Representation of the Translational Hull of a Proper Type A Semigroup

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Abstract
The structure of proper type A semigroup constructed by Lawson (1986) is visited and some results obtained. The structure is married up with the structure maps of type A semigroup and an alternative structure of proper type A semigroup is obtained in line with Armstrong (1988) analysis of concordant semigroups. By using Munn semigroup, the representation of the left translational hull of the alternative proper type A semigroup is obtained.

Keywords: ∗-Prehomomorphism, ∗-Prehomomorphism triple, Translational Hull, Representation.

I. INTRODUCTION

*-Prehomomorphism triple is an alternative name we have, in this paper, chosen to call the structure of proper type A semigroup due to the pivotal position of ∗-prehomomorphism in the structure as constructed by Lawson (1986). The structure, whose operation is the direct product, takes the form $T = T(S, C, \psi)$, where $C$ is a cancellative monoid, $\psi$ is a ∗-prehomomorphism from $C$ to a type A semigroup $W$, and $S$ is a type $A^*$-subsemigroup of of $W$.

Let $W$ be a type A semigroup and $C$ a cancellative monoid. A mapping $\psi: C \to W$ is called a ∗-Prehomomorphism if the following conditions are satisfied:

i. $\forall g, h \in C, \psi(g)\psi(h) \leq \psi(gh)$

ii. $\psi(1)$ is an idempotent

iii. If $s \in W$ and $s \leq \psi(g)$ then $s^*, s^+ \leq \psi(1)$

where $\leq$ is the natural partial order on $W$.

Now, let $\psi: C \to W$ be a ∗-prehomomorphism and let $S$ be a type $A^*$-subsemigroup of $W$. We define ∗-prehomomorphism triple by $T = T(S, C, \psi) = \{(s, g) \in S \times C | s \leq \psi(g)\}$ endowed with direct product.

If $(s, g), (t, h) \in T$, then $s \leq \psi(g)$ and $t \leq \psi(h)$ and therefore $st \leq \psi(g)\psi(h) \leq \psi(gh)$. Hence, $(st, gh) \in T$. Thus, if $T$ is non-empty, $T$ is a subsemigroup of $S \times C$.

If $S$ is type $A$ with semilattice $E$ of idempotents and assuming for $e \in E$, $a \in S$, $a \leq e$. Then $\exists f \in E$ such that $a = fe = ef \in E$. That is $a \leq e \Rightarrow a \in E$. For references, we can formally state this fact.

Lemma 1.1: If $S$ is type $A$ with semilattice $E$ of idempotents and assuming for $e \in E$, $a \in S$, then $a \leq e \Rightarrow a \in E$.

Still on $S$ a type $A$ semigroup, if $a, b \in S$ such that $a \leq b$. Then $\exists e \in E$ such that $a = be$. This implies that $ae = be = a$. Therefore, $ae = a$.

If $aL^*b$, then $a^* = b^*$. Now, assuming $a \leq b$ and $aL^*b$, then $ae = a$ and $\forall x, y \in S^1$, $ax = ay \Leftrightarrow bx = by$. If we take $x = e$, $y = b^* = a^*$, then we have that $[ae = aa^* = a] \Leftrightarrow [be = bb^* = b]$ and since $be = a$, $ae = a \Leftrightarrow a = b$.

So that if $a \leq b$ and $aL^*b$, then $a = b$. Similarly or dually, if $a \leq b$ and $aR^*b$, then $a = b$.

If $C$ is a cancellative monoid with set of idempotents $E(C)$ and suppose $e \in E(C)$. Then $e.e = e = 1.e$ and by cancellation, $e = 1$. That is, the only idempotent in a cancellative monoid is $1$. If $(s, 1) \in T = T(S, C, \psi)$, then $s \leq \psi(1)$ and since by definition $\psi(1)$ is an idempotent, $s \in E(S)$.
Assuming \((s,g) \in T\) is an idempotent, then \((s,g)^2 = (s^2, g^2) = (s,g)\). That is \(s^2 = s\) and \(g^2 = g\). \(C\) being a cancellative monoid implies that \(g = 1\).

Thus, the set of idempotents of \(T\) is given by \(E(T) = \{ (e,1) \mid e \leq \psi(1), e \in E(S) \}\).

That is \(E(T) = \{ o(\psi(1)) \cap E(S) \} \times \{ 1 \}\).

We know that direct product of two semigroups is again a semigroup. Now, what about when the semigroups are type \(A\)?

**Proposition 1.2:** The direct product of two type \(A\) semigroups is again type \(A\).

Proof: Let \(S\) and \(T\) be type \(A\) semigroups and assuming \((s,t) \in S \times T\), \(s \in S, t \in T\).

For all \(s_1, s_2 \in S, ss_1 = ss_2 \Leftrightarrow s's_1 = s's_2\)

Similarly, \(\forall t_1, t_2 \in T, tt_1 = tt_2 \Leftrightarrow t't_1 = t't_2\)

So that, \((ss_1, tt_1) = (ss_2, tt_2) \Leftrightarrow (s's_1, t't_1) = (s's_2, t't_2)\)

That is, \((s,t)(s_1,t_1) = (s,t)(s_2,t_2) \Leftrightarrow (s',t')(s_1,t_1) = (s',t')(s_2,t_2)\)

Therefore, \(\forall (s,t) \in S \times T\), \((s,t)\L^*(S \times T)(s',t')\)

Similarly, \(\forall (s,t) \in S \times T\), \((s,t)\R^*(S \times T)(s',t')\).

Thus, every \(\L^*\) and \(\R^*\) contains an idempotent.

If \(e_1 \in E(S)\) and \(e_2 \in E(T)\), then \((e_1,e_2) \in E(S \times T)\). Let also \((f_1,f_2) \in E(S \times T)\), then \((e_1,e_2)(f_1,f_2) = (e_1f_1, e_2f_2) = (f_1e_1, f_2e_2) = (f_1,f_2)(e_1,e_2) \in E(S \times T)\). Therefore, \(E(S \times T)\) is a semilattice.

