On The Lattice of Trail Sets of a Connected Graph

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Abstract— In this paper Trail set is defined for a finite connected graph G and it is found that the set of all trail sets $\tau(G)$ together with empty set partially ordered by set inclusion relation forms a lattice. Also derived graph of G, denoted by $G_d$ is defined such that lattices $\tau(G)$ and $\tau(G_d)$ are isomorphic. Some of the properties of the lattices so obtained are studied. The definition of trail sets is extended to directed graphs and studied.

Keywords — connected graph, lattice, tree, pendant vertex, Connected digraph, Path, Circuit.

I. INTRODUCTION

In many research papers, lattices are constructed using graphs and their properties are studied. In fact in [9], it is shown that the set of all convex subgraphs of a directed graph G together with empty set partially ordered by set inclusion relation forms a complete semimodular, A-regular lattice. In [7], it is shown that the set of all convex sets of a finite connected graph G together with empty set partially ordered by set inclusion relation forms a lattice. In [8], lattice of path sets of a connected graph is discussed. In [1], [2], lattice of convex edge sets of a connected directed graph is discussed.

Motivated by the above studies, in this paper we have defined trail sets of a finite connected graph G and found that the set of all trail sets of G together with empty set partially ordered by set inclusion relation forms a lower semi modular lattice. In section 2 after introducing some basic concepts and notations we have shown that the set of all trail sets of a finite connected graph G together with empty set forms a lattice with respect to the partial order set inclusion and is denoted by $<\tau(G), \subseteq>$. In section 3 some of the properties of such lattices are studied.

In section 4, derived graph $G_d$ is defined such that $<\tau(G), \subseteq>$ and $<\tau(G_d), \subseteq>$ are isomorphic. Also we found that for any connected graph G, $G_d$ is a tree. Some of the properties of $<\tau(G_d), \subseteq>$ are also studied.

In section 5, the definition of trail sets is extended to directed graphs and it is shown that the set of all trail sets of a finite connected digraph G together with empty set also forms a lattice with respect to the partial order set inclusion. The conditions under which $<\tau(G), \subseteq>$ forms a chain and $<\tau(G), \subseteq>$ is a complemented lattice are studied for a connected digraph G. Necessary and sufficient condition for an element A in $<\tau(G), \subseteq>$ to be doubly irreducible is established. Also it is observed that for a digraph G there is unique $<\tau(G), \subseteq>$ depending upon its underlying graphs.

II. PRELIMINARIES

A walk of a graph G is an alternating sequence of vertices and edges $v_0,e_1,v_1,…,v_n,e_n,v_n$ beginning and ending with vertices, in which each edge is incident with the two vertices immediately preceding and following it. This walk joins $v_0$ and $v_n$, and may also be denoted by $v_0,v_1,…,v_n$, called $v_0$-$v_n$ walk. It is closed if $v_0=v_n$ and is open otherwise. An open walk in which no edge is repeated is a trail. A closed walk in which no edge is repeated is a circuit. A circuit containing all the vertices of G is called a spanning circuit. A walk in which no vertex is repeated is a path. A closed path is called a cycle. A path containing n vertices is denoted by $P_n$. A graph is acyclic, if it has no cycles. A tree is a connected acyclic graph.

An element ‘a’ of a lattice L is join irreducible if $a=b \lor c$ implies that $a=b$ or $a=c$. ‘a’ is meet irreducible if $a=b \land c$ implies that $a=b$ or $a=c$. An element which is both meet and join irreducible is called doubly irreducible. A lattice L is defined such that the lower covering condition if for $a,b \in L$ $a \land b < b$ implies $a < a \lor b$. A lattice L is lower semimodular(LSM) if $a \lor b$ covers both a and b implies that both a and b cover $a \land b$.
Let G be a finite connected graph. V(G) be the vertex set of G. A set T \subseteq V(G) is said to be trail set in G if it satisfies the following two conditions (i) for every two vertices u, v \in T, the vertex set of every u-v trail is contained in T, (ii) for every vertex v \in T, if there exist a circuit containing v, then the set of all vertices of a maximal circuit containing v is contained in T.

