions in bipolar metric spaces. We also given illustrate the validity of the hypotheses of our results.

Definition 1.1: ([1]) Let A, B be two non-empty sets. Suppose that d: A×B → [0, ∞) be a mapping satisfying the below properties:

(B_1) If d(a, b) = 0, if and only if a=b for all (a, b) ∈ A×B,
(B_2) If d(a, b) =d(b, a), for all a, b ∈ A∩B
(B_2) If d(a_1, b_2) ≤ d(a_1, b_1) + d(a_2, b_1) + d(a_2, b_2) for all a_1, a_2 ∈ A, and b_1, b_2 ∈ B.

Then the mapping d is termed as Bipolar-metric of the pair (A, B) and the triple (A, B, d) is termed as Bipolar-metric space.

Definition 1.2: ([1]) Assume (A_1, B_1) and (A_2, B_2) as two pairs of sets and a function as F: A_1 \cup B_1 → A_2 \cup B_2 is said to be a covariant map. If F (A_1) \subseteq A_2 and F (B_1) \subseteq B_2, and denote this with F: (A_1, B_1) \rightrightarrows (A_2, B_2). And the mapping F: A_1 \cup B_1 → A_2 \cup B_2 is said to be a contravariant map. If F (A_1) \subseteq B_2 and F (B_1) \subseteq A_2 and write F: (A_1, B_1) \lefteff orphan{(A_2, B_2)}. In particular, if d_1 and d_2 are bipolar metric on (A_1, B_1) and (A_2, B_2) respectively, we sometimes use the notation F: (A_1, B_1, d_1) \rightrightarrows (A_2, B_2, d_2) and F: (A_1, B_1, d_1) \lefteff orphan{(A_2, B_2, d_2)}. 

ISSN: 2231 – 5373 http://www.ijmttjournal.org
Definition 1.3: ([1]) Assume (A, B, d) is a bipolar metric space. A point v∈A U B is termed as a left point if v∈A, a right point if v∈B and a central point if both. Similarly, a sequence {a_n} on the set A and a sequence {b_n} on the set B are called a left sequence and right sequence respectively. In a bipolar metric space, sequence is the simple term for a left or right sequence. A sequence {v_n} is considered convergent to a point v, if and only if {v_n} is the left sequence, v is the right point and lim n→∞ d(v_n, v) = 0; or {v_n} is a right sequence, v is a left point and lim n→∞ d(v, v_n) = 0. A bi-sequence ({{a_n}}, {{b_n}}) on (A, B, d) is said to be convergent. ({{a_n}}, {{b_n}}) is called Cauchy sequence, if lim n→∞ d(a_n, b_n) = 0. In a bipolar metric space, every convergent Cauchy bi-sequence is bi-convergent. A bipolar metric space is called complete, if every Cauchy bi-sequence is convergent hence bi-complete.

Definition 1.4: ([1]) Let (A_1, B_1, d_1) and (A_2, B_2, d_2) be bipolar metric spaces.
(i) A map F: (A_1, B_1, d_1) → (A_2, B_2, d_2) is called left-continuous at a point a_0 ∈ A_1, if for every ε>0, there is a δ>0 such that d_1(a, b) < δ implies d_2(F(a), F(b)) < ε for all b∈B_1.
(ii) A map F: (A_1, B_1, d_1) → (A_2, B_2, d_2) is called right-continuous at a point b_0 ∈ B_1, if for every ε>0, there is a δ>0 such that d_1(a, b) < δ implies d_2(F(a), F(b)) < ε for all a∈A_1.
(iii) A map F is considered continuous, if it is left continuous at each point a∈A and right continuous at each point b∈B.
(iv) A contravariant map F: (A_1, B_1, d_1) → (A_2, B_2, d_2) is continuous if and only if F: (A_1, B_1, d_1) → (A_2, B_2, d_2) it is continuous as a covariant map.

II. MAIN RESULTS

In this section, we give some common coupled fixed point theorems for two covariant mappings satisfying Geraghty type contractive conditions using weakly compatible property in bipolar metric spaces.

Definition 2.1: Let (A, B, d) be a bipolar metric space, F: (A^2, B^2) → (A, B) and S:(A, B) → (A, B) be two covariant mappings.
(i) If F(a, b)=a and F(b, a)=b for (a, b)∈A^2 U B^2 then (a, b) is called a coupled fixed point of F.
(ii) If F(a, b)=Sa and F(b, a)=Sb for (a, b)∈A^2 U B^2 then (a, b) is called a coupled coincidence point of F and S.
(iii) If F(a, b)=Sa=a and F(b, a)=Sb=b for (a, b)∈A^2 U B^2 then (a, b) is called a common coupled point of F and S.

Definition 2.2: Let (A, B, d) be a bipolar metric space, F: (A^2, B^2) → (A, B) and S:(A, B) → (A, B) be two covariant mappings. Then
(i) (F, S) is called weakly compatible if S(F(a, b))=F(Sa, Sb) and S(F(b, a))=F(Sb, Sa) whenever for all (a, b)∈A^2 U B^2 such that F(a, b)=Sa and F(b, a)=Sb.
(ii) (F, S) is called compatible if lim n→∞ d(F(Sa_n, Sb_n), SF(p_n, q_n))=lim n→∞ d(SF(a_n, b_n), F(Sp_n, Sq_n))=0 and lim n→∞ d(F(Sb_n, Sa_n), SF(q_n, p_n))=lim n→∞ d(SF(b_n, a_n), F(Sq_n, Sp_n))=0 for bi-sequences ({{a_n}}, {{p_n}}) and ({{b_n}}, {{q_n}}) in (A, B) such that 
lim n→∞ F(a_n, b_n)=lim n→∞ Sb_n=lim n→∞ Sp_n=lim n→∞ F(p_n, q_n) and
lim n→∞ F(b_n, a_n)=lim n→∞ Sa_n=lim n→∞ Sq_n=lim n→∞ F(q_n, p_n).