Now, \((e_1,e_2)(s,t) = (e_1e_2s, e_1e_2t) = (s,t)(e_1s', e_1t') = (s,t)((e_1s', e_1t'))^* = (s,t)((e_1,s')(s,t))^* = (s,t)(e_1s', e_1t')^* = (s,t)(e_1s', e_1t')(s,t)^* = (s,e_1)(t,e_1)^*(s,t)^* = ((s,e_1)(t,e_1))^*(s,t)^* = ((s,t)(e_1,e_2))^*(s,t)^*\)

Hence, \(S \times T\) is type \(A\).

**Lemma 1.3** (Lawson 1986): Let \(S\) be a type \(A\) semigroup and let \(T\) be an abundant subsemigroup of \(S\) with \(E(T)\) an order ideal of \(E(S)\). Then, \(T\) is type \(A\).

**Proposition 1.4** (Lawson 1986): Let \(T = T(S,C,\psi)\). Then \(T\) is a type \(A\) semigroup.

**Proposition 1.5** (Lawson 1986): \(T = T(S,C,\psi)\) is a proper type \(A\) semigroup.

Now, the projection \(\alpha: T = T(S,C,\psi) \to S\), defined by \(\alpha: (s,g) \mapsto s\) is clearly a homomorphism as we can see that for \((s,g)\alpha = s, gh = s\alpha\) since \(g^2 = g\). \(\alpha\) is one-to-one on the idempotents of \(T\) since \((e,1)(f,1) \in E(T), (e,1)\alpha = (f,1)\alpha \Rightarrow e = f \Rightarrow (e,1) = (f,1)\)

Thus, \(\alpha\) is idempotent – separating.

We can easily show that \(\alpha\) is good. For \((s,g),(t,h) \in T\), \((s,g)\L^*(T)(t,h) \Rightarrow \exists (s_1,g_1), (s_2,g_2) \in T\) such that \(ss_1g_1 = ss_2g_2 \Leftrightarrow (t,h)(s_1,g_1) = (t,h)(s_2,g_2)\)

\(ss_1g_1 = ss_2g_2 \Rightarrow (ts_1, hg_1) = (ts_2, hg_2)\)

\(ss_1 = ss_2 \Rightarrow ts_1 = ts_2\). This implies that \(s\L^*(T)s = (s,g)\alpha\L^*(T)(s,h)\)

Similarly, \((s,g)\R^*(T)(t,h) \Rightarrow (s,g)\alpha\R^*(T)(t,h)\)

\(ker\alpha = \{(s,g) \mid (t,h) \in T \times T \mid s = t\}\)

The projection \(\beta: T = T(S,C,\psi) \to C\), defined by \(\beta: (s,g) \mapsto g\) is equally a homomorphism.

\(ker\beta = \{(s,g) \mid (t,h) \in T \times T \mid g = h\}\).

**Proposition 1.6:** The kernel of the projection \(\beta: T \to C\) coincides with the minimum cancellative monoid congruence on \(T\).

Proof: Let \((s,g),(t,h) \in ker\beta\). This implies that \(g = h\) and \(s \leq \psi(g)\) and \(t \leq \psi(h) = \psi(g)\).

This implies that \(\exists u,v \in E(W)\) such that \(s = \psi(g)u\) and \(t = \psi(g)v\).

Therefore, \(svu = \psi(g)vu\) and \(tuv = \psi(g)vu\)

Hence, \(svu = tvu\) and since \(E(W)\) is a semilattice, \(\exists e \in E(W)\) such that \(vu = e\). So that \(se = te\). Since \(S\) is a *-subsemigroup of \(W, e \in S\).

Now, we have: \(se = te\) and \(g = h\). That is \((s,g)(e,1) = (t,h)(e,1); \ (e,1) \in E(T)\)

This implies that \((s,g)\sigma = (t,h)\). Reversing the argument gives the converse.
Theorem 1.7: The composition of the kernels, $\beta \circ \text{ker} \alpha \circ \text{ker} \beta$, is transitive.

Proof: Let $[(s,g),(t,h)], [(t,h),(x,k)] \in \text{ker} \beta \circ \text{ker} \alpha \circ \text{ker} \beta$. Then, we wish to show that $[(s,g),(x,k)] \in \text{ker} \beta \circ \text{ker} \alpha \circ \text{ker} \beta$.

To be more explicit, we need to show that if there exist $(s_1, g_1), (t_1, h_1), (t_2, h_2), (x_1, k_1) \in T$ such that

\[
[(s,g),(s_1,g_1)] \in \text{ker} \beta, \quad [(s_1,g_1),(t_1,h_1)] \in \text{ker} \alpha, \quad [(t_1,h_1),(t_2,h_2)] \in \text{ker} \beta
\]

and

\[
[(t_2,h_2),(x_1,k_1)] \in \text{ker} \alpha, \quad [(x_1,k_1),(x,k)] \in \text{ker} \beta
\]

Then, we can find $u, v \in T$ such that

\[
[(s,g),u] \in \text{ker} \beta, \quad [u,v] \in \text{ker} \alpha, \quad [v,(x,k)] \in \text{ker} \beta.
\]

If we accept the premise, then we know from the definitions of $\text{ker} \beta$ and $\text{ker} \alpha$ that $g = g_1$, $s_1 = t_1$, $h = h_1$, $t_2 = x_1$ and $k = k_1$.