For a finite connected graph G, let the set of all trail sets in G together with empty set be denoted by \( \tau(G) \). Define a binary relation \( \leq \) on \( \tau(G) \) by, for A, B \in \tau(G), A \leq B if and only if A \subseteq B. Then clearly \( \leq \) is a partial order on \( \tau(G) \). Moreover \( < \tau(G), \leq > \) forms a lattice where for A, B \in \tau(G), A \wedge B = A \cap B and A \vee B = < A \cup B >, where < A \cup B > is the trail set generated by A \cup B or equivalently the smallest trail set containing A \cup B.

For example, the lattice given in Fig 2.2 represents the lattice \( < \tau(G), \leq > \) of the connected graph G given in Fig 2.1.

Throughout this paper we consider G as a non trivial graph and we use the notation \( \tau(G) \) to represent the lattice \( < \tau(G), \leq > \). The undefined terms and notations used in this paper are from [3], [4] and [5].

### III. ON THE LATTICE \( \tau(G) \)

**Remark 3.1:** The smallest element of \( \tau(G) \) is empty set and the largest element is V(G).

**Theorem 3.2:** A singleton set \( \{v_i\} \) is a trail set if and only if \( v_i \) does not belong to a circuit.

Proof: Let A = \( \{v_i\} \) be a trail set. If \( v_i \) belong to a circuit, then all the vertices of that circuit also belong to A, a contradiction. Conversely, If \( v_i \) does not belong to a circuit, then A = \( \{v_i\} \) is a trail set.

**Remark 3.3:** \( \tau(G) \) is atomic where atoms are as follows.

(i) Trail sets containing single vertex.

(ii) Trail sets containing all vertices of maximal circuits of G.

**Theorem 3.4:** \( \tau(G) \) is a two element chain if and only if either G is P_2 or G contains a spanning circuit.

Proof: Let \( \tau(G) \) be a two element chain. Then there are only two trail sets \( \emptyset \) and \( V(G) \) such that \( \emptyset < V(G) \), which implies \( V(G) \) is an atom. Therefore G contains a spanning circuit. Conversely, let G contains a spanning circuit, then \( \emptyset, V(G) \) are the only two trail sets, such that \( \emptyset < V(G) \). Thus \( \tau(G) \) is a two element chain.

**Remark 3.5:** \( \tau(G) \) is a two element chain for all graphs containing spanning circuit. Moreover \( \tau(G) \) is a chain only in this case. In fact if G does not contain spanning circuit, then G contains at least one bridge. Therefore \( \tau(G) \) will contain at least two atoms. Thus \( \tau(G) \) will contain a sublattice as shown in Fig 3.1. Hence we can conclude that \( \tau(G) \) is a chain if and only if G has a spanning circuit.
Example 3.6: We note that the graphs shown in figures 2.1 and 3.2 are not isomorphic but their lattices shown in figures 2.2 and 3.3 are isomorphic.

Theorem 3.7: \( G \) contains \( n-1 \) bridges if and only if \( \tau(G) \) contains \( n \) atoms.

Proof: Let \( G \) contains \( n-1 \) bridges. Since \( G \) is a connected graph, the removal of these \( n-1 \) bridges results into a disconnected graph with \( n \) components. Each component is either a single vertex or a circuit. Therefore trail sets of vertices of these components form atoms of \( \tau(G) \). Hence \( \tau(G) \) contains \( n \) atoms. Conversely if \( \tau(G) \) contains \( n \) atoms, then each atom is either a set with single vertex or a set containing vertices of a circuit. But then since \( G \) is connected, clearly \( G \) must contain \( n-1 \) bridges.

