Ideals and Separation Axioms in LSTBCH-Algebras

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Abstract
There has been some recent interest in applying topological notions to non-mainstream algebras [3, 4]. In our paper [12], we define the notion of topological BCH-algebras, give some examples and prove some important theorems. In this paper we discuss some algebraic properties such as ideals and topological properties such as $T_0, T_1$ and $T_2$ - axioms in LS topological BCH-algebra.

Keywords: Positive implicative TBCH-algebra, LST$_2$BCH, LST$_1$BCH, LST$_2$BCH-algebras.

I. INTRODUCTION

Y. Imai and K. Iseki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras [8, 9, 10, 11]. In [5,6] Q. P. Hu and X. Li introduced a wider class of abstract algebras: BCH-algebras. In [2], the authors studied topological BL-algebras and proved some theorems that determine the relationship between them. There has been some recent interest in applying topological notions to non-mainstream algebras [3,4]. In [1], Ahn and Kwon addresses the issue of attaching topologies to BCC-algebras in a natural manner.

In our paper [12], we define the notion of topological BCH-algebras, give some examples and prove some important theorems. Also we introduce the notion of topological BCH-algebras in the set of all left maps. In this paper we discuss ideals and separation axioms in LS-topological BCH-algebras.

II. PRELIMINARIES

In this section, we recall some basic definitions and results that are required for our work.

**Definition 2.1:**[6] A BCH-algebra which is not a BCI-algebra, then it is called a proper BCH-algebra.

**Definition 2.2:**[10] Let $(X, \ast)$ be a BCH-algebra and a non empty subset $I$ of $X$ is called an ideal of $X$, if it satisfies the following conditions:
1. $0 \in I$
2. $(x \ast y) \in I \land y \in I \Rightarrow x \in I$.

**Definition 2.3:**[12] Let $(X, \ast)$ be a BCH-algebra and $\tau$ a topology on $X$. Then $X = (X, \ast, \tau)$ is called a topological BCH-algebra, if the operation ‘$\ast$’ is continuous, or equivalently, for any $x, y \in X$ and for any $x, y \in X$ and for any open set $W$ of $x\ast y$ there exist two open sets $U$ and $V$ respectively such that $U \ast V$ is a subset of $W$.

**Definition 2.4:**[12] Let $X$ be a TBCH-algebra. If $X$ satisfies the condition,$(x \ast y) \ast z = (x \ast y) \ast (y \ast z)$, for all $x, y, z$ in $X$
Then $X$ is called positive implicative TBCH-algebra.

**Definition 2.5:** [12] Let $(X, \ast, \tau)$ be a TBCH-algebra, and $a \in X$. Define a left map $L_a : X \rightarrow X$ by, $L_a(x) = a \ast x$, for all $x \in X$.

**Definition 2.6:**[12] Let $(X, \ast, \tau)$ be a TBCH-algebra. The set of all left maps on $X$ is defined as, $L(X)$.

**Definition 2.7:**[12] Let $X$ be a positive implicative BCH-algebra and $A$ be any nonempty subset of $L(X)$, then $L_a = \{ L_a e L(X), a \in A \}$
Definition 2.8: [12] Let $(X, *, \tau)$ be a positive implicative TBCH-algebra and the collection of subsets of $\mathbb{L}(X)$, $\tau' = \Phi(G) \subseteq \mathbb{L}(X) \mid G \in \tau$ is called a LS-topology on the set $\mathbb{L}(X)$. Where $\Phi(G) = \{L_x \mid x \in G\}$ and the collection $(\mathbb{L}(X), \circ, \tau')$ is called the LS-topological BCH-algebra or LSTBCH-algebra.

III. IDEALS AND SEPARATION AXIOMS IN LSTBCH-ALGEBRAS

In this section we discuss about the ideals and separation axioms on a TBCH-algebra. Also we define ideals and separation axioms in LSTBCH-algebra. Finally, we discuss some properties on LSTBCH-algebra.

Definition 3.1: Let $X$ be a positive implicative TBCH-algebra. A nonempty subset $I$ of $\mathbb{L}(X)$ is called an ideal in $\mathbb{L}(X)$ if it satisfies the following condition:

1. $L_0 \in I$
2. $L_a \odot L_b \in I \land L_b \in I \Rightarrow L_a \in I$.