And since $\text{ker} \beta = \sigma_T$, there exist $(e,1)$ and $(n,1) \in E(T)$ such that

\[
(e,1)(s,g) = (e,1)(s_1,g) \quad \text{(1)}
\]

\[
(n,1)(x,k) = (n,1)(x_1,k) \quad \text{(2)}
\]

For $[(s,g),u] \in \text{ker} \beta$, we need to find $e \in E(T)$ such that

\[
e_1(s,g) = e_1 u \quad \text{(3)}
\]

Similarly, for $[v,(x,k)] \in \text{ker} \beta$, we need some $f_1 \in E(T)$ such that

\[
f_1 v = f_1(x,k) \quad \text{(4)}
\]

We know that our $e$, $f$, $m$, and $n$ lie in $E(S)$ and since $E(S)$ is a semilattice, $efm$, $nfm \in E(S)$.

If we take $e_1 = (efm,1) \in E(T)$ and $f_1 = (nfm,1) \in E(T)$, then equations (3) and (4) will respectively give

\[
(efm,1)(s,g) = (efm,1)u \quad \text{(5)}
\]

\[
(nfm,1)v = (nfm,1)(x,k) \quad \text{(6)}
\]

Now, if we choose $u = (fms, g)$ where $s_1 = fs$ and $e \leq f$, and if we choose $v = (fmx, k)$ where $x_1 = mx$ and $n \leq m$, then our equations are satisfied and therefore our goal achieved.

II. THE STRUCTURE MAPS OF TYPE A SEMIGROUP

The structure maps of a semigroup $S$ are maps between $L$-classes and $R$-classes of the idempotents $E(S)$ of the semigroup. Interestingly, Armstrong (1988) broadened the definition to maps between $L^*$-classes and $R^*$-classes of the idempotents $E(S)$ for abundant semigroups. Lawson (1986) characterised type $A$ semigroup as proper if and only if these structure maps are injective. Our goal this section is to characterize type $A$ semigroup in structure maps framework in line with Armstrong’s (1988) analysis of concordant semigroups in terms of their traces and structure maps. A concordant semigroup is an idempotent connected abundant semigroup in which the idempotents generate a regular semigroup.

Let $S$ be a type $A$ semigroup and $e, f \in E(S)$ with $f \omega e$. Suppose $a \in R_e^*$, then $(f a)^\dagger = (f a^\dagger)^\dagger = (f e)^\dagger = f^\dagger = f$

Thus, $fa \in R_f^*$. So that corresponding to each pair of $R^*$-classes $R_e^*$ and $R_f^*$ with $f \omega e$, we can associate a map:

\[
\phi_{e,f} : R_e^* \to R_f^* \text{ defined by } a \mapsto fa
\]

Similarly, for $L^*$-classes $L_e^*$ and $L_f^*$ with $f \omega e$ we have $\psi_{e,f} : L_e^* \to L_f^*$ defined by $a \mapsto af$

The maps $\phi_{e,f}$ and $\psi_{e,f}$, given that $f \omega e$, are called the structure maps of $S$.

Suppose $S$ is a type $A$ semigroup endowed with one-one structure maps and assuming $(a,b) \in \sigma \cap R^*$, so that for some $f \in E(S)$, $fa = fb$ and $a^\dagger = b^\dagger$. We therefore have $fa = fb = f b^\dagger b = fa^\dagger b$.

We know that $f a^\dagger a a^\dagger (f a a^\dagger)^\dagger = (fa)^\dagger (fa)^\dagger = (fa^\dagger)^\dagger$ and therefore, $faR^*fa^\dagger$. Let us put $e = a^\dagger$ and $h = fa^\dagger$. Then we have a structure map $\phi_{e,h} : R_e^* \to R_h^*$ with $a \phi_{e,h} = ha$, $a \in R_e^*$.

Thus, $a \in R_e^*$ implies that $e = a^\dagger = b^\dagger$. So that $b \in R_e^*$ as well.

$a \phi_{e,h} = ha = fa a^\dagger a = fa = fa^\dagger = ha = b h = b \phi_{e,h}$ and since by assumption, $\phi_{e,h}$ is one-one, $a = b$. Thus, $\sigma \cap R^* = \iota$. Similar argument produces $\sigma \cap L^* = \iota$. 

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On the other hand, assuming the type A semigroup $S$ is proper with $\phi_{e,f}: \mathcal{R}_\omega^e \to \mathcal{R}_\omega^f$, $e,f \in E(S)$. Suppose that for $a,b \in \mathcal{R}_\omega^+$, $a\phi_{e,f} = b\phi_{e,f}$. This implies that $a\mathcal{R}b$ and $fa = fb$. That is, $(a,b) \in \sigma \cap \mathcal{R}^e$ and since $S$ is proper, $a = b$.

Thus, a type A semigroup is proper if and only if its structure maps are one-one.

Let us recall that $\omega = \omega^\omega \cup \omega^1$. Now, suppose $f\omega^\omega \subset \omega^1$. Then we have $f\omega^\omega e$ or $f\omega^1 e$. $f\omega^1 e$ implies that $ef = f$.

So that $f e f = f$ if $f e f = fe$ and therefore, $f e f \mathcal{R} f$. In the same vein, $f \omega^1 e$ produces $fe = f$ and $ef \mathcal{L} f$.

Now, let $\Phi$ and $\Psi$ be collections of structure maps on $S$, given by $\Phi = \{\phi_{e,f} : \mathcal{R}_\omega^e \to \mathcal{R}_\omega^f \mid f \omega^e\}$ and $\Psi = \{\psi_{e,f} : \mathcal{L}_\omega^e \to \mathcal{L}_\omega^f \mid f \omega^e\}$.

With $g \in E(S)$, suppose $g\omega^e$, then for $a \in \mathcal{R}_\omega^e$. $a\phi_{e,f} \phi_{f,g} = f a \phi_{f,g} = g f a = gfe a \ [\text{since } f \omega^e]$ $= a \phi_{e,g} f e$

and with $\text{dom}[\phi_{e,f} \phi_{f,g}] = \text{dom}[\phi_{e,g} f e]$, we have $\phi_{e,f} \phi_{f,g} = \phi_{e,g} f e$.