IV. DERIVED GRAPH

Let \( G \) be a connected graph with \( n-1 \) bridges, say \( e_1, e_2, \ldots, e_{n-1} \) and \( G^1 \) be the graph obtained from \( G \) by deleting all bridges of \( G \). Then \( G^1 \) is a disconnected graph with \( n \) components. These components are either single vertices or sub graphs of \( G \) containing circuit. Let us denote the vertex set of components as \( T_1, T_2, \ldots, T_n \). These \( n \) components are connected by \( n-1 \) edges which are bridges of \( G \). If there is an edge connecting one vertex of \( T_1 \) to one vertex of \( T_j \), then they are adjacent. Let \( G_d \) be the graph with vertex set \( \{T_1, T_2, \ldots, T_n\} \) and edge set \( \{e_1, e_2, \ldots, e_{n-1}\} \). We call \( G_d \) as the derived graph of \( G \), which is a connected acyclic graph with \( n \) vertices and \( n-1 \) edges. Hence it is a tree.

For example, the graph given in Fig 4.2 represents the derived graph \( G_d \) of the graph \( G \) given in Fig 4.1.

We use the notation \( \tau(G_d) \) to represent the lattice \( \langle \tau(G_d) \rangle \subseteq \).

Remark 4.1: If \( G \) itself is a tree, then \( G \) is isomorphic to \( G_d \).
**Remark 4.2:** Let $G_d$ be the derived graph of $G$ and $\mathcal{T}(G_d)$ be the lattice of trail sets of $G_d$. Then the lattices $\mathcal{T}(G)$ and $\mathcal{T}(G_d)$ are isomorphic.

**Remark 4.3:** Following statements are equivalent.
1) G has spanning circuit
2) $G_d$ is a trivial graph
3) $\mathcal{T}(G)$ is a two element chain.

**Theorem 4.4:** $\mathcal{T}(G)$ is planar if and only if $G_d$ is a path.

**Proof:** Let $\mathcal{T}(G)$ be planar. If $G_d$ is not a path, then there exist a vertex with degree at least three as shown in Fig 4.3. Then $\mathcal{T}(G)$ contains a subposet as shown in Fig 4.4, which implies $\mathcal{T}(G)$ cannot be planar (see [6]).

Conversely, if $G_d$ is a path $T_1T_2...T_n$ as shown in Fig 4.5, then $\mathcal{T}(G)$ will be as shown in Fig 4.6 and hence it is planar.

**Theorem 4.5:** An element $A \in \mathcal{T}(G_d)$ is doubly irreducible if and only if $A = \{T\}$ where $T$ is a pendant vertex of $G_d$.

**Proof:** Let $A \in \mathcal{T}(G_d)$ be doubly irreducible. If $A = \{T\}$ where $T$ is a vertex with two edges incident on it, say $TT_1$ and $TT_2$. Then $A = \{T, T_1\} \cup \{T, T_2\}$, a contradiction. On the other hand if $A$ contains more than one element, say $A = \{T_1, T_2, ..., T_n\} \in \mathcal{T}(G_d)$, then $A = \bigvee_{i=1}^{n} \{T_i\}$. Therefore $A = \{T_i\}$ for some $i$, since $A$ is join irreducible.

Conversely, for any vertex $T$ in $G_d$, $A = \{T\} \in \mathcal{T}(G_d)$ is an atom of $\mathcal{T}(G_d)$. Therefore $A$ is join irreducible. If $A$ is meet reducible, say $A = B \wedge C = B \cap C$ for some $B, C \in \mathcal{T}(G_d)$ such that $A \neq B, A \neq C$, then $T \in B \cap C$. Consider $U \in B, V \in C$ where $T \neq U, T \neq V$. Let $T, U_1, U_2, ..., U$ be a
shortest path connecting T and U in B. Also let \( T, V_1, V_2, \ldots, V \) be a shortest path connecting T and V in C. If \( U_1 = V_1 \), then \( U_1 \in B \cap C \), contradiction to \( B \cap C = \{ T \} \). Also, if \( U_1 \neq V_1 \), then \( \{ T, U_1 \} \) and \( \{ T, V_1 \} \) are two edges incident on T, contradiction to T is pendant vertex of \( G_d \). Hence \( A \) must be irreducible.

