Growth Properties of Composition of Two Meromorphic Functions

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Abstract

In this paper, we have proved few important results on relative growth properties of entire functions, meromorphic functions and their compositions.

Key Words : Entire function, Order of entire function, Meromorphic function, Order of meromorphic function, Relative order of entire function, Relative order of meromorphic function.


1 Introduction

Meromorphic function is a function whose singularities are only poles in the finite plane and an entire function is a function which is analytic in the entire finite complex plane.

The maximum modulus of an entire function $f(z)$ is defined by

$$M_f(r) = \sup\{|f(z)| : |z| = r\}$$

If $f$ is non constant then $M_f(r)$ is strictly increasing and continuous function of $r$ and the inverse function

$$M_f^{-1} : (|f(0)|, \infty) \rightarrow (0, \infty)$$

exists and $\lim_{r \rightarrow \infty} M_f^{-1}(r) = \infty$

**Definition 1.1** The order of an entire function $f$ is defined as

$$\rho_f = \limsup_{r \rightarrow \infty} \frac{\log \log M_f(r)}{\log r}$$
Definition 1.2 If $f$ and $g$ are two entire functions then the relative order of $f$ with respect to $g$ is defined as

$$
\rho_g(f) = \inf\{\mu > 0 : M_f(r) < M_g(r^{\mu}) \text{ for all } r > r_0(\mu) > 0\}
$$

$$
= \limsup_{r \to \infty} \frac{\log M_g^{-1} M_f(r)}{\log r}
$$

Now let $h(z)$ be a non constant meromorphic function in the complex plane $\mathbb{C}$. Let us denote the number of roots of the equation $h(z) = a$ in $|z| \leq r$, with due count of multiplicity by $n(r, a)$ for any complex number $a$ and number of poles of $h(z)$ in $|z| \leq r$ by $n(r, \infty)$ or $n(r, h)$. Let us take

$$
N(r, a) = \int_{0}^{r} \frac{|n(t, a) - n(0, a)|}{t} dt + n(0, a) \log r,
$$

$$
N(r, h) = \int_{0}^{r} \frac{n(t, h)}{t} dt
$$

$$
N(r, \frac{1}{h}) = \int_{0}^{r} \frac{n(t, \frac{1}{h})}{t} dt
$$

$$
m(r, h) = \frac{1}{2\pi} \int_{0}^{2\pi} \log^+ |h(re^{i\theta})| d\theta
$$

where $\log^+ x = \max\{0, \log x\}$ for all $x > 0$

Now we write

$$
T_h(r) = T(r, h) = m(r, h) + N(r, h)
$$

(1.1)

Thus we understand that $m(r, h)$ is a sort of averaged magnitude of $\log |h|$ on arcs of $|z| = r$ where $|h|$ is large. The term $N(r, h)$ relates to the number of poles. The function $T_h(r)$ is called the characteristic function of the meromorphic function $h(z)$.

2 Definitions and Lemmas

In this section we state few important definitions and important lemmas.

Definition 2.1 The order of a meromorphic function $h$ is defined as

$$
\rho_h = \limsup_{r \to \infty} \frac{\log T_h(r)}{\log r}
$$
Definition 2.2 The relative order of a meromorphic function $h$ with respect to an entire function $f$ is defined as [6]
\[ \rho_f(h) = \inf \{ \lambda > 0 : T_h(r) < T_f(r^\lambda) \text{ for all } r > r_0(\lambda) > 0 \} \]
\[ = \limsup_{r \to \infty} \frac{\log T_f^{-1}T_h(r)}{\log r} \]

Definition 2.3 The relative order of meromorphic function $f$ with respect to another meromorphic function $h$ ([1], [2]) is defined as
\[ \rho_h(f) = \limsup_{r \to \infty} \frac{\log T_f(r)}{\log T_h(r)} \]

Lemma 2.1 (P.18 [5]) Let $g$ be an entire function then for all large $r$
\[ T_g(r) \leq \log M_g(r) \leq 3T_g(2r) \] (2.2)