Lemma 2.3: Let (A, B, d) be a bipolar metric space, F: (A^2, B^2) → (A, B) and S:(A, B) → (A, B) be two covariant mappings. If (F, S) is compatible then (F, S) is ω-compatible.
Proof. Let F(a, b)=Sa, F(b, a)=Sb and F(p, q)=Sp, F(q, p)=Sq for some a, b∈A, p, q∈B. Consider the constant sequences a_n≡a, b_n≡b and p_n≡p, q_n≡q for all n∈N.
It is obvious that F(a_n, b_n)=Sa_n→Sa, F(b_n, a_n)=Sb_n→Sb as n→∞ and F(p_n, q_n)=Sp_n→Sa, F(q_n, p_n)=Sq_n→Sb as n→∞. Since (F, S) is compatible, d(F(Sa_n, Sb_n), SF(p_n, q_n)) → 0.
\[ d(SF(a_n, b_n), F(Sp_n, Sq_n)) \to 0 \quad \text{and} \quad d(F(Sb_n, Sa_n), SF(q_n, p_n)) \to 0, \quad d(SF(b_n, a_n), F(Sq_n, Sp_n)) \to 0. \]

Thus \( SF(p, q) = F(Sa, Sb), F(Sp, Sq) = SF(a, b) \) and \( SF(q, p) = F(Sb, Sa), F(Sq, Sp) = SF(b, a) \).

On the other hand
\[ d(a_n, p_n) = d\left( \lim_{n \to \infty} a_n, \lim_{n \to \infty} p_n \right) = \lim_{n \to \infty} d(a_n, p_n) = 0 \]
and
\[ d(b_n, q_n) = d\left( \lim_{n \to \infty} b_n, \lim_{n \to \infty} q_n \right) = \lim_{n \to \infty} d(b_n, q_n) = 0. \]
Hence \((F, S)\) is weakly compatible. But converse is need not be true. For example, let \( A = (0, \infty) \) and \( B = [-1, 1] \), define \( d : A \times B \to [0, \infty) \) as \( d(a, b) = |a^2 - b^2| \) for all \((a, b) \in A \times B\).

Then obviously, \((A, B, d)\) is bipolar metric space.

Define two covariant mappings \( F : A^2 \cup B^2 \to A \cup B \) and \( S : A \cup B \to A \cup B \) as follows;
\[ F(a, b) = \begin{cases} \frac{1}{2} \cdot a \in \left[ \frac{1}{2}, 2 \right], & b \in \left[ \frac{1}{2}, 5 \right] \\ \frac{1}{2} \cdot a \in \left( 0, 2 \right), & b \in (-1, 2) \end{cases} \]
\[ S(a, a) = \begin{cases} \frac{1}{2} \cdot a \in \left[ \frac{1}{2}, 2 \right] \\ \frac{1}{2} \cdot a \in \left( 0, 2 \right) \end{cases} \]

and
\[ F(p, q) = \begin{cases} \frac{1}{2} \cdot p \in \left[ -1, \frac{1}{2} \right], & q \in \left[ \frac{1}{2}, 1 \right] \\ \frac{1}{2} \cdot p \in \left( 0, \frac{1}{2} \right), & q \in \left[ \frac{1}{2}, \infty \right) \end{cases} \]
\[ S(p, p) = \begin{cases} \frac{1}{2} \cdot p \in \left[ -1, \frac{1}{2} \right] \\ \frac{1}{2} \cdot p \in \left( 0, \frac{1}{2} \right) \end{cases} \]

Now we define the bi-sequences \((a_n, p_n)\) and \((b_n, q_n)\) as \(a_n = \frac{1}{n} \quad b_n = 1 + \frac{1}{n} \quad p_n = 1 \quad q_n = 1 \quad n\to \infty\), then
\[ F(a_n, b_n) = 1 + \frac{1}{2n} \to 1 \quad \text{as} \quad n \to \infty, \quad S(a_n, a_n) = \frac{1}{2} \quad b_n = 1 + \frac{1}{2n} \to 1 \quad \text{as} \quad n \to \infty, \quad F(b_n, q_n) = \frac{1}{2} \quad q_n = 1 \quad \text{as} \quad n \to \infty. \]

But \( \lim_{n \to \infty} d(F(Sa_n, b_n), SF(p_n, q_n)) = \lim_{n \to \infty} d\left( F\left( 1 + \frac{1}{2n} \right), S\left( 1 + \frac{1}{2n} \right) \right) = \lim_{n \to \infty} d\left( \frac{1}{2}, \frac{1}{2} \right) = 0. \]

Also, we have \( \lim_{n \to \infty} \max \{d(SF(a_n, b_n), SF(p_n, q_n))\} = 0 \), \( \lim_{n \to \infty} \min \{d(SF(a_n, b_n), SF(p_n, q_n))\} = 0 \), \( \lim_{n \to \infty} \min \{d(F(p_n, q_n), F(Sp_n, Sq_n))\} = 0 \), \( \lim_{n \to \infty} \max \{d(F(p_n, q_n), F(Sp_n, Sq_n))\} = 0 \).

Thus the pair \((F, S)\) is not compatible. Also, the coupled coincidence point of \( F \) and \( S \) is \( \left( \frac{1}{2}, \frac{1}{2} \right) \). It is namely that
\[ \theta(a, b) = 0 \quad \forall \quad a, b \in [0, \infty) \]

(i) \( \theta(a, b) = 0 \quad \forall \quad a, b \in [0, \infty) \)

(ii) For any two sequences \( \{a_n\} \) and \( \{b_n\} \) of non-negative real numbers \( \theta(a_n, b_n) \to 1 \Rightarrow a_n, b_n \to 0 \)

**Theorem 2.4:** Let \((A, B, d)\) be a complete bipolar metric space, \( F : A^2 \cup B^2 \to A \cup B \) and \( S : A \cup B \to A \cup B \) be two covariant mappings satisfying the following conditions

\( (\psi) \quad d(F(a, b), F(p, q)) \leq \theta(d(Sa, Sp), d(Sb, Sq)) \ max\{d(Sa, Sp), d(Sb, Sq)\} \)

Where \( \theta \in \Theta \) and \( a, b \in A, p, q \in B \).

\( (\psi_1) \quad F(A^2 \cup B^2) \subseteq S(A \cup B) \)

\( (\psi_2) \quad \text{The pair} \quad (F, S) \text{\ is compatible.} \)

\( (\psi_3) \quad S \text{ is continuous.} \)

Then the mappings \( F : A^2 \cup B^2 \to A \cup B \) and \( S : A \cup B \to A \cup B \) have unique common fixed point.