**Theorem 4.6:** For any connected graph \( G \), \( \tau(G) \) is lower semimodular.

**Proof:** Let \( G_d \) be the derived graph of \( G \), with vertex set \( \{ T_1, T_2, \ldots, T_n \} \).

Since \( G_d \) is a tree, if \( A \cup B \) covers both A and B, then \( A, B, A \cup B \) are of the form \( \{ T_1, T_2, \ldots, T_k \} \), \( \{ T_1, T_2, \ldots, T_k, T_m \} \), respectively. Where \( \{ T_1, T_2, \ldots, T_k \} \) are common elements in A and B. Clearly \( A \cap B = \{ T_1, T_2, \ldots, T_k \} \) and it is covered by both \( A \) and \( B \). Hence \( \tau(G) \) lower semi modular.

**Remark 4.7:** The following statements are equivalent.

i) \( G \) is a graph with single bridge.

ii) \( G_d \) is \( P_2 \).

iii) \( \tau(G) \) will be of the form as shown in Fig 4.7

\[
\begin{align*}
\{T_1, T_2\} \\
\{T_1\} \\
\{T_2\} \\
\{T_1, T_2, T_3\}
\end{align*}
\]

Fig 4.7

\[
\begin{align*}
\{T_1, T_2, T_3\} \\
\{T_1, T_2\} \\
\{T_2, T_3\} \\
\{T_1\} \\
\{T_2\} \\
\{T_3\} \\
\{\emptyset\}
\end{align*}
\]

Fig 4.8

**Remark 4.8:** The following statements are equivalent.

i) \( G \) is a graph with two bridges.

ii) \( G_d \) is \( P_3 \).

iii) \( \tau(G) \) will be of the form as shown in Fig 4.8.

\[
\begin{align*}
\{T_1, T_2, T_3\} \\
\{T_1, T_2\} \\
\{T_2, T_3\} \\
\{T_1\} \\
\{T_2\} \\
\{T_3\} \\
\{\emptyset\}
\end{align*}
\]

Fig 4.8

**Theorem 4.9:** \( \tau(G) \) contains \( n \) atoms if and only if \( G \) contains \( n \) - 1 bridges.

**Proof:** If \( \tau(G) \) contains \( n \) atoms, then \( \tau(G_d) \) also contains \( n \) atoms. Therefore \( G_d \) is a tree with \( n \) vertices and hence it contains exactly \( n \) - 1 edges which are nothing but bridges of \( G \). Conversely, if \( G \) contains \( n \) - 1 bridges, then \( G_d \) contains \( n \) vertices. Therefore \( \tau(G_d) \) as well as \( \tau(G) \) will contain \( n \) atoms.

**Theorem 4.10:** A chain containing more than two elements cannot be realized as the lattice of trail sets.

Proof: If \( G_d \) is a trivial graph, then \( \tau(G_d) \) is a two element chain. If \( G_d \) is not a trivial graph, then \( G_d \) contains at least one edge. Therefore \( \tau(G_d) \) contains a subposet as shown in Fig 4.7.

**Theorem 4.11:** A lattice shown in Fig 4.9 cannot be realized as a lattice of trail sets [for \( n \geq 3 \)]

**Proof:** For \( n \geq 3 \), \( G_d \) will contain a subgraph \( P_3 \). Therefore \( \tau(G_d) \) will contain a subposet as shown in Fig 4.8. Thus lattice shown in Fig 4.9 cannot be realized as a lattice of trail sets [for \( n \geq 3 \)].
Remark 4.12: Lattices shown in Fig 4.10, Fig 4.11 cannot be realizable as lattice of trail sets as they are not lower semi modular.

Theorem 4.13: Let T be any vertex of Gd. Degree of T = k in Gd if and only if \{T\} is covered by k elements in \(\tau(G_d)\).