Lemma 2.2 ([8]) Let $f$ and $g$ be two entire functions. Then for a sequence of values of $r$ tending to infinity
\[ T_{fof}(r) \geq \frac{1}{3} \log M_f(\frac{1}{8} M_g(\frac{r}{4}) + o(1)) \] (2.3)

Lemma 2.3 ([4]) Let $f$ and $g$ be two entire functions. Then for all sufficiently large values of $r$
\[ M_f(\frac{1}{8} M_g(\frac{r}{2}) - |g(0)|) \leq M_{fof}(r) \leq M_f(M_g(r)) \] (2.4)

Lemma 2.4 [3] Let $g$ be an entire function and $\alpha > 1$, $0 < \beta < \alpha$. Then for all large $r$
\[ M_g(\alpha r) > \beta M_g(r) \] (2.5)

Lemma 2.5 [3] Let $g$ be an entire function with property (A). Then for any positive integer $n$ and for all $\sigma > 1$
\[ \{M_g(r)\}^n < M_g(r^\sigma) \] (2.6)

Lemma 2.6 [10] Let $f$ be meromorphic and let $g$ be entire and suppose that $0 < \mu < \rho_g \leq \infty$. Then for a sequence of values of $r$ tending to infinity,
\[ T_{fof}(r) \geq T_f(\exp(\mu)) \] (2.7)
3 Theorems and results

In this section we have obtained theorems and results which we have proved.

Theorem 3.1 Let \( f, h \) be two entire functions of respective finite orders \( \rho_f, \rho_h \) such that \( \rho_f \neq 0 \) and \( g \) be a polynomial of degree \( m \). The relative order of \( h \) with respect to \( f \circ g \) satisfies the inequality:
\[
\rho_{f \circ g}(h) \geq \frac{\rho_h}{m \rho_f}.
\]
The sign of equality occurs if \( |g(0)| = 0 \).

Proof: We know by the definition of order of entire function [3]
\[
\rho_f = \limsup_{r \to \infty} \frac{\log \log M_f(r)}{\log r}
\]
Therefore for any \( \epsilon > 0 \) there exists \( r_0(\epsilon) > 0 \) such that
\[
\frac{\log \log M_f(r)}{\log r} < \rho_f + \epsilon \quad \text{for all } r > r_0(\epsilon)
\]

or
\[
M_f(r) < \exp\{r^{\rho_f + \epsilon}\} \quad \text{for all } r > r_0(\epsilon) \tag{3.8}
\]

Let \( \exp\{r^{\rho_f + \epsilon}\} = r_1 \) or \( r^{\rho_f + \epsilon} = \log r_1 \)
\[
\text{or } \log r = \frac{1}{\rho_f + \epsilon} \log \log r_1 \tag{3.9}
\]
This implies,
\[
M_f(\exp\{\frac{1}{\rho_f + \epsilon} \log \log r_1\}) < r_1
\]

or
\[
M_f^{-1}(r) > \exp\{\frac{1}{\rho_f + \epsilon} \log \log r\} \quad \text{for all } r > r_0(\epsilon) \tag{3.10}
\]

Also there exists a sequence \( \{r_n\} \) strictly increasing and increases to \( \infty \) such that
\[
M_f(r_n) > \exp\{r_n^{\rho_f - \epsilon}\} \tag{3.11}
\]

Following the same steps as shown in equation (3.9) we get
\[
M_f^{-1}(r_n) < \exp\{\frac{1}{\rho_f - \epsilon} \log \log r_n\} \tag{3.12}
\]

Similarly for the entire function \( h \), for any \( \epsilon > 0 \) there exists \( r_1(\epsilon) > 0 \) such that
\[
M_h(r) < \exp\{r^{\rho_h + \epsilon}\} \quad \text{for all } r > r_1(\epsilon) \tag{3.13}
\]

and for strictly increasing sequence \( \{u_n\} \), increasing to \( \infty \)
\[
M_h(u_n) > \exp\{u_n^{\rho_h - \epsilon}\} \tag{3.14}
\]