Thus, if $g\omega^e$, then $\phi_{e,f} \phi_{f,g} = \phi_{e,g} f e$.

**Lemma 2.1:** $\phi_{e,f} = 1_{\mathcal{R}_\omega^e}$ if $f \in \mathcal{R}_\omega^+ \cap E(S)$.

**Proof:** Suppose $f \in \mathcal{R}_\omega^+ \cap E(S)$, $e \in E(S)$. Since $S$ is adequate, $f = e$. With $a \in \mathcal{R}_\omega^+$, $a\phi_{e,f} = fa = ea = a$.

The following theorem was adapted from lemma 8.1 of Armstrong(1988) where it basically had to do with concordant semigroup. We recast the proof to particularly suit type A semigroup.

**Theorem 2.2:** Given that $e \mathcal{L}^* a \mathcal{R}^* f$, $(a \in S)$ $[e,f \in E(S)]$. Then there exist isomorphisms $\theta_{f,a,e}: \omega(f) \to \omega(e)$ satisfying $a h f = a \psi_{f,a,e} h$ for $h \in \omega(f)$ $\theta_{e,a,f}: \omega(e) \to \omega(f)$ satisfying $a \psi_{e,a,f} = a \phi_{f,a,e} \psi_{e,a}$ for $g \in \omega(e)$ such that $\theta_{e,a,f} \theta_{f,a,e} = 1_{\text{dom} \theta_{e,a,f}}$ and $\theta_{f,a,e} \theta_{e,a,f} = 1_{\text{dom} \theta_{f,a,e}}$.

**Proof:** Type A semigroup is idempotent – connected. Hence, for all $a \in S$, and for all $a^+, a^* \in S$, there exists isomorphisms $\alpha: \langle a^+ \rangle \to \langle a^* \rangle$ satisfying $xa = a(xa)$ for all $x \in \langle a^+ \rangle$ and $a^{-1}: \langle a^* \rangle \to \langle a^+ \rangle$ satisfying $ay = (ya^{-1})a$ for all $y \in \langle a^* \rangle$.

We recall that $\langle a^+ \rangle = \{x \mid x \omega a^+\} = \omega(a^+)$.

$e \mathcal{L}^* a \mathcal{R}^* f$ implies that $a^+ = f$ and $a^* = e$ since $S$ is adequate. We can therefore give our connecting isomorphisms as $\alpha: \omega(f) \to \omega(e)$ such that $h a = a(h \alpha)$ for $h \in \omega(f)$ $\beta: \omega(e) \to \omega(f)$ such that $a g = (g \beta)a$ with $g \in \omega(e)$.

Since we are talking about type A semigroup here, $h a = (h a)^*$ and $g \beta = (g \beta)^+$.

We can see that $h a = (h a)^*$ implies that $a \phi_{f,a,e} = a \psi_{f,a,e}$ and $g \beta = (g \beta)^+$ implies that $a \psi_{e,a,f} = a \phi_{e,a,f}$.

To satisfy the demands, we can now take $\alpha = \theta_{f,a,e}$ and $\beta = \theta_{e,a,f}$.

$\beta a: \omega(e) \to \omega(e)$ and $a (g \beta a) = a((g \beta) a) = a(a(a)^{\uparrow} a) = (a)^{\uparrow} a = a g$.

So that $a (g \beta a) = a g$. The fact that $a \psi_{e,a} = a \phi_{e,a,f}$ tells us that $a \in \mathcal{L}^*$.

Therefore, $a \beta a = a g \Rightarrow e(g \beta a) = e g$.

Now, $g \beta a \in \omega(e)$ so that $e(g \beta a) = g \beta a$. $g \in \omega(e)$, therefore $eg = g$.

Hence, with $e(g \beta a) = eg$, we have $g \beta a = g$. Thus, $\theta_{e,a,f} \theta_{f,a,e} = 1_{\text{dom} \theta_{e,a,f}}$.

Furthermore, $a \beta: \omega(f) \to \omega(f)$. $(h a \beta a) = [(h a)^{\uparrow} a] = (h a)^{\uparrow} a = ha^{\uparrow} a = ha$.

From $a \phi_{f,a,e} = a \psi_{f,a,e}$, we notice that $a \in \mathcal{L}^*_f$. Therefore, $(h a \beta) a = ha \Leftrightarrow (h a \beta) f = h f$.

$h a \beta \in \omega(f)$ so that $(h a \beta) f = h a \beta$, $h \in \omega(f)$ so that $hf = h$. Hence, $ha = h$.

Thus, $\theta_{f,a,e} \theta_{e,a,f} = 1_{\text{dom} \theta_{f,a,e}}$. All as required.

Let $a, b \in S$, and suppose $L_a^* \cap R_b^* \cap E(S) \neq \emptyset$. Then $\exists h \in L_a^* \cap R_b^* \cap E(S)$, and since $L^*$ is a right congruence and $h$ is left identity in $R_b^*$, $a b \mathcal{R} h b = b$. Similarly, $\alpha = ah \mathcal{R} \alpha b$.

Thus, $L_a^* \cap R_b^* \cap E(S) \neq \emptyset$ implies that $a \mathcal{R} \alpha b L^* b$. In fact, this is obviously true of all semigroups.
Suppose, \( h\omega f \). This implies that \( fh = h \) and therefore \( (h, hf) \in R \subseteq R^* \). Our \( S \) here is adequate and \( h, hf \in E(S) \), so that \( h = hf \). Thus, if \( h\omega f \), then \( \phi_{h, hf} = \phi_{f, h} \). Dually, if \( k\omega f \), then \( \psi_{f, fk} = \psi_{f, k} \).

Lawson(1985) described the structure of proper type \( A \) semigroup in terms of \( \ast \)-prehomomorphism. Now, we wish to give an alternative description of this structure in terms of the structure maps of type \( A \) semigroup. We make use of the product given by Armstrong(1988) on the structure maps of concordant semigroup.