Proof: If degree of T = k, then there are k edges incident on T. Let \(T_1, T_2, ..., T_k\) be the adjacent vertices of T. Then \(\{T, T_1\}, \{T, T_2\}, ..., \{T, T_k\}\) are the elements which cover \{T\}.

Conversely, Let \{T\} be covered by k elements, say \(\{T, T_1\}, \{T, T_2\}, ..., \{T, T_k\}\). Therefore vertex T is incident with k edges. Hence degree of T = k

Theorem 4.14: \(\tau(G_d)\) is complemented if and only if either Gd is a single vertex (trivial graph) or Gd is \(P_2\).

Proof: Let \(\tau(G_d)\) be complemented. If Gd is not a single vertex or \(P_2\), then Gd must contain \(P_3\) as a subgraph. Let \(T_1, T_2, T_3\) be the subgraph of Gd. Then \(\{T_1\}\) is not complemented. And hence \(\tau(G_d)\) is not complemented, a contradiction. Conversely, if Gd is a trivial graph, then \(\tau(G_d)\) is a two element chain. If Gd is \(P_2\), then \(\tau(G_d)\) will be as shown in Fig 4.7. Clearly both are complemented.

Theorem 4.15: \(l(\tau(G_d)) = |V(G_d)|\)

Proof: Let \(V(G_d) = \{T_1, T_2, ..., T_n\}\). Then the longest chain in \(T(G_d)\) is \(\emptyset < \{T_1\} < \{T_1, T_2\} < ... < \{T_1, T_2, ..., T_{n-1}\} < \{T_1, T_2, ..., T_n\}\) which is of length n.

Theorem 4.16: If Gd is \(P_n\), then \(|\tau(G_d)| = \frac{n(n+1)}{2} + 1\)

Proof: Let Gd be a path with n vertices. Then there are n trail sets with single element, n-1 trail sets with two elements and so on continuing like this finally one trail set with n elements. Including empty set, \(|\tau(G_d)| = n + (n - 1) + (n - 2) ... + 1 + 1 = \frac{n(n+1)}{2} + 1\)

Theorem 4.17: Following statements are equivalent for any derived graph Gd.

1. Gd is \(P_1\) or \(P_2\).
2. \(\tau(G_d)\) is distributive.
3. \(\tau(G_d)\) is modular.
4. \(\tau(G_d)\) satisfies lower covering condition.

Proof: (1) \(\Rightarrow\) (2): If Gd is \(P_1\), then \(\tau(G_d)\) is a two element chain. If Gd is \(P_2\) then \(\tau(G_d)\) is of the form as shown in Fig 4.7. Both are distributive.
(2) ⇒ (3) ⇒ (4) is true for all lattices.
To prove (4) ⇒ (1) : Let \( \tau(G_d) \) satisfy lower covering condition. If \( G_d \) is not \( P_1 \) or \( P_2 \), then \( G_d \) contains \( P_3 \), say \( T_j, T_k \). Clearly \( \emptyset = \{ T_j \} \land \{ T_k \} < \{ T_i \} \). But \( \{ T_k \} < \{ T_j, T_k \} < \{ T_i \} \lor \{ T_k \} \). Contrdiction to \( \tau(G_d) \) satisfies lower covering condition. Hence \( G_d \) must be \( P_1 \) or \( P_2 \).

**Remark 4.18:** \( \tau(G_d) \) is dually atomic. Dual atom \( D_i = V(G_d) - T_i \), where \( T_i \) is a pendant vertex of \( G_d \). Thus the number of dual atoms of \( \tau(G_d) \) is the number of pendant vertices of \( G_d \).

**Remark 4.19:** For any graph \( G \), \( \tau(G) \) satisfies Jordan dedekind chain condition since it is lower semi modular.