**Proof.** Let \( a_n, b_n \in A \) and \( p_n, q_n \in B \) and from \((\psi_1)\), we construct the bi-sequences \((\{a_n\}, \{p_n\}), (\{b_n\}, \{q_n\})\), \((\{\omega_n\}, \{\xi_n\})\) and \((\{\omega_n\}, \{\xi_n\})\) in \((A, B)\) as

\[ F(a_n, b_n) = Sp_n + 1 = \xi_n, \quad F(p_n, q_n) = Sp_{n+1} = \omega_n, \]
\[ F(b_n, a_n) = Sp_n + 1 = \xi_n, \quad F(q_n, p_n) = Sp_{n+1} = \omega_n, \]
for \( n = 0, 1, 2, \ldots \),

Now from \((\psi_2)\), we have\( d(\omega_n, \xi_{n+1}) \leq \theta(d(S\omega_n, S\xi_{n+1}), d(S\xi_n, S\omega_{n+1})) \ max\{d(S\omega_n, S\xi_{n+1}), d(S\xi_n, S\omega_{n+1})\} \)

and \( d(\xi_n, \omega_{n+1}) \leq \theta(d(S\xi_n, S\omega_{n+1}), d(S\omega_n, S\xi_{n+1})) \ max\{d(S\xi_n, S\omega_{n+1}), d(S\omega_n, S\xi_{n+1})\} \)

as desired.
\[
\begin{align*}
&\leq \theta(d(S_{a_{n+1}}, S_{b_{n+1}}), d(S_{b_{n+1}}, S_{q_{n+1}})) \max\{d(S_{a_{n+1}}, S_{p_{n+1}}), d(S_{b_{n+1}}, S_{q_{n+1}})\} \\
&\leq \max\{d(S_{a_{n+1}}, S_{p_{n+1}}), d(S_{b_{n+1}}, S_{q_{n+1}})\} \\
&\leq \theta(d(S_{a_{n+1}}, S_{p_{n+1}}), d(S_{b_{n+1}}, S_{q_{n+1}})) \max\{d(S_{a_{n+1}}, S_{p_{n+1}}), d(S_{b_{n+1}}, S_{q_{n+1}})\} \\
&\leq \max\{d(S_{a_{n+1}}, S_{p_{n+1}}), d(S_{b_{n+1}}, S_{q_{n+1}})\}.
\end{align*}
\]

Combining (6) and (7), we get
\[
\max\{d(S_{a_{n+1}}, S_{p_{n+1}}), d(S_{b_{n+1}}, S_{q_{n+1}})\} \leq \theta(d(S_{a_{n+1}}, S_{p_{n+1}}), d(S_{b_{n+1}}, S_{q_{n+1}})) \max\{d(S_{a_{n+1}}, S_{p_{n+1}}), d(S_{b_{n+1}}, S_{q_{n+1}})\}.
\]

Letting \( n \to \infty \), it follows that \( \theta(d(S_{a_{n+1}}, S_{p_{n+1}}), d(S_{b_{n+1}}, S_{q_{n+1}})) \to 1 \).

By the property of \( \theta \in \Theta \), we obtain \( d(S_{a_{n+1}}, S_{p_{n+1}}) \to 0 \) and \( d(S_{b_{n+1}}, S_{q_{n+1}}) \to 0 \) as \( n \to \infty \).

Therefore, \( \max\{ d(S_{a_{n+1}}, S_{p_{n+1}}), d(S_{b_{n+1}}, S_{q_{n+1}}) \} \to 0 \) as \( n \to \infty \).

Moreover,
\[
\begin{align*}
&\leq \theta(d(S_{a_{n+1}}, S_{p_{n+1}}), d(S_{b_{n+1}}, S_{q_{n+1}})) \max\{d(S_{a_{n+1}}, S_{p_{n+1}}), d(S_{b_{n+1}}, S_{q_{n+1}})\} \\
&\leq \theta(d(S_{a_{n+1}}, S_{p_{n+1}}), d(S_{b_{n+1}}, S_{q_{n+1}})) \max\{d(S_{a_{n+1}}, S_{p_{n+1}}), d(S_{b_{n+1}}, S_{q_{n+1}})\} \\
&\leq \theta(d(S_{a_{n+1}}, S_{p_{n+1}}), d(S_{b_{n+1}}, S_{q_{n+1}})) \max\{d(S_{a_{n+1}}, S_{p_{n+1}}), d(S_{b_{n+1}}, S_{q_{n+1}})\} \\
&\leq \theta(d(S_{a_{n+1}}, S_{p_{n+1}}), d(S_{b_{n+1}}, S_{q_{n+1}})) \max\{d(S_{a_{n+1}}, S_{p_{n+1}}), d(S_{b_{n+1}}, S_{q_{n+1}})\}.
\end{align*}
\]

Combining (11) and (12), we get
\[
\max\{d(S_{a_{n+1}}, S_{p_{n+1}}), d(S_{b_{n+1}}, S_{q_{n+1}})\} \leq \theta(d(S_{a_{n+1}}, S_{p_{n+1}}), d(S_{b_{n+1}}, S_{q_{n+1}})) \max\{d(S_{a_{n+1}}, S_{p_{n+1}}), d(S_{b_{n+1}}, S_{q_{n+1}})\}.
\]

Letting \( n \to \infty \), it follows that \( \theta(d(S_{a_{n+1}}, S_{p_{n+1}}), d(S_{b_{n+1}}, S_{q_{n+1}})) \to 1 \).

By the property of \( \theta \in \Theta \), we obtain \( d(S_{a_{n+1}}, S_{p_{n+1}}) \to 0 \) and \( d(S_{b_{n+1}}, S_{q_{n+1}}) \to 0 \) as \( n \to \infty \).

Therefore, \( \max\{ d(S_{a_{n+1}}, S_{p_{n+1}}), d(S_{b_{n+1}}, S_{q_{n+1}}) \} \to 0 \) as \( n \to \infty \).