To avoid ambiguity, we hereby replace our \( \ast \)-prehomomorphism \( \psi \) by \( \pi \) and therefore the structure \( T(S, C, \psi) \) becomes \( T(S, C, \pi) \). We need to do this because we have also used \( \psi \) as one of the structure maps of type \( A \) semigroup and right here, the \( \ast \)-prehomomorphism and the structure maps are coming together.

We have given \( \Phi \) and \( \Psi \) as collections of structure maps on \( S \). That is: 
\[
\Phi = \{ \Phi_{e,f}: R^*_e \rightarrow R^*_f \mid f \omega e \} \quad \text{and} \quad \Psi = \{ \Psi_{e,f}: L^*_e \rightarrow L^*_f \mid f \omega e \}
\]

Now, define a product \( ** \) on \( T \) by:
For all \((s, g), (t, h) \) in \( T \), \( (s, g)_{\circ}(t, h) = \left[ (s, g)\Psi_{(e,1),(k,1)} \right] \left[ (t, h)\Phi_{(f,1),(k,1)} \right] \)
Where \( (e, 1) \in L^*_e(T) \cap E(T) \). \( (f, 1) \in R^*_f(T) \cap E(T) \). \( (k, 1) \in S[[e, 1], (f, 1)] \)
\[
S[[e, 1], (f, 1)] = \{ (k, 1) \in E(T): (ke, 1) = (fk, 1) = (k, 1); \ (ekf, 1) = (ef, 1) \}
\]

We denote the structure by \( \hat{T} = \hat{T}(T, \Phi, \Psi) \) and show that the binary operation \( ** \) in \( \hat{T} \) coincides with the direct product in \( T = T(S, C, \pi) \).

**Theorem 2.3:** \( \hat{T} = \hat{T}(T, \Phi, \Psi) \) is a proper type \( A \) semigroup with only one \( D^* \)-class.

We first show that the operation in \( \hat{T} = \hat{T}(T, \Phi, \Psi) \) coincides with the direct product in \( T = T(S, C, \pi) \).

**Proof:** \((s, g), (t, h) \in T \). This implies that \((s, g), (t, h) \in T \). 
\[
(s, g)_{\circ}(t, h) = \left[ (s, g)\Psi_{(e,1),(k,1)} \right] \left[ (t, h)\Phi_{(f,1),(k,1)} \right] = (s, g)(k, 1)(t, h) = (skt, gh)
\]
From the product, we notice that \((s, g) \in L^*_e[[e, 1], (k, 1)] \) and therefore \( sL^*e \).

So that \( se = s \) ……………… (i)

We also have that \((k, 1) \in S[[e, 1], (f, 1)] \) which implies that \( ke = k \) ……………… (ii)

Equations (i) and (ii) give that \( sL^*k \) ……………… (iii)

Hence, \( sk = s \).

In the same vein, \((t, h) \in R^*_f[[f, 1], (k, 1)] \) implying that \( tR^*f \). So that \( ft = t \) and \((k, 1) \in S[[e, 1], (f, 1)] \) implies that \( fk = k \) and therefore \( tR^*k \) ……………… (iv)

That is \( kt = t \).

Now, \((skt, gh) = (st, gh) \) which is the direct product of \((s, g) \) and \((t, h) \) in \( T = T(S, C, \pi) \).

Evidently, \( \hat{T} \) is a subsemigroup of \( T \) and therefore a proper type \( A \) semigroup.

(iii) and (iv) give \( sD^*t \) ……………… (v)

With \((s, g) \in L^*_e[[e, 1], \ g \in L^*_1 \) and with \((t, h) \in R^*_f[[f, 1], \ h \in R^*_1 \) we have \( h \in R^*_1 \). So that \( gd^*h \) ……………… (vi)

Thus, (v) and (vi) give \((s, g)D^*\hat{T}(t, h) \).

### III. THE TRANSLATIONAL HULL

#### 3.1 The Translational Hull of a Semigroup

A map \( \lambda \) from a semigroup \( S \) to itself is a **left translation** of \( S \) if for all elements \( a, b \in S \), \( \lambda(ab) = (\lambda a)b \). A map \( \rho \) from a semigroup \( S \) to itself is a **right translation** of \( S \) if \( (ab)\rho = a(b\rho) \) for all elements \( a, b \in S \). A left translation \( \lambda \) and a right translation \( \rho \) are linked if \( a(\lambda b) = (ab)\rho \) for all \( a, b \in S \). The set of all linked pairs \((\lambda, \rho)\) of left and right translations is called the **translational hull** of \( S \) and it is denoted by \( \Omega(S) \). We denote the set of all the idempotents of \( \Omega(S) \) by \( E_{\Omega(S)} \). The set of the left translations of \( S \) is denoted by \( A(S) \) and the set of the right translations of \( S \) is denoted by \( P(S) \). \( \Omega(S) \) is a subsemigroup of the direct product \( A(S) \times P(S) \). For \((\lambda, \rho)(\lambda', \rho') \in \Omega(S) \), the multiplication is given by \((\lambda, \rho)(\lambda', \rho') = (\lambda\lambda', \rho\rho') \) where \( \lambda\lambda' \) denotes the composition of the left maps \( \lambda \) and \( \lambda' \) (that is, first \( \lambda' \) and then \( \lambda \)) and \( \rho\rho' \) denotes the composition of the right maps \( \rho \) and \( \rho' \) (that is, first \( \rho \) and then \( \rho' \)). For each \( a \in S \), there is a linked pair \((\lambda_a, \rho_a)\)
Theorem 3.2: The translational hull of proper type \( A \) semigroup is a proper type \( A \) semigroup

Proof:

Suppose \( S \) is a proper type \( A \) semigroup; let \((\lambda, \rho_1), (\lambda, \rho_2) \in \Omega(S)\) such that

\[
(\lambda, \rho_1) (\lambda, \rho_2) \in L^* \cap \sigma \quad \text{.........(i)}
\]