V. ON THE LATTICE OF TRAIL SETS OF A CONNECTED DIRECTED GRAPH

A (directed) walk in a digraph is an alternating sequence of vertices and edges \( v_0, e_1, v_1, ..., e_n, v_n \) in which each edge \( e_i \) is \( v_{i-1}v_i \). A closed walk has the same first and last vertices, and a spanning walk contains all the vertices. A path is a walk in which all vertices are distinct, and a closed path is called a cycle. If there is a path from \( u \) to \( v \), then \( v \) is said to be reachable from \( u \). Each walk is directed from the first vertex \( v_0 \) to the last vertex \( v_n \). A semiwalk is an alternating sequence \( v_0, e_1, v_1, ..., e_n, v_n \) of vertices and edges in which each edge \( e_i \) is \( v_{i-1}v_i \) or \( v_{i+1}v_i \). Similarly a semipath, semicycle are also defined.

A digraph is strongly connected, if every two vertices are mutually reachable, it is unilaterally connected, if for any two points at least one is reachable from the other, and it is weakly connected if every two points are joined by a semipath. A digraph is disconnected if it is not even weak. For any digraph \( G \), the undirected graph obtained by removing the directions is called it’s underlying graph.

Let \( G \) be a connected directed graph. Let \( E(G) \) be the edge set of \( G \). A subset \( A \) of \( E(G) \) is said to be Trail set of \( G \) if it satisfies the following conditions,

i) For every \( e_i \in A \), all edges which lie on the trail connecting end vertices of \( e_i \in A \). Also if there is any circuit from any end vertex of \( e_i \) to itself, then all the edges of that circuit belong to \( A \).

ii) For any trail \( e_i, e_j, ..., e_k \in A \) all the edges connecting end vertices of this trail belong to \( A \) and also all edges which connect end vertices of a trail contained in the above trail belong to \( A \).

Let \( \tau(G) \) be the set of all trail sets of \( G \) together with the empty set. Define a binary relation \( \leq \) on \( \tau(G) \) by, for \( A, B \in \tau(G) \), \( A \leq B \) if and only if \( A \subseteq B \). Then clearly \( \leq \) is a partial order on \( \tau(G) \). Moreover \( \tau(G) \), \( \leq \) forms a lattice where for \( A, B \in \tau(G) \), \( A \land B = A \cap B \) and \( A \lor B = \{ A \cup B \} \) is the smallest convex edge set containing \( A \cup B \).

For example, the lattice given in Fig 5.2 represents the lattice \( \tau(G) \), \( \leq \) of the connected digraph \( G \) given in Fig 5.1.

Hereafter we use \( \tau(G) \) to denote \( < \tau(G), \leq > \) where \( G \) is a connected directed graph.

**Remark 5.1:** The smallest element of \( \tau(G) \) is empty set and the largest element is \( E(G) \).

**Remark 5.2:** \( \tau(G) \) is atomic where atoms are as follows

(i) sets containing edges of a maximal strongly connected subgraphs of \( G \).

(ii) singleton sets \( \{ e_i \} \) if there is no directed trail connecting end vertices of \( e_i \) and also there is no directed circuit from any end vertex of \( e_i \) to itself.
Theorem 5.3: \( \tau(G) \) is a two element chain if and only if either \( G \) is a digraph containing single edge or \( G \) is a strongly connected digraph.

**Proof:** Let \( \tau(G) \) be a two element chain, then \( \emptyset \) and \( E(G) \) are the only trail sets in \( G \) such that \( \emptyset < E(G) \). Therefore \( E(G) \) itself is an atom of \( \tau(G) \). Hence \( G \) is a digraph containing a single edge or \( G \) is a strongly connected digraph.

Conversely, let \( G \) be a strongly connected digraph. Then every pair of vertices are connected by two directed trails. Therefore \( \emptyset \) and \( E(G) \) are the only trail sets in \( G \) such that \( \emptyset < E(G) \).

**Corollary 5.4:** If \( G \) is as given in theorem 5.3, then \( |\tau(G)| = 2, l(\tau(G)) = 1 \) and \( \tau(G) \) is a Boolean algebra.

