Introduction to D-Space & C-Space

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Abstract
To reference in my paper, at first, I would like a text of ‘Introduction to General Topology’ that addressed convergence of a sequence in topological spaces. The concept seems plausible. Quite interested to consider the convergence of real sequence \( \frac{1}{n} \) with respect to different topology on \( \mathbb{R} \). All of us knew that \( \frac{1}{n} \) converges to 0 with respect to usual topology on \( \mathbb{R} \). Here I pursue my work on discussing the space on which \( \frac{1}{n} \) divergent and stronger than usual topology on \( \mathbb{R} \). In consideration of this I am trying to define a new concept of D-Space and C-Space.

Keywords
\(( \mathbb{R}, \tau_1 )\) : Indiscrete Topology
\(( \mathbb{R}, \tau_2 )\) : Discrete Topology
\(( \mathbb{R}, \tau_3 )\) : Co-Finite Topology
\(( \mathbb{R}, \tau_4 )\) : Co-Countable Topology
\(( \mathbb{R}, \tau_5 )\) : Usual Topology
\(( \mathbb{R}, \tau_6 )\) : Upper-Limit Topology
\(( \mathbb{R}, \tau_7 )\) : Lower-Limit Topology
\(( \mathbb{R}, \tau_8 )\) : Scattering Topology
\(( \mathbb{R}, \tau_9 )\) : Ray Topology
\(\emptyset\) : Null Set
\(\in\) : Belongs To
\(\forall\) : For All
\(\exists\) : There Exists

I. INTRODUCTION
As we all knew, sequences can have different behavior with respect to different topological spaces. When we give our attention to Convergence and Divergence, arrives at a new concept of D-Space and C-Space. Consider a sequence \( \{x_n\} \) which is convergent with respect to some topology, it is well known that \( \{x_n\} \) convergent with respect to weaker topology \( \tau' \subseteq \tau \). So we are interested to find out a topology stronger than \( \tau \) and on which the sequence \( \{x_n\} \) is divergent. Here I am discussing some ideas by introducing D-Space and C-Space.

II. TOPOLOGICAL SPACE
Let \( X \) be any set and \( \tau \) be a family of its subset. Then \( \tau \) is said to be a topology if it satisfies the following conditions
1. \( \emptyset, X \in \tau \)
2. \( \tau \) is closed under arbitrary union (ie, if \( A_\alpha \in \tau \) \( \forall \alpha \) then \( \bigcup_\alpha A_\alpha \in \tau \))
3. \( \tau \) is closed under finite intersection (ie, if \( F_i \in \tau \) for \( i = 1, 2, 3, ..., n \) then \( \bigcap_{i=1}^n F_i \in \tau \))

If \( \tau \) is a topology on \( X \) then \( (X, \tau) \) is said to be a topological space and members of \( \tau \) are called open sets in \( (X, \tau) \). Then 1, 2, 3 can be restated as
1. \( \emptyset, X \) are open sets in \( (X, \tau) \).
2. Arbitrary union of open sets is open.
3. Finite intersection of open sets is open.

A. Convergence and Divergence
A sequence \( \{x_n\} \) in a topological space \( (X, \tau) \) is said be convergent and converges to a point \( x \in X \) if for every open set \( U \) in \( X \) with \( x \in U, \exists \) a natural number \( N \) such that \( x_n \in U \forall n \geq N \). Otherwise we say that \( \{x_n\} \) diverges to \( x \).
**B. Remarks**
1. Eventually constant sequences are convergent with respect to any topological space. And converges to repeating term.
2. Any sequence is convergent with respect to indiscrete topology.
3. Eventually constant sequence is the only convergent sequence with respect to discrete topological space.
4. A sequence in co-finite topological spaces is convergent if and only if there is at most one term which repeats infinitely many often.
5. Eventually constant sequence is the only convergent sequence with respect to co-countable topological space.

**III. D- SPACE AND C- SPACE**

1. Let \((X,\tau)\) be any topological space, Let \(\{x_n\}\) be any sequence (other than eventually constant sequence) converges to \(x\) in \((X,\tau)\). Then **D- Space** is a topological space \((X,\tau')\) on which \(\{x_n\}\) is diverges to \(x\) and \(\tau' \subset \tau\). And is denoted by \(D(x_n, \tau, x)\).

2. Let \((X,\tau)\) be any topological space, Let \(\{x_n\}\) be any sequence(other than eventually constant sequence) diverges to \(x\) in \((X,\tau)\). Then **C- Space** is a topological space \((X,\tau)\) on which \(\{x_n\}\) is converges to \(x\) and \(\tau' \subset \tau\). And is denoted by \(C(x_n, \tau, x)\).

*Throughout this paper we will proscribe Eventually Constant Sequence.

**A. Remarks**
1. Indiscrete topological space is a C-space for any sequence.
2. Discrete topological space is a D-space for any sequence.

**B. Convergence with respect to Base**

A sequence \(\{x_n\}\) is converges to \(x\) with respect to Base \(\mathcal{B}\) of \((X,\tau)\) if for every base point \(B\) in \(\mathcal{B}\) with \(x \in B\), \(\exists\) a natural number \(N\) such that \(x_n \in B \forall n \geq N\).

1) **Theorem(a):**

A sequence \(\{x_n\}\) is convergent with respect to \((X,\tau)\) iff the sequence \(\{x_n\}\) is convergent with respect to Base \(\mathcal{B}\) of \((X,\tau)\). .

**Proof:**

**Necessary part**

Suppose \(\{x_n\}\) be any sequence converges to \(x \in X\) with respect to \((X,\tau)\)

Let \(x \in B : B \in \mathcal{B}\)

Since \(B\) is a base point, it is open with respect to \((X,\tau)\).