Then, \((\lambda, \rho_1) (\lambda, \rho_2) \Rightarrow \exists (\lambda, \rho) \in E_{\Omega(S)}\) such that \((\lambda, \rho)(\lambda, \rho_1) = (\lambda, \rho)(\lambda, \rho_2)\)

\[
\Rightarrow \lambda \lambda_1 = \lambda \lambda_2 \quad \text{and therefore, } \forall e \in E(S), \lambda_1(e) = \lambda_2(e)
\]

Theorem 3.3: Let \( \Lambda(\hat{T}) \)

Proof:

Let \( \lambda \sigma_{\Lambda(\hat{T})} = (\lambda \sigma_{\Lambda(\hat{T})}, \hat{\lambda}) \), \( \lambda, \hat{\lambda} \in \Lambda(\hat{T}), \hat{\lambda} \in E_{\Lambda(\hat{T})} \)

\[
\Rightarrow \lambda \sigma_{\Lambda(T)} = \lambda \sigma_{\Lambda(T)} \quad \text{and since } \Lambda(\hat{T}) \text{ is left } E_{\Lambda(\hat{T})} \quad \text{cancellable, } \lambda \Lambda^* \hat{\lambda} \Lambda^* \lambda
\]

Thus, \( (\lambda, \lambda) \in \sigma_{\Lambda(T)} \cap \Lambda^* \Lambda(\hat{T}) \)

\[
\Lambda(\hat{T}) \text{ is proper and therefore } (\lambda, \lambda) \in \sigma_{\Lambda(T)} \cap \Lambda^* \Lambda(\hat{T}) = \tau_{\Omega(T)}
\]

IV. THE REPRESENTATION

Let \( X \) be a set, and denote by \( T_X \) the set of all functions \( \alpha: X \rightarrow X \). \( T_X \) is called the full transformation semigroup on \( X \) with the operation of composition of functions. A homomorphism \( \phi: S \rightarrow T_X \) is a representation of the semigroup \( S \).
In this section, we use the Munn semigroup to obtain the representation of $A(\tilde{T})$

### 4.1 The Munn Semigroup

Given a semilattice $E$, the Munn semigroup $T_E$ consists of all isomorphisms between principal ideals of $E$. It is an inverse subsemigroup of the symmetric inverse semigroup $\mathcal{I}_E$ which is a subsemigroup of the full transformation semigroup. Munn showed that for any inverse semigroup $S$, there is a homomorphism from $S$ into $T_E(S)$ which maps $E(S)$ isomorphically onto $E(T_E(S))$ and induces the maximum idempotent separating congruence on $S$. The maximum idempotent separating congruence on an inverse semigroup is the largest congruence contained in Green’s relation $\mathcal{H}$ (details in Howie 1995).

Fountain (1979) extended Munn’s result to type $A$ semigroup $S$ to give a homomorphism from $S$ into $T_E(S)$ mapping $E(S)$ isomorphically onto $E(T_E(S))$ and inducing the largest congruence contained in $\mathcal{H}^*$.

By analogy with this Fountain’s result, we wish to obtain a representation of $A(\tilde{T})$ on a full subsemigroup of a type $A$ semigroup of mapping using the Munn semigroup.

So, for an arbitrary semilattice $A(E)$, $T_A(E)$ consists of all isomorphisms between principal ideals of $A(E)$, and $T_A(E)$ is an inverse semigroup.

For $\lambda \in A(\tilde{T})$, define the maps $\theta_\lambda$ and $Y_\lambda$ as follows:

$\theta_\lambda : \langle \lambda' \rangle \rightarrow \langle \lambda^* \rangle$ and $Y_\lambda : \langle \lambda^* \rangle \rightarrow \langle \lambda' \rangle$ by $\lambda' \theta_\lambda = (\lambda \lambda)^*$ and $\lambda Y_\lambda = (\lambda \lambda)^*$  

For $\lambda' \in \langle \lambda' \rangle$, $\lambda' \theta_\lambda Y_\lambda = [\lambda(\lambda \lambda)^*]^*$

Now, $\lambda' \cdot [\lambda' \theta_\lambda Y_\lambda] = \lambda' [\lambda(\lambda \lambda)^*]^* = [\lambda' \lambda(\lambda \lambda)^*]^* = (\lambda' \lambda)^* = \lambda' \lambda^* = \lambda^*$ since $\lambda' \in \langle \lambda' \rangle$.

Therefore, $\forall \lambda' \in \langle \lambda' \rangle$, $\lambda' \leq \lambda' \theta_\lambda Y_\lambda$. In the same vein, for $\lambda^* \in \langle \lambda^* \rangle$, $\lambda^* \leq \lambda^* Y_\lambda \theta_\lambda$

For $\lambda' \in \langle \lambda' \rangle$, suppose $\lambda' \theta_\lambda = \lambda'' \theta_\lambda$. Then we have $(\lambda' \lambda)^* = (\lambda'' \lambda)^*$. Since $A(\tilde{T})$ is type $A$, $\lambda' \lambda = (\lambda' \lambda)^* = \lambda(\lambda' \lambda)^* = \lambda'' \lambda$. Therefore, $\lambda' = \lambda' \lambda^* = (\lambda' \lambda)^* = \lambda'' \lambda^* = \lambda''$.

Thence, $\theta_\lambda$ is one-one for each $\lambda \in A(\tilde{T})$. Similarly, $Y_\lambda$ is one-one for each $\lambda \in A(\tilde{T})$.