Example 3.2: Consider a positive implicative TBCH-algebra $(X = \{0,1,2\}, *, \tau)$ with the following cayley table,

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And the topology $\tau = \{\phi, X, \{0,1\}, \{2\}\}$. The ideals of $\mathbb{L}(X)$ are $\mathbb{L}(X), \{L_0\}$ & $\{L_0, L_1\}$.

Theorem 3.3: Let $X$ be a positive implicative BCH-algebra, A subset $I \subseteq X$ is an ideal in $X$ if and only if $L_I$ is an ideal in $\mathbb{L}(X)$.

Proof: Let $I$ be an ideal in $X$. Clearly, $0 \in X \Rightarrow L_0 \in L_I$.

For $x, y \in X$. Let $x \ast y$ and $y \in I$. Since $I$ is an ideal in $X$, and $x \ast y, y \in I \Rightarrow x \in I$.

Then $L_{x \ast y}, L_y \in L_I \Rightarrow L_x \in L_I$. That is, $L_x \odot L_y, L_x \in L_I \Rightarrow L_x \in L_I, \Rightarrow L_I$ is an ideal in $\mathbb{L}(X)$.

Conversely, assume that $L_I$ is an ideal in $\mathbb{L}(X)$. Then $L_0 \in \mathbb{L}(X)$, showing that $0 \in X$.

Assume that if $x \ast y \in I$ and $y \in I$. Then $L_{x \ast y} \in L_I$ and $L_y \in L_I, \Rightarrow L_x \odot L_y \in L_I \land L_y \in L_I$.

Since $L_I$ is an ideal in $\mathbb{L}(X)$, $L_x \in L_I \Rightarrow x \in I.$ Therefore, $I$ is an ideal in $X$.

Definition 3.4: Let $(X, *, \tau)$ be a positive implicative TBCH-algebra and $A$ be any subset of $X$. Then $x \in A$ is called an interior point of $A$ if there is an open set $U \in \tau$ such that, $U \subseteq A$.

Example 3.5: Consider the positive implicative TBCH-algebra $(X = \{0,1,2\}, *, \tau)$ with the following cayley table,

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And the topology $\tau = \{\phi, X\}$.

If $A = \{0,1\}$, no point of $A$ is an interior point. If $A = \{0,1,2\}$, every point of $A$ is an interior point of $A$.

Theorem 3.6: Let $(X, *, \tau)$ be a positive implicative TBCH-algebra and $A$ be any ideal of $X$. If $0$ is an interior point of $A$ then $L_A$ is open in LSTBCH-algebra.

Proof:
For every $x \in A$, $L_x \odot L_x = L_0$. Since 0 is an interior point of A then there exist an open set $U \in \tau$ such that $U \subseteq A$. Then $\Phi(U) \subseteq \Phi(A)$.

Since $L(X)$ is a LSTBCH-algebra, there exist a neighbourhood $\Phi(V)$ of $L_x$ such that $\Phi(V \ast V) \subseteq \Phi(U) \subseteq L_A$. Now we claim that $V \subseteq A$.

If not, there is an element $a \in V \ast A \Rightarrow y \ast a \in V \ast V \subseteq A \Rightarrow a \in A$, which is contradicts to our assumption. Therefore, $V \subseteq A$.

**Definition 3.7:** Let $L(X)$ be a LSTBCH-algebra. Then it is called a $LST_2 BCH$-algebra, if for every $x, y \in X \ast x \neq y$, there exist atleast one open neighbourhood U that contains one point of the pair (either x or y) excluding the point.

**Definition 3.8:** Let $L(X)$ be a LSTBCH-algebra. Then it is called a $LST_1 BCH$-algebra, if for every $x, y \in X \ast x \neq y$, there exist open neighbourhoods $U_1 \ast U_2$ such that $U_1$ contains x but not y and $U_2$ contains y but not x.

**Definition 3.9:** Let $L(X)$ be a LSTBCH-algebra. Then it is called a $LST_0 BCH$-algebra, if for every $x, y \in X \ast x \neq y$, both have disjoint open neighbourhoods U & V such that $x \in U \ast y \in V$.