Theorem 5.5: \( \tau(G) \) is a three element chain if and only if \( G = S \cup A \) where \( S \) is the maximal strongly connected subgraph of \( G \) and \( A \) is a nonempty set of edges whose initial(terminal) vertices belong to \( S \) and they share a common terminal(initial) vertex.

**Proof:** Let \( \tau(G) \) be a three element chain of the form \( \emptyset < A_1 < E(G) \). Where \( A_1 \) is either singleton set, say \( A_1 = \{e_1\} \) or set of edges of a maximal strongly connected subgraph of \( G \). If \( A_1 = \{e_1\} \) is an atom and \( A_1 < E(G) \), then \( \tau(G) \) contains a subposet as shown in Fig 5.4. Therefore \( A_1 \) must be a set of edges of a maximally strongly connected subgraph of \( G \). Also \( G = A_1 \cup A \) where \( A \) is a nonempty set of edges whose initial(terminal) vertices belong to \( A_1 \) and they share a common terminal(initial) vertex. We take \( A_1 = S \).

Conversely, let \( G = S \cup A \) where \( S \) is the maximal strongly connected subgraph of \( G \) and \( A \) is a nonempty set of edges whose initial(terminal) vertices belong to \( S \) and they share a common terminal(initial) vertex. Then \( \emptyset, E(S) \) and \( E(G) \) are the only trail sets in \( G \) such that \( \emptyset < E(S) < E(G) \).

**Corollary 5.6:** If \( G \) is as given in theorem 5.5, then \( |\tau(G)| = 3, l(\tau(G)) = 2 \) and \( \tau(G) \) is a Boolean algebra.

Theorem 5.7: Depending on the nature of underlying graph, \( \tau(G) \) will be as follows.

Figure 5.4, 5.6, 5.8, 5.10, 5.12 represent lattices \( \tau(G) \) corresponding to underlying graphs Figures 5.3, 5.5, 5.7, 5.9, 5.11 respectively.
**Theorem 5.8:** If underlying graph of digraph \( G \) is of the form as shown in Fig 5.13, then \( \tau(G) \) is a Boolean algebra.

**Proof:** If \( G \) is of the form as shown in Fig 5.13, then \( \tau(G) \) is a lattice for \( A, B \in \tau(G) \).

\[ A \wedge B = A \cap B \quad \text{and} \quad A \vee B = A \cup B. \]

Clearly \( \tau(G) \) is a complemented distributive lattice. Hence \( \tau(G) \) is a Boolean algebra.

**Theorem 5.9:** If \( G \) is a directed path with \( n \) edges, then

(i) \[ |\tau(G)| = \frac{n(n+1)}{2} + 1 \]

(ii) \[ l(\tau(G)) = n. \]

**Proof:** (i) If \( G \) is a directed path with \( n \) edges, then there are \( n \) trail sets with single edge, \( n - 1 \) trail sets with two edges, \( n - 2 \) trail sets with three edges and so on, finally one trail set with \( n \) edges. Including empty set,

\[ |\tau(G)| = n + (n - 1) + (n - 2) + \cdots + 1 + 1 = \frac{n(n+1)}{2} + 1 \]

(ii) Let \( e_1, e_2, \ldots, e_n \) be the directed path.

Then \( \emptyset < \{e_1\} < \{e_1, e_2\} < \cdots < \{e_1, e_2, \ldots, e_n\} \) is the maximum chain.

Hence \( l(\tau(G)) = n \)

**Theorem 5.10:** Let \( G \) be a digraph containing Euler trail. An element \( A \in \tau(G) \) is doubly irreducible if and only if \( A = \{e_i\} \) where \( e_i \) is a pendant edge without containing circuits at its end vertex OR \( A = S \), where \( S \) is a set of edges of a maximal strongly connected subgraph of \( G \) such that \( A \) is adjacent to at most one vertex of \( G \).

**Proof:** Let \( G \) be a digraph containing Euler trail. Let \( A \in \tau(G) \) be doubly irreducible. Then \( A = \{e_i\} \) or \( A = S \). Otherwise \( A \) will be join reducible.