Thus, $\theta_\lambda, Y_\lambda \in T_{A(\tilde{T})}$ $\forall \lambda \in A(\tilde{T})$. For $\lambda \in \langle \lambda' \rangle$, assume $\lambda \theta_\lambda Y_\lambda = \lambda'$  

where $\lambda' \in \langle \lambda' \rangle$. Then, $\lambda \lambda' = \lambda [\lambda \theta_\lambda Y_\lambda] = \lambda [\lambda (\lambda \lambda)^*]^* = (\lambda \lambda)^* = (\lambda \lambda)^* = \lambda \lambda^* = \lambda$

So that $\lambda \leq \lambda' = \lambda \theta_\lambda Y_\lambda$, $\forall \lambda \in \langle \lambda' \rangle$  

The domain of $\theta_\lambda$, $\lambda' \in \langle \lambda \lambda^* \rangle$  

$\lambda' \theta_\lambda = (\lambda' \lambda \lambda)^* = [(\lambda' \lambda \lambda)^* \lambda]^*$  

$\theta_\lambda \theta_\lambda = (\lambda' \lambda \lambda)^* = (\lambda' \lambda \lambda)^*$

So that $\theta_\lambda \theta_\lambda = \theta_\lambda \theta_\lambda |_{\langle \lambda' \rangle}$.  

Similarly, $Y_\lambda \lambda' = Y_\lambda \lambda' |_{\langle (\lambda \lambda')^* \rangle}$. Therefore,  

$\lambda \theta_\lambda Y_\lambda = \lambda \lambda' \theta_\lambda Y_\lambda = (\lambda' \lambda \lambda)^*$

$\lambda \theta_\lambda Y_\lambda = \lambda \lambda' \theta_\lambda Y_\lambda = \lambda \lambda' \theta_\lambda Y_\lambda$ since $\lambda \leq \lambda'$ from (ii)

For $\lambda \lambda' \in \langle \lambda \lambda' \rangle$, $\lambda \theta_\lambda Y_\lambda = \lambda \theta_\lambda Y_\lambda |_{\langle \lambda \lambda' \rangle}$

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That is \( \lambda' \theta_{\lambda', \lambda} Y_{\lambda', \lambda} = \lambda \theta_{\lambda', \lambda} Y_{\lambda', \lambda} \)
\[\Rightarrow \lambda' = \lambda \text{ since } \theta_{\lambda', \lambda} \text{ & } Y_{\lambda', \lambda} \text{ are one – one.}\]
So from (i), we see that \( \theta_{\lambda} Y_{\lambda} = 1_{(\lambda^+)} \), the identity map on \( (\lambda^+) \). In the same vein, \( Y_{\lambda} \theta_{\lambda} = 1_{(\lambda^+)} \).
Thus, \( \theta_{\lambda} \) and \( Y_{\lambda} \) are mutually inverse isomorphisms for each \( \lambda \in \Lambda(\tilde{T}) \).

**Theorem 4.2:** \( \mu = (\lambda, \lambda) \in A(\tilde{T}) \times A(\tilde{T}) \): \( \theta_{\lambda} = \theta_{\lambda}, Y_{\lambda} = Y_{\lambda} \) is the largest idempotent separating congruence contained in \( \mathcal{H}^* \), and there is a homomorphism from \( A(\tilde{T}) \) onto a full subsemigroup of \( T_{E_A(\tau)} \) whose kernel equals \( \mu \), with semilattice of idempotents of \( T_{E_A(\tau)} \) isomorphic to \( E_A(\tau) \).

Proof: The straightforward definition clearly makes \( \mu \) an equivalence. To show compatibility of \( \mu \) with the operations on \( A(\tilde{T}) \), let \( (\lambda_1, \lambda_1), (\lambda_2, \lambda_2) \in \mu \)
\[\Rightarrow \theta_{\lambda_1} \theta_{\lambda_2} Y_{\lambda_1} = Y_{\lambda_1}, \theta_{\lambda_2} \theta_{\lambda_2} Y_{\lambda_2} = Y_{\lambda_2} \]
\[\Rightarrow \theta_{\lambda_1} \theta_{\lambda_2} Y_{\lambda_1} \theta_{\lambda_2} = \theta_{\lambda_1} \theta_{\lambda_2} \text{ and } Y_{\lambda_1} Y_{\lambda_2} = Y_{\lambda_1} Y_{\lambda_2} \]
From (iii) above, we have \( \theta_{\lambda_1, \lambda_2} = \theta_{\lambda_1, \lambda_2} \) and \( Y_{\lambda_1, \lambda_2} = Y_{\lambda_1, \lambda_2} \)
\[\Rightarrow (\lambda_1 \lambda_2, \lambda_1 \lambda_2) \in \mu. \text{ Thus, } \mu \text{ is a congruence.} \]
Furthermore, let \( \lambda', \lambda' \in E_A(\tau) \) with \( \lambda' \mu \lambda' \)
\[\Rightarrow \theta_{\lambda'} \theta_{\lambda'} Y_{\lambda'} = Y_{\lambda'} \]
\[\Rightarrow (\lambda') = (\lambda \times \cdot \lambda') \] and evidently, \( \lambda' = \lambda' \). Therefore, \( \mu \) is idempotent separating.
Now, let \( (\lambda, \lambda) \in \mu \). This implies that \( \theta_{\lambda} = \theta_{\lambda} \) and \( Y_{\lambda} = Y_{\lambda} \)
\[\Rightarrow (\lambda^+) = (\lambda^+) \text{ and } (\lambda^+) = (\lambda^+) \]
\[\Rightarrow \lambda^+ = \lambda^+ \text{ and } \lambda^+ = \lambda^+ \] so that \( \lambda \mathcal{R}^* \lambda \) and \( \lambda \mathcal{L}^* \lambda \)
and therefore \( \lambda \mathcal{H}^* \lambda \). Hence \( \mu \in \mathcal{H}^* \).
Suppose \( \tau \) is a congruence on \( A(\tilde{T}) \) with \( \mu \in \mathcal{H}^* \) and suppose \( (\lambda_1, \lambda_1) \in \tau. \)
Then \( \forall \lambda' \in (\lambda^+) \) and \( \forall \lambda'' \in (\lambda^+) \text{, } (\lambda' \lambda_1, \lambda' \lambda_1) \in \tau \text{ and } (\lambda_1 \lambda'', \lambda_1 \lambda'') \in \tau \)
\[\Rightarrow \lambda' \lambda_1 \mathcal{H} \lambda_1 \lambda_1 \text{ and } \lambda_1 \lambda'' \mathcal{H}^* \lambda_1 \lambda_1 \]
That is \( (\lambda' \lambda_1)^* = (\lambda_1 \lambda')^* \) and \( (\lambda_1 \lambda')^* = (\lambda_1 \lambda')^* \)
\[\Rightarrow \forall \lambda' \in (\lambda^+) \text{, } \lambda' \theta_{\lambda_1} = \lambda' \theta_{\lambda_1} \text{ and } \forall \lambda'' \in (\lambda^+) \text{, } \lambda'' Y_{\lambda_1} = \lambda'' Y_{\lambda_1} \]
\[\Rightarrow (\lambda_1, \lambda_1) \in \mu. \text{ So that, } \tau \subseteq \mu. \]
Define a map \( \xi: A(\tilde{T}) \to T_{E_A(\tau)} \) by \( \lambda \xi = \theta_{\lambda} \) and suppose \( \lambda, \lambda \in A(\tilde{T}) \).
\[dom \theta_{\lambda} = (im \theta_{\lambda} \cap dom \theta_{\lambda}) \theta_{\lambda}^{-1} \] (Howie 1995, pg 148)
Where $\vartheta_\lambda: \langle \lambda^+ \rangle \to \langle \lambda^* \rangle$, $\vartheta_\mu: \langle \mu^+ \rangle \to \langle \mu^* \rangle$. So that $\text{dom} \vartheta_\lambda \vartheta_\mu = (\langle \lambda^* \rangle \cap \langle \mu^+ \rangle) \vartheta_\lambda^{-1}$