Now we shall show \( \{\omega_{n_k}\}, \{\chi_{n_k}\} \) and \( \{\xi_{n_k}\}, \{\kappa_{n_k}\} \) are Cauchy bi-sequences in \( (A, B) \).

Suppose to the contrary that \( \{\omega_{n_k}\}, \{\chi_{n_k}\} \) and \( \{\xi_{n_k}\}, \{\kappa_{n_k}\} \) are not Cauchy bi-sequences. Then there exists \( \varepsilon > 0 \), for which we can find subsequences \( \{\omega_{n_{k_m}}\}, \{\omega_{n_{k_n}}\} \) of \( \{\omega_{n_k}\} \), \( \{\chi_{n_{k_m}}\} \) of \( \{\chi_{n_k}\} \), \( \{\xi_{n_{k_m}}\}, \{\xi_{n_{k_n}}\} \) of \( \{\xi_{n_k}\}, \{\kappa_{n_k}\} \) of \( \{\kappa_{n_k}\} \) with \( n_{k_m} < n_{k_n} \geq k \) such that
\[
\begin{align*}
R_{k_m} &= \max\{ d(\omega_{n_{k_m}}, \omega_{m_2}), d(\xi_{n_{k_m}}, \xi_{m_2}) \} \geq \varepsilon \\
\max\{ d(\omega_{n_{k_m}}, \chi_{n_{k_m}}), d(\xi_{n_{k_m}}, \kappa_{n_{k_m}}) \} \leq \varepsilon \\
\end{align*}
\]

and
\[
\begin{align*}
R_{k_n} &= \max\{ d(\omega_{n_{k_n}}, \omega_{n_{k_n}}), d(\chi_{n_{k_n}}, \kappa_{n_{k_n}}) \} \geq \varepsilon \\
\max\{ d(\omega_{n_{k_n}}, \chi_{n_{k_n}}), d(\xi_{n_{k_n}}, \kappa_{n_{k_n}}) \} \leq \varepsilon \\
\end{align*}
\]

By view of (16) and triangle inequality, we get
\[
\varepsilon \leq R_{k_n} \max\{ d(\omega_{n_{k_n}}, \chi_{n_{k_n}}), d(\xi_{n_{k_n}}, \kappa_{n_{k_n}}) \} \\
\leq \max\{ d(\omega_{n_{k_n}}, \chi_{n_{k_n}}), d(\xi_{n_{k_n}}, \kappa_{n_{k_n}}) \} + \max\{ d(\omega_{n_{k_n}}, \chi_{n_{k_n}}), d(\xi_{n_{k_n}}, \kappa_{n_{k_n}}) \} \\
+ \max\{ d(\omega_{n_{k_n}}, \chi_{n_{k_n}}), d(\xi_{n_{k_n}}, \kappa_{n_{k_n}}) \} + \varepsilon
\]

Letting \( k \to \infty \), we obtain
\[
\begin{align*}
R_{k_n} &= \max\{ d(\omega_{n_{k_n}}, \chi_{n_{k_n}}), d(\xi_{n_{k_n}}, \kappa_{n_{k_n}}) \} \to \varepsilon \\
\end{align*}
\]

Again by means of triangle inequality, we have
\[
R_{k_m} \max\{ d(\omega_{n_{k_m}}, \chi_{n_{k_m}}), d(\xi_{n_{k_m}}, \kappa_{n_{k_m}}) \}
\]
\[ \begin{align*}
&< \max \{ d(\omega_{mk}, \chi_{n+1}), \delta(\varepsilon_{mk}, \kappa_{n+1}) \} + \max \{ d(\omega_{mk}, \chi_{n+1}), d(\varepsilon_{mk}, \kappa_{n+1}) \} \\
&+ \max \{ d(\omega_{mk}, \chi_{mk}), d(\varepsilon_{mk}, \kappa_{mk}) \} \\
&< \max \{ d(\omega_{mk}, \chi_{n+1}), \delta(\varepsilon_{mk}, \kappa_{n+1}) \} + \max \{ d(\omega_{mk}, \chi_{n+1}), d(\varepsilon_{mk}, \kappa_{n+1}) \} \\
&+ \theta \{ d(\omega_{mk}, \chi_{mk}), d(\varepsilon_{mk}, \kappa_{mk}) \} \max \{ d(\omega_{mk}, \chi_{mk}), d(\varepsilon_{mk}, \kappa_{mk}) \}. 
\end{align*} \]

Letting \( n \to \infty \), it yields that \( \theta \{ d(\omega_{mk}, \chi_{n+1}), d(\varepsilon_{mk}, \kappa_{n+1}) \} \to 1 \). Now by means of property of \( \theta \), it follows \( \max \{ d(\omega_{mk}, \chi_{n+1}), d(\varepsilon_{mk}, \kappa_{n+1}) \} \to 0 \) as \( k \to \infty \), which implies that
\begin{equation}
\lim_{k \to \infty} R_e = \lim \max \{ d(\omega_{mk}, \chi_{mk}), d(\varepsilon_{mk}, \kappa_{mk}) \} = 0. \quad (19)
\end{equation}

Similarly, we can prove
\begin{equation}
\lim_{k \to \infty} \eta_f = \lim \max \{ d(\omega_{mk}, \chi_{mk}), d(\varepsilon_{mk}, \kappa_{mk}) \} = 0. \quad (20)
\end{equation}

Which are contracts with (16) and (17). Thus \( \{(\omega_n), (\chi_n)\} \) and \( \{(\varepsilon_n), (\kappa_n)\} \) are Cauchy bi-sequences in \( (A, B) \).

Therefore, \( \lim_{n \to \infty} \omega_n = \lim \delta(\varepsilon_{mk}, \kappa_{mk}) = 0 \). Since \( (A, B, d) \) is complete, there exist \( u, v \in A \) and \( w, z \in B \) with
\[ d(Su, S\bar{u}) = d(Sv, S\bar{v}) = d(Sw, S\bar{w}) = 0. \]

Now by means of property of \( \theta \), it follows
\[ \max \{ d(\omega_{mk}, \chi_{n+1}), d(\varepsilon_{mk}, \kappa_{n+1}) \} \to 0 \] as \( k \to \infty \), which implies that
\[ \lim_{k \to \infty} R_e = \lim \max \{ d(\omega_{mk}, \chi_{mk}), d(\varepsilon_{mk}, \kappa_{mk}) \} = 0. \quad (19) \]

Similarly, we can prove
\[ \lim_{k \to \infty} \eta_f = \lim \max \{ d(\omega_{mk}, \chi_{mk}), d(\varepsilon_{mk}, \kappa_{mk}) \} = 0. \quad (20) \]

Which are contracts with (16) and (17). Thus \( \{(\omega_n), (\chi_n)\} \) and \( \{(\varepsilon_n), (\kappa_n)\} \) are Cauchy bi-sequences in \( (A, B) \).