Hence by definition of convergence with respect to \((X,\tau)\), \(\exists\) a natural number \(N\) such that \(x_n \in B \forall n \geq N\).

**Sufficient part**

Suppose \(\{x_n\}\) is converges to \(x\) with respect to base \(\mathcal{B}\) of \((X,\tau)\).

Let \(U\) be any open set containing \(x\).

By definition of base there exists a base point \(B\) in \(\mathcal{B}\) such that \(x \in B \subset U\).

But then we can find a natural number \(N\) such that \(x_n \in B \forall n \geq N\) (hypothesis)

\(\exists a\) natural number \(N\) such that \(x_n \in U \forall n \geq N\)

Since \(U\) is an arbitrary open set, \(\{x_n\}\) converges to \(x\) with respect to \((X,\tau)\).

**C. Convergence with respect to Sub-Base**

A sequence \(\{x_n\}\) is converges to \(x\) with respect to Sub-Base \(\mathcal{S}\) of \((X,\tau)\) if for every Sub-Base point \(S\) in \(\mathcal{S}\) with \(x \in S\), \(\exists a\) natural number \(N\) such that \(x_n \in S \forall n \geq N\).

1) **Theorem (b):**
A sequence \( \{x_n\} \) is convergent with respect to \((X,\tau)\) iff the sequence \( \{x_n\} \) is convergent with respect to sub base \( \mathcal{S} \) of \((X,\tau)\).

Proof-

**Necessary part**

Suppose \( \{x_n\} \) be any sequence converges to \( x \) with respect to \((X,\tau)\)

Let \( x \in S, S \in \mathcal{S} \)

Since \( S \) is a sub-base point, it is open with respect to \((X,\tau)\).

Hence by definition of convergence with respect to \((X,\tau)\), \( \exists \) a natural number \( N \) such that \( x_n \in S \forall \ n \geq N \).

**Sufficient part**

Suppose \( \{x_n\} \) is converges to \( x \) with respect to sub-base \( S \) of \((X,\tau)\).

Let \( U \) be any open set containing \( x \)

By definition of base there exists a base point \( B \) such that \( x \in B \subset U \).

We know that \( B = S_1 \cap S_2 \cap S_3 \ldots \cap S_n \).

Hence \( x \in S_n \forall \ n \).

But then we can find,

A natural number \( N_1 \) such that \( x_n \in S_1 \forall \ n \geq N_1 \)

A natural number \( N_2 \) such that \( x_n \in S_2 \forall \ n \geq N_2 \)

A natural number \( N_n \) such that \( x_n \in S_n \forall \ n \geq N_n \)

Let \( N = \text{Max}\{ N_1, N_2, \ldots, N_n \} \)

Then we can say that \( x_n \in B = S_1 \cap S_2 \cap S_3 \cap \ldots \cap S_n \forall \ n \geq N \)

\( \text{i.e.,} \exists a \) a natural number \( N \) such that \( x_n \in B \forall \ n \geq N \)

\( \text{i.e.,} \exists a \) a natural number \( N \) such that \( x_n \in U \forall \ n \geq N \)

Since \( U \) is an arbitrary open set \( \{x_n\} \) converges to \( x \) with respect to \((X,\tau)\).

**D. Remarks**

1. We can always find a C-Space corresponding to any sequence.

Proof-

We knew that any collection \( \mathcal{S} \) generates a topology.

And \( \mathcal{S} \) will be a sub base for that topology.

Let us take any collection, on which the sequence is convergent,

Then by Theorem (b) we can find a C-Space

2. We can always find a D-Space corresponding to any sequence.

Proof-

Consider a sequence \( \{x_n\} \) converges to \( x \).

We knew that any collection \( \mathcal{S} \) generates a topology.

And \( \mathcal{S} \) will be a sub base for that topology.

Let \( \mathcal{S} = \tau \cup \{x\} \), and \( \tau' \) be the topology generated by \( \mathcal{S} \).

Clearly \( \tau \subset \tau' \)..........................(1)

Since \( \{x\} \) is a sub base point, \( \{x_n\} \) diverges to \( x \) with respect to sub base \( \mathcal{S} \) of \( \tau' \).

\( \therefore \) By Theorem (b)

\( \{x_n\} \) diverges to \( x \) with respect to \( \tau' \) ...........(2)

(1) And (2) implies that \( \tau' \) is a D-Space.

**E. Examples**

1. Consider the sequence \( \{1/n\} \), Then

\( \circ \) \( D(1/n, \tau_1, 0) = (R, \tau_6) \)

\( \circ \) \( D(1/n, \tau_2, 0) = (R, \tau_8) \)

\( \circ \) \( D(1/n, \tau_2, 0) = (R, \tau_2) \)
2. Consider the sequence \( \{n\} \), Then

\[
\begin{align*}
&\circ D(1/n, \tau_4, 0) = (\mathbb{R}, \tau_3) \\
&\circ D(1/n, \tau_5, 0) = (\mathbb{R}, \tau_6) \\
&\circ D(1/n, \tau_6, 0) = (\mathbb{R}, \tau_5) \\
&\circ D(1/n, \tau_7, 0) = (\mathbb{R}, \tau_2) \\
&\circ D(1/n, \tau_8, 0) = (\mathbb{R}, \tau_2) \\
&\circ D(1/n, \tau_9, 0) = (\mathbb{R}, \tau_6)
\end{align*}
\]

IV. CONCLUSIONS

1. We can always find a C-space corresponding to any sequence.
2. We can always find a D-space corresponding to any sequence.

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REFERENCE