Every type $A$ semigroup $S$ with semilattice $E$ is characterized by $eS^1 \cap aS^1 = eaS^1$ and $S^1e \cap S^1a = S^1ae \forall a \in S, \forall e \in E$. (Fountain 1979).

Therefore, $\text{dom} \vartheta_\lambda \vartheta_\mu = (\lambda^* \lambda^1) \vartheta_\lambda^{-1}$ and since $\vartheta_\mu$ and $Y_\mu$ are mutually inverse isomorphisms

$\text{dom} \vartheta_\lambda \vartheta_\mu = (\lambda^* \lambda^1) Y_\mu = (\langle \lambda^* \lambda^1 \rangle Y_\mu) = (\langle \lambda \lambda^* \lambda^1 \rangle) = (\langle \lambda \lambda^* \rangle) = \text{dom} \vartheta_\lambda \vartheta_\mu$

For $\lambda' \in (\langle \lambda \lambda^* \rangle)$, $\lambda' \vartheta_\lambda \vartheta_\mu = (\lambda \lambda^*) \vartheta_\mu = (\lambda \lambda^* \lambda^*) = (\lambda \lambda^*) = \lambda \lambda^* \lambda^*$

Hence, $\vartheta_\lambda \vartheta_\mu = \vartheta_\lambda \vartheta_\mu$. So that, $\lambda \lambda^* = \lambda \lambda^* \lambda^*$ and thus $\lambda^*$ is a homomorphism.

For each $\lambda' \in E_A(\overline{\Gamma})$, $\vartheta_\lambda' \vartheta_\nu = \vartheta_\nu$ and $(\lambda')^* = \lambda'$ so that $\vartheta_\lambda': \langle \lambda' \rangle \to \langle \lambda' \rangle$. For $\lambda \in \langle \lambda', \lambda' \rangle \vartheta_\lambda \lambda^* = \lambda \lambda^* = \lambda$.

Thus, the idempotents of $T_{E_A(\overline{\Gamma})}$ have the form $1_{\langle \lambda', \lambda' \rangle}$ - the identical map of $\langle \lambda' \rangle$ onto itself, $\lambda' \in E_A(\overline{\Gamma})$.

Thus, the one – one translation $\lambda' \mapsto 1_{\langle \lambda', \lambda' \rangle}$ is an isomorphism since $1_{\langle \lambda', \lambda' \rangle} = \vartheta_\lambda', \lambda^* \vartheta_\lambda' = \vartheta_\lambda', \lambda^* \vartheta_\lambda' = 1_{\langle \lambda', \lambda' \rangle}$. That is, $E_A(\overline{\Gamma})$ is isomorphic to the semilattice of idempotents of $T_{E_A(\overline{\Gamma})}$.

Moreover, $\lambda' \xi = \vartheta_\lambda' = 1_{\langle \lambda' \rangle}$ and therefore, the $im \xi$ is a full subsemigroup of $T_{E_A(\overline{\Gamma})}$.

Kernel of $\xi$ is given by $\xi \circ \xi^{-1}$. For $\lambda, \lambda \in A(\overline{\Gamma})$,

$[(\lambda, \lambda) \in \mu] \iff [\vartheta_\lambda = \vartheta_\lambda$ and $Y_\lambda = Y_\lambda] \iff [\vartheta_\lambda = \vartheta_\lambda$ and $\vartheta_\lambda^{-1} = \vartheta_\lambda^{-1}] \iff [\vartheta_\lambda = \vartheta_\lambda$]

$[(\lambda, \lambda) \in \xi \circ \xi^{-1}].$ Thus, $\ker \xi = \mu$.

V. CONCLUSION

In this article, we married up the structure of proper type $A$ semigroup constructed by Lawson(1986) with the structure maps of type $A$ semigroup and we obtained an alternative structure of proper type $A$ semigroup in line with Armstrong(1988) analysis of concordant semigroups. We obtained the representation of the left translational hull of the alternative proper type $A$ semigroup using the Munn semigroup. Some other results associated proper type $A$ semigroup were also obtained.

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