Therefore, \( \lim_{n \to \infty} \omega_n = \lim \delta(\varepsilon_{mk}, \kappa_{mk}) = 0 \). Since \( (A, B, d) \) is complete, there exist \( u, v \in A \) and \( w, z \in B \) with
\[ d(Su, S\bar{u}) = d(Sv, S\bar{v}) = d(Sw, S\bar{w}) = 0. \]

Now by means of property of \( \theta \), it follows
\[ \max \{ d(\omega_{mk}, \chi_{n+1}), d(\varepsilon_{mk}, \kappa_{n+1}) \} \to 0 \] as \( k \to \infty \), which implies that
\[ \lim_{k \to \infty} R_e = \lim \max \{ d(\omega_{mk}, \chi_{mk}), d(\varepsilon_{mk}, \kappa_{mk}) \} = 0. \quad (19) \]

Similarly, we can prove
\[ \lim_{k \to \infty} \eta_f = \lim \max \{ d(\omega_{mk}, \chi_{mk}), d(\varepsilon_{mk}, \kappa_{mk}) \} = 0. \quad (20) \]

Which are contracts with (16) and (17). Thus \( \{(\omega_n), (\chi_n)\} \) and \( \{(\varepsilon_n), (\kappa_n)\} \) are Cauchy bi-sequences in \( (A, B) \).

Therefore, \( \lim_{n \to \infty} \omega_n = \lim \delta(\varepsilon_{mk}, \kappa_{mk}) = 0 \). Since \( (A, B, d) \) is complete, there exist \( u, v \in A \) and \( w, z \in B \) with
\[ d(Su, S\bar{u}) = d(Sv, S\bar{v}) = d(Sw, S\bar{w}) = 0. \]

Now by means of property of \( \theta \), it follows
\[ \max \{ d(\omega_{mk}, \chi_{n+1}), d(\varepsilon_{mk}, \kappa_{n+1}) \} \to 0 \] as \( k \to \infty \), which implies that
\[ \lim_{k \to \infty} R_e = \lim \max \{ d(\omega_{mk}, \chi_{mk}), d(\varepsilon_{mk}, \kappa_{mk}) \} = 0. \quad (19) \]
Then (u, v) is \( A^2 \cap B^2 \) is unique common coupled fixed point of covariant mappings \( F \) and \( S \).

Finally, we will show \( u = v \).

\[
\begin{align*}
   d(u, v) &= d(F(u, v), v, u) \leq \theta(d(F(Su, Sv), d(Sv, Su))) \\
   &= \max\{d(Su, Sv), d(Sv, Su)\}
\end{align*}
\]

Combining (25) and (26), we get \( \max\{d(u, u^*), d(v, v^*)\} = \theta(d(Su, Sv), d(Sv, Su)) \max\{d(Su, Sv), d(Sv, Su)\} \). Therefore, \( d(u, u^*) = 0 \) and \( d(v, v^*) = 0 \) implies that \( u = u^* \) and \( v = v^* \). Similarly, if \( (u^*, v^*) \in B^2 \), then we have \( u = u^* \) and \( v = v^* \). Then \( (u, v) \) is a unique common coupled fixed point of covariant mappings \( F \) and \( S \).

**Theorem 2.5:** Let \( (A, B, d) \) be a complete bipolar metric space, \( (A, B, d) \) be a covariant mapping satisfying the condition

\[
   d(F(g, h)) \leq \theta(d(a, p), d(b, q)) + \max\{d(a, p), d(b, q)\}
\]

where \( \theta \in \Theta \) and for all \( a, b, p, q \in (A, B) \) \( F \) has a unique fixed point.

**Theorem 2.6:** Let \( (A, B, d) \) be a complete bipolar metric space, \( (A, B, d) \) be two covariant mappings satisfying the following conditions

\[
   \begin{align*}
   (p_1): & \quad d(F(a, b), F(p, q)) \leq \theta(d(Sa, Sp), d(Sb, Sq)) + \max\{d(Sa, Sp), d(Sb, Sq)\} \\
   (p_2): & \quad \theta \in \Theta, S(A) \subseteq S(B)
   \end{align*}
\]

Then the mappings \( F: A^2 \cup B^2 \rightarrow A \cup B \) and \( S: A \rightarrow B \) have unique common fixed point.

**Proof.** Let \( a_n, b_n \in (A, B, d) \) and from Theorem 2.4, we construct the bi-sequences, \( \{(\omega_n)\}_{n=0}^{\infty} \) and \( \{(\xi_n)\}_{n=0}^{\infty} \) in \( (A, B, d) \) are Cauchy bi-sequences. Since \( (A, B, d) \) is complete, \( \{(\omega_n)\}_{n=0}^{\infty} \) and \( \{(\xi_n)\}_{n=0}^{\infty} \) are converges sequences and its sub-sequences converges as follows

\[
   \begin{align*}
   \lim_{n \to \infty} \omega_{n+1} &= \lim_{n \to \infty} F(a_{n+1}, b_{n+1}) = \lim_{n \to \infty} Sa_{n+2} = w, \\
   \lim_{n \to \infty} \xi_{n+1} &= \lim_{n \to \infty} F(b_{n+1}, a_{n+1}) = \lim_{n \to \infty} Sb_{n+2} = z.
   \end{align*}
\]

Since \( S(A) \subseteq S(B) \) is closed in \( (A, B, d) \), then \( \omega_{n+1}, \xi_{n+1} \in S(A) \subseteq S(B) \) are converges in the complete bipolar metric spaces \( (S(A), S(B), d) \), therefore, there exist \( u, v \in S(A), w, z \in S(B) \) such that

\[
   \begin{align*}
   \lim_{n \to \infty} \omega_{n+1} &= \lim_{n \to \infty} Sa_{n+2} = w, \\
   \lim_{n \to \infty} \xi_{n+1} &= \lim_{n \to \infty} F(b_{n+1}, a_{n+1}) = \lim_{n \to \infty} Sb_{n+2} = z.
   \end{align*}
\]

Since \( S: A \rightarrow B \) is \( S(A) \subseteq S(B) \) and \( u, v \in S(A), w, z \in S(B) \), there exist \( l, m \in (A, B, d) \) such that \( Sl = u, Sm = v \) and \( Sr = w, Sz = z \).