**Example 3.10:** Consider the positive implicative LSTBCH-algebra $(X = \{0, 1, 2\}, \ast, \tau)$ with the following Cayley table,

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And the topology $\tau = \{\phi, X, \{0, 1\}, \{2\}, \{2, 0\}, \{0\}\}$ & $\tau_2 = \{\phi, L(X), \{L_0, L_2\}, \{L_2\}, \{L_2, L_0\}, \{L_0\}\}$ and $\tau_2 = \{\phi, L(X), \{L_0, L_1\}, \{L_1\}, \{L_0, L_1\}, \{L_1\}, \{L_1\}, \{L_1, L_1\}\}$.

Then $(L(X), \odot, \tau_2)$ is called $LST_0 BCH$-algebra and $(L(X), \odot, \tau_2)$ is called a $LST_1 BCH$-algebra and $LST_2 BCH$-algebra.

**Theorem 3.11:** Let X be a positive implicative TBCH-algebra. If $\{0\}$ is closed in X if and only if $(L(X), \odot, \tau)$ is a $LST_2 BCH$-algebra.

**Proof:** Let $x, y \in X \ast x \neq y$. Then we have $L_x \odot L_y \neq L_0 \ast L_x \odot L_y \neq L_0$.

Assume that, $L_x \odot L_x \neq L_0$, that is $L_x \odot L_x \neq L_0$.

Then there exist some neighbourhood $\Phi(U)$ & $\Phi(V)$ of $L_x \ast L_y$ respectively such that, $\Phi(U \ast V) \subseteq L_0 \ast L_0$.

Clearly, $\Phi(U) \cap \Phi(V) \neq \phi$. Hence, $L(X)$ is $LST_2 BCH$.

Conversely, we claim that, $X \setminus \{0\}$ is open in X. Let $x \in X \setminus \{0\}$.

Then there exist some neighbourhood $\Phi(U) \ast \Phi(V)$ of $L_x \ast L_0$ respectively such that, $\Phi(U) \cap \Phi(V) = \phi$.

$\Rightarrow 0 \in U \Rightarrow U \subseteq X \setminus \{0\} \Rightarrow X \setminus \{0\}$ is open.

Thus $\{0\}$ is closed in X.

**Theorem 3.12:** Let X be a proper positive implicative TBCH-algebra, then the following are equivalent

1. $LST_2 BCH$-algebra
2. $LST_1 BCH$-algebra
3. $LST_0 BCH$-algebra.

**Proof:** From theorem 3.11, $LST_2 BCH \iff LST_1 BCH$. We need only show (1) $\Rightarrow$ (2).

Suppose that $L_0 = L_2 \ast L_2 \neq L_0 \Rightarrow L_0 \ast L_0 = L_0$. 

ISSN: 2231 – 5373 http://www.ijmttjournal.org Page 88
We may assume that $L_{x+y} \neq L_y$.
If there exist an open set $\Phi(U)$ such that $L_{x+y} \in \Phi(U)$ & $L_0 \notin \Phi(U)$, then we can find open neighbourhoods $\Phi(V)$ of $L_x$ & $\Phi(W)$ of $L_y$ such that $\Phi(V * W) \subseteq \Phi(U)$.
Then $L_x \in \Phi(W)$ & $L_y \notin \Phi(V)$.

On the other hand, if there exist an open set $\Phi(U)$ such that $L_{x+y} \in \Phi(U)$ & $L_0 \notin \Phi(U)$, then we can find open neighbourhoods $\Phi(V)$ of $L_x$ & $\Phi(W)$ of $L_y$ such that $\Phi(V * W) \subseteq \Phi(U)$.
Then $L_x \notin \Phi(W)$ & $L_y \in \Phi(V)$.

REFERENCES


IV. CONCLUSION

In this paper, we worked on some algebric properties such as ideals and topological properties such as $T_0, T_1, T_2, T_3, T_4$ in LS topological BCH-algebra. If our BCH-algebra is associative, the same properties are true in RS-topological BCH-algebra, that is set of all right mapping in TBCH-algebras with binary operation $\odot$. Also we extend this paper by using other separation axioms like us $T_2, T_4$ & normal spaces.