If \( A = \{e_i\} \) where \( e_i \) is a pendant edge containing circuits at its end vertex, then \( A = \{e_i\} \) can not be a trail set. If \( A = \{e_i\} \) is not a pendant edge, then either \( e_i \) belongs to a directed circuit or there exist a directed path \( e_j e_i e_k \). If \( e_i \) belongs to a directed circuit, then \( A = \{e_i\} \) can not be a trail set. If there exist a directed path \( e_j e_i e_k \), then \( \{e_i\} = \langle e_j, e_i \rangle \land \langle e_k, e_i \rangle \) is a contradiction. Let \( A = S \), where \( S \) is a set of edges of a maximal strongly connected subgraph of \( G \). If \( A \) is adjacent to two or more vertices of \( G \) say \( e_j, e_k \) be the corresponding edges, then \( A = \langle A \cup \{e_j\} \rangle \land \langle A \cup \{e_k\} \rangle \), a contradiction.

Conversely, if \( A = \{e_i\} \) where \( e_i \) is a pendant edge without containing circuits at its end vertex OR \( A = S \), where \( S \) is a set of edges of a maximal strongly connected subgraph of \( G \) such that \( A \) is adjacent to at most one vertex of \( G \), then \( A \) is an atom of \( \tau(G) \). Therefore \( A \) is join irreducible. Let \( A = \{e_i\} \) where \( e_i \) is a pendant edge without containing circuits at its end vertex. If \( A \) is meet reducible say \( A = B \land C = B \cap C \) for some \( B, C \in \tau(G) \), then \( e_i \in B \cap C \). Consider \( e_j \in B \), \( e_k \in C \) where \( e_j \neq e_i \), \( e_k \neq e_i \). Let \( e_i f_1 f_2 \cdots e_j \) be the shortest path connecting \( e_i, e_j \in B \). Also let \( e_i g_1 g_2 \cdots e_k \) be the shortest path connecting \( e_i, e_k \in C \). If \( f_1 = g_1 \), then \( f_1 \in B \cap C \) a contradiction to \( B \cap C = \{e_i\} \). Also if \( f_1 \neq g_1 \), then \( e_i \) contains circuit at its end vertex since \( G \) contains an euler trail a contradiction. If \( A = S \) and \( A = B \land C = B \cap C \) for some \( B, C \in \tau(G) \) such that \( A \neq B, A \neq C \), then \( A \) is adjacent to atleast two vertices. Hence a contradiction.
Theorem 5.11 Let G be a directed graph containing Euler trail. Then $\tau(G)$ is complemented if and only if G is any of the following three forms.

i) G contains a spanning circuit.

ii) G is a directed path containing atmost two edges.

iii) G is of the form as shown in Fig 5.14, Fig 5.15 where $C_1$ and $C_2$ are directed circuits.

Proof: Let $\tau(G)$ be complemented. If G is not of the form as given in the theorem, then either G contains a directed path containing three edges $e_j \in E(G)$ OR $G = S \cup A$ where S is the maximal strongly connected subgraph of G and A is a nonempty set of edges whose initial(terminal) vertices belong to S and they share a common terminal(initial) vertex. If G contains a directed path containing three edges $e_j \in E(G)$ then $\tau(G)$ is not complemented. If $G = S \cup A$, then $\tau(G)$ is a three element chain. Hence $\tau(G)$ is not complemented.

Conversely, if G contains a spanning circuit or G is a directed path containing one edge, then $\tau(G)$ is a two element chain. If G is a directed path with two edges or of the form as shown in Fig 5.14 where $C_1$ and $C_2$ are directed circuits, then $\tau(G)$ is of the form as shown in Fig 5.15. If G is of the form as shown in Fig 5.15, then $\tau(G)$ is of the form as shown in Fig 5.16. In all these cases $\tau(G)$ is complemented. Hence the result.

REFERENCES


[5] Harary F: Graph theory, Addison-wesley 1969