Putting \( a = a_n, b = b_n \) in the inequality \( (p_3) \), we get

\[
   \begin{align*}
   d(F(a_n, b_n), F(r, s)) &\leq \theta(d(Sa_n, Sr), d(Sb_n, Ss)) + \max\{d(Sa_n, Sr), d(Sb_n, Ss)\} \\
   &\leq \max\{d(Sa_n, Sr), d(Sb_n, Ss)\}.
   \end{align*}
\]

Letting \( n \to \infty \), it yields that \( \lim_{n \to \infty} d(F(a_n, b_n), F(r, s)) = 0 \). It follows that \( F(r, s) = w = S(w, z) \) and \( F(s, r) = z = S(z, w) \). Since \( F, S \) are \( \Theta \)-compatible mappings, we have \( F(u, v) = Su, F(v, u) = Sv, F(w, z) = Sw, F(z, w) = Sz \). Now we shall prove that \( u = w, v = z \).

Consider,

\[
   d(Su, x_n) = d(F(u, v), F(p_n, q_n)) \leq \theta(d(Su, p_n), d(Sv, q_n)) = \max\{d(Su, p_n), d(Sv, q_n)\}.
\]

Letting \( n \to \infty \), we get \( d(Su, w) = d(Su, v) \) (29)

and similarly, we shall show \( d(Sv, w) = d(Sv, z) \).

Therefore, from (29) and (30), we get \( \max\{d(Su, w), d(Sv, z)\} = \max\{d(Su, v), d(Sv, w)\} \) implies that \( Su = w = Sv = z \). Similarly, we can prove \( Sw = w = Sz = z \).

Therefore,

\[
   d(F(r, s) = Sw = Sw, F(w, z), F(s, r) = Ss = z, Sz = F(z, w),
\]
and
\[ F(l, m) = Sl = u = S(u, v), \quad F(m, l) = Sm = v = S(v, u), \]

On the other hand
\[ d(Sl, Sr) = d(u, w) = d\left( \lim_{n \to \infty} \chi_n, \lim_{n \to \infty} \omega_n \right) = \lim_{n \to \infty} d(\omega_n, \chi_n) = 0 \]
and
\[ d(Sm, Ss) = d(v, z) = d\left( \lim_{n \to \infty} \xi_n, \lim_{n \to \infty} \eta_n \right) = \lim_{n \to \infty} d(\eta_n, \xi_n) = 0. \]

So \( u = w \) and \( v = z \). Therefore, \((u, v)\) is a coupled fixed point of \( F \) and \( S \).

As in the proof of Theorem 2.4, uniqueness of the coupled fixed point and unique common fixed point of \( F \) and \( S \) can be shown easily.

**Example 2.7:** Let \( A = \{ U_m(R) / U_m(R) \} \) be an upper triangular matrices over \( R \)
and \( B = \{ L_m(R) / L_m(R) \} \) be a lower triangular matrices over \( R \).
Define \( d : A \times B \to \mathbb{R} \) as \( d(P, Q) = \sum_{i=1}^{m} |p_{ij} - q_{ij}| \) for all \( P = (p_{ij})_{m \times m} \in U_m(R) \)
and \( Q = (q_{ij})_{m \times m} \in L_m(R) \). Then obviously, \((A, B, d)\) is a bipolar metric space.

Let the covariant maps \( F : (A, B) \to (A, B) \) be defined as \( F(P, Q) = (\frac{p_{ij} + q_{ij}}{2}, \frac{p_{ij} + q_{ij}}{2}) \) for all \( P = (p_{ij})_{m \times m} \in U_m(R) \)
and \( Q = (q_{ij})_{m \times m} \in L_m(R) \). Then obviously, \((A, B, d)\) is a bipolar metric space.

**Theorem 3.1:** Let \( A = \{ U_m(R) / U_m(R) \} \) be an upper triangular matrices over \( R \)
and \( B = \{ L_m(R) / L_m(R) \} \) be a lower triangular matrices over \( R \).
Define \( d : A \times B \to \mathbb{R} \) as \( d(P, Q) = \sum_{i=1}^{m} |p_{ij} - q_{ij}| \) for all \( P = (p_{ij})_{m \times m} \in U_m(R) \)
and \( Q = (q_{ij})_{m \times m} \in L_m(R) \). Then obviously, \((A, B, d)\) is a bipolar metric space.

**Theorem 3.2:** Let \((A, B, d)\) be a complete bipolar metric space, \( F : (A, B) \to (A, B) \) be defined as \( F(P, Q) = (\frac{p_{ij} + q_{ij}}{2}, \frac{p_{ij} + q_{ij}}{2}) \) for all \( P = (p_{ij})_{m \times m} \in U_m(R) \)
and \( Q = (q_{ij})_{m \times m} \in L_m(R) \). Then obviously, \((A, B, d)\) is a bipolar metric space.

Clearly \( F \) and \( S \) are satisfies all the conditions of Theorem 2.6 and \((O_{m \times m}, O_{m \times m})\) is the coupled fixed point.

**Definition 2.8:** Let \((A, B, d)\) be a bipolar metric space, \( F : (A \times B) \to (A, B) \) be a covariant mapping. If \( F(a, p) = a \) and \( F(p, a) = p \) for all \( a \in A \) and \( p \in B \) then \((a, p)\) is called a coupled fixed point of \( F \).

**Theorem 2.9:** Let \((A, B, d)\) be a complete bipolar metric space, \( F : (A \times B) \to (A, B) \) and \( S : (A, B) \to (A, B) \) be two covariant mappings satisfying the following conditions
\[ \Phi_0 : d(F(a, p), F(q, b)) \leq \theta (d(Sa, Sq), d(Sb, Sp)) \max \{d(Sa, Sq), d(Sb, Sp)\} \]
where \( \theta \in \Theta \) and \( a, b \in A \), \( p, q \in B \).

\( \Phi_1 \) is the pair \((F, S)\) is compatible.
\( \Phi_2 \) is the pair \((F, S)\) is \( \omega \)-compatible.

Then the mappings \( F : (A \times B) \cup (B \times A) \to A \cup B \) and \( S : A \cup B \to A \cup B \) have unique common fixed point.

**Theorem 2.10:** Let \((A, B, d)\) be a complete bipolar metric space, \( F : (A \times B) \to (A, B) \) and \( S : (A, B) \to (A, B) \) be two covariant mappings satisfying the following conditions
\[ \Phi_0 : d(F(a, p), F(q, b)) \leq \theta (d(Sa, Sq), d(Sb, Sp)) \max \{d(Sa, Sq), d(Sb, Sp)\} \]
where \( \theta \in \Theta \) and \( a, b \in A \), \( p, q \in B \).

\( \Phi_1 \) is the pair \((F, S)\) is compatible.
\( \Phi_2 \) is the pair \((F, S)\) is \( \omega \)-compatible.

Then the mappings \( F : (A \times B) \cup (B \times A) \to A \cup B \) and \( S : A \cup B \to A \cup B \) have unique common fixed point.

**III. APPLICATION TO HOMOTOPY**

**Theorem 3.1:** Let \((A, B, d)\) be a complete bipolar metric space, \((U, V)\) be an open subset of \((A, B)\) and \((\overline{U}, \overline{V})\) be closed subset of \((A, B)\) such that \((U, V) \subseteq \overline{U}, \overline{V})\). Suppose \( H : (\overline{U} \cup \overline{V}) \times [0, 1] \to A \cup B \) be an operator with following conditions
\[ \Omega_1 : u \in H(u, v, x) \text{ and } v \in H(v, u, x) \text{ for each } u, v \in \mathbb{R} \cup \mathbb{V} \text{ and } x \in [0, 1] \]
\[ \Omega_2 : \exists M > 0 \text{ such that } d(H(u, v, x), H(x, y, z)) \leq M |x - y| \text{ for every } u, v, x, y, z \in \overline{U}, \overline{V} \text{ and } x, y, z \in [0, 1] \].
Then \( H(., 0) \) has a coupled fixed point \( \Leftrightarrow \ H(., 1) \) has a coupled fixed point.

**Proof.** Let the set

\[
X = \{ x \in [0, 1] : u = H(u, v, x), \quad v = H(v, u, x) \text{ for some } u, v \in U \} \text{ and } \\
Y = \{ y \in [0, 1] : u = H(x, y, \xi),\quad y = H(y, x, \zeta) \text{ for some } x, y \in V \}.
\]

Since \( H(., 0) \) has a coupled fixed point in \( U^2 \cup V^2 \), we have that \((0, 0) \in X \cap Y \). So that \( X \cap Y = \{ 0, 1 \} \) is non-empty set.

Now we show that \( X \cap Y \) is both closed and open in \([0, 1]\) and hence by the connectedness \( X=\{0,1\} \). Let \((\xi_n, \eta_n) \to (\xi, \eta) \in [0, 1] \) as \( n \to \infty \).

We must show that \( \xi, \eta \in X \cap Y \). Since \((\xi_n, \eta_n) \in X \cap Y \) for \( n=0, 1, 2, \ldots \), there exist bi-sequences \((u_n, x_n)\) and \((v_n, y_n)\) such that \( u_n+ v_n= H(u_n, v_n, x_n) \) and \( x_{n+1}= H(x_n, u_n, x_n) \). Thus, \( u_0=x_0\) and \( v_0=y_0\).

Consider,
\[
d(u_n, x_{n+1}) \leq \theta(d(u_{n-1}, x_n), d(v_{n-1}, y_n)) \leq \max\{ d(u_{n-1}, x_n), d(v_{n-1}, y_n) \}
\]
and
\[
d(v_n, y_{n+1}) \leq \theta(d(v_{n-1}, y_n), d(u_{n-1}, x_n)) \leq \max\{ d(v_{n-1}, y_n), d(u_{n-1}, x_n) \}
\]
Combining (31) and (32), we get
\[
\max\{ d(u_{n-1}, x_n), d(v_{n-1}, y_n) \} \leq \theta(\min\{ d(u_{n-1}, x_n), d(v_{n-1}, y_n) \})<1.
\]
Letting \( n \to \infty \), we get \( d(u_{n-1}, x_n), d(v_{n-1}, y_n) \to 0 \). By the property of \( \theta \), we obtain that \( d(u_{n-1}, x_n), d(v_{n-1}, y_n) \to 0 \) as \( n \to \infty \).

Therefore, \[ \text{max} \{ d(u_n, x_n), d(v_n, y_n) \} \to 0 \] as \( n \to \infty \).

Similarly, \[ \text{max} \{ d(u_n, x_n), d(v_n, y_n) \} \to 0 \] as \( n \to \infty \).

For each \( n, m \in \mathbb{N} \), \( m<n \). Using the property \( B_2 \), we have
\[
d(u_n, x_m) \leq \theta(d(u_{n-1}, x_n), d(v_{n-1}, y_n)) \leq \max\{ d(u_{n-1}, x_n), d(v_{n-1}, y_n) \}
\]
and
\[
d(v_n, y_m) \leq \theta(d(v_{n-1}, y_n), d(u_{n-1}, x_n)) \leq \max\{ d(v_{n-1}, y_n), d(u_{n-1}, x_n) \}
\]
Combining (36) and (37), we get
\[
\max\{ d(u_n, x_m), d(v_n, y_m) \} \leq \text{max} \{ d(u_{n-1}, x_n), d(v_{n-1}, y_n) \} + M \| x_{n+1} - x_n \| + M \| x_{n-1} - \zeta_n \|
\]
and
\[
\max\{ d(u_n, x_m), d(v_n, y_m) \} \leq \text{max} \{ d(u_{n-1}, x_n), d(v_{n-1}, y_n) \} + M \| x_{n+1} - x_n \| + M \| x_{n-1} - \zeta_n \|
\]

It follows that \( \lim_{n \to \infty} \max\{ d(u_n, x_m), d(v_n, y_m) \} = 0 \). Similarly, we can prove that

\[
\lim_{n \to \infty} \max\{ d(u_n, x_m), d(v_n, y_m) \} = 0.
\]

Now consider,
\[
d(H(\xi, v, k, \delta)) \leq \theta(d(H(\xi, v, k, x_{n+1}), d(v_{n+1}, \delta)) \leq \theta(d(H(\xi, v, k), H(x_n, y_n, \zeta_n)) = \max\{ d(H(\xi, v, k), H(x_n, y_n, \zeta_n)) \}
\]
and
\[
d(H(\xi, v, k, \delta)) \leq \theta(d(H(\xi, v, k), H(x_n, y_n, \zeta_n)) = \max\{ d(H(\xi, v, k), H(x_n, y_n, \zeta_n)) \}
\]

It follows that \( d(H(\xi, v, k, \delta)) = 0 \) implies \( H(\xi, v, k) = \delta \). Similarly, we obtain \( H(v, \xi, k) = \eta \) and \( H(\delta, \eta, \zeta) = \zeta \), \( H(\eta, \delta, \zeta) = v \).
On the other hand 
\[ d(\xi, \eta) = d\left( \lim_{n \to \infty} x_n, \lim_{n \to \infty} u_n \right) = \lim_{n \to \infty} d(u_n, x_n) = 0 \]
and 
\[ d(\nu, \eta) = d\left( \lim_{n \to \infty} y_n, \lim_{n \to \infty} v_n \right) = \lim_{n \to \infty} d(v_n, y_n) = 0. \]
Therefore, \( \xi = \eta \) and \( \nu = \eta \) and hence \( \kappa = \zeta \). Thus \( (\kappa, \zeta) \in X^2 \cap Y^2 \). Clearly, \( X^2 \cap Y^2 \) is closed in \([0, 1]\).

Let \((x_0, y_0) \in (X, Y)\), then there exist bi-sequences \((u_0, x_0)\) and \((v_0, y_0)\) with \(u_0 = H(u_0, v_0, k_0)\) and \(v_0 = H(v_0, u_0, k_0)\) and \(x_0 = x(H(x_0, y_0, k_0), y_0) = H(y_0, x_0, k_0)\). Since \( U^2 \cup V^2 \) is open, then there exist \( r > 0 \) such that \( X_d(u_0, r) \subseteq U^2 \cup V^2 \) and \( X_d(x_0, r) \subseteq U^2 \cup V^2 \). Choose \( \kappa = (\kappa - \epsilon, \kappa + \epsilon) \), \( \delta = (\delta - \epsilon, \delta + \epsilon) \), then \( |\kappa - \delta| \leq \frac{\epsilon}{M+1} \) and \( |\kappa_3 - \delta_3| \leq \frac{\epsilon}{M+1} \). Then for \( x = B_{X_U}(u_0, r) = \{ x, x_0 \} \in (X, Y) \), \( u_0 \in V/d(u_0, x) \leq r + d(u_0, x_0) \), and \( v_0 \in V/d(v_0, y) \leq r + d(v_0, y_0) \).

Also,
\[ d(H(u, v, k), x_0) = d\left( H(x_0, y_0, k_0) \right) \leq d(H(u, v, k), H(x, y, \zeta_0)) + d(H(u, v, k), H(x, y, k_0)) + d(H(u, v, k), H(x, y, k_0)) \]
\[ < \frac{1}{M+1} \theta (d(u_0, x_0), d(y_0, y)) \max \{ d(u_0, x_0), d(v_0, y_0) \}. \]

Letting \( n \to \infty \), we get \( d(H(u, v, k), x_0) \leq \max \{ d(u_0, x_0), d(v_0, y_0) \} \).

Similarly, we show that \( d(H(u, v, k), y_0) \leq \max \{ d(u_0, x_0), d(v_0, y_0) \} \).

Combining (39) and (40), we get
\[ \max \{ d(H(u, v, k), x_0), d(H(u, v, k), y_0) \} \leq \max \{ d(u_0, x_0), d(v_0, y_0) \}. \]

On the other hand
\[ d(u_0, x_0) = d(H(u_0, v_0, k_0), H(x_0, y_0, k_0) \leq M|k_3 - \zeta_3| \leq M \frac{1}{M+1} \to 0 \text{ as } n \to \infty \]
and
\[ d(v_0, y_0) = d(H(v_0, u_0, k_0), H(y_0, x_0, k_0) \leq M|k_3 - \zeta_3| \leq M \frac{1}{M+1} \to 0 \text{ as } n \to \infty. \]
So \( u_0 \to x_0 \) and \( v_0 \to y_0 \) and hence \( \kappa = \zeta \). Thus for each fixed \( k \in [k_0 - \epsilon, k_0 + \epsilon] \),
\[ H(\cdot, k) : B_{X_U}(u_0, r) \to B_{X_U}(u_0, r) \text{ and } H(\cdot, k) : B_{X_U}(v_0, r) \to B_{X_U}(v_0, r). \]
Then all the conditions of Theorem 3.1 are satisfied. Thus we conclude that \( H(\cdot, k) \) has a coupled fixed point in \( U^2 \cap V^2 \). But this must be in \( U^2 \cap V^2 \).

Therefore, \( (\kappa, \zeta) \in X^2 \cap Y^2 \) for \( \kappa \in [k_0 - \epsilon, k_0 + \epsilon] \). Hence \( (k_0 - \epsilon, k_0 + \epsilon) \subseteq X^2 \cap Y^2 \). Clearly, \( X^2 \cap Y^2 \) is open in \([0, 1]\).
To prove the reverse, we can use the similar process.

IV. CONCLUSIONS

In this paper we conclude some applications to homotopy theory by using coupled fixed point theorems in Bipolar metric spaces.

ACKNOWLEDGMENT

The authors are very thanks to the reviewers and editors for valuable comments, remarks and suggestions for improving the content of the paper.

REFERENCE