Now let \( g(z) = a_0 + a_1 z + \ldots + a_m z^m \). Then \( M_g(r) \sim |a_m| r^m \).
Therefore for any $\epsilon > 0$ there exists $r_2(\epsilon) > 0$ such that

$$|a_m|r_m(1-\epsilon) < M_g(r) < |a_m|r_m(1+\epsilon) \quad \text{for all } r > r_2(\epsilon) \quad (3.15)$$

Hence for all $r > r_2(\epsilon)$

$$M_g^{-1}(r) > \left\{ \frac{r}{|a_m|(1+\epsilon)} \right\}^\frac{1}{m} \quad (3.16)$$

and

$$M_g^{-1}(r) < \left\{ \frac{r}{|a_m|(1-\epsilon)} \right\}^\frac{1}{m} \quad (3.17)$$

By Lemma (2.3,[4]), for all sufficiently large values of $r$,

$$M_{fog}(r) \leq M_f(M_g(r)) \quad (3.18)$$

That implies, for all large $r$

$$M_g^{-1}(M_f^{-1}(r)) \leq M_{fog}^{-1}(r) \quad (3.19)$$

Now by definition of relative order of entire function with respect to another entire function, we have

$$\rho_{fog}(h) = \lim_{r \to \infty} \sup r \log \frac{M_{fog}^{-1}(M_h(r))}{\log r} \geq \lim_{r \to \infty} \sup \frac{\log M_g^{-1}(M_f^{-1}(M_h(r)))}{\log r} \quad [\text{by equation (3.19)}]$$

$$\geq \lim_{r \to \infty} \sup \frac{\log M_g^{-1}(exp((\frac{1}{\rho_f+\epsilon} \log \log M_h(r))))}{\log r} \quad [\text{by equation (3.10)}]$$

$$\geq \lim_{u_n \to \infty} \sup \frac{\log M_g^{-1}(exp(\frac{\rho_h-\epsilon}{\rho_f+\epsilon} \log u_n)))}{\log u_n} \quad [\text{by equation (3.14)}]$$

$$= \lim_{u_n \to \infty} \sup \frac{\log M_g^{-1}(u_n^{\frac{\rho_h-\epsilon}{\rho_f+\epsilon}})}{\log u_n}$$

$$\geq \lim_{u_n \to \infty} \sup \frac{\log \left( \frac{u_n^{\frac{\rho_h-\epsilon}{|a_m|(1+\epsilon)}}}{|a_m|(1+\epsilon)} \right)^\frac{1}{m}}{\log u_n} \quad [\text{by equation (3.16)}]$$

$$= \lim_{u_n \to \infty} \sup \frac{1}{m} \left\{ (\frac{\rho_h-\epsilon}{\rho_f+\epsilon} \log u_n - \frac{\log |a_m|(1+\epsilon)}{\log u_n}) \right\}$$

$$= \frac{1}{m} \left( \frac{\rho_h-\epsilon}{\rho_f+\epsilon} \right)$$
Since $\epsilon$ is arbitrarily small,

$$\rho_{fog}(h) \geq \frac{\rho_h}{m \rho_f} \quad (3.20)$$

By Lemma (2.3,[4]), for all sufficiently large values of $r$,

$$M_f\left(\frac{1}{8}M_g\left(\frac{r}{2}\right) - |g(0)|\right) \leq M_{fog}(r)$$

If $|g(0)| = 0$ then

$$M_f\left(\frac{1}{8}M_g\left(\frac{r}{2}\right)\right) \leq M_{fog}(r)$$

That implies for all sufficiently large values of $r$,

$$M_{fog}^{-1}(r) \leq 2M_{f}^{-1}(8M_{f}^{-1}(r)) \quad (3.21)$$

Now

$$\rho_{fog}(h) = \limsup_{r \to \infty} \frac{\log M_{fog}^{-1}(M_h(r))}{\log r}$$

$$\leq \limsup_{r \to \infty} \frac{\log(2M_g^{-1}(8M_f^{-1}(M_h(r))))}{\log r} \quad [\text{by equation (3.21)}]$$

$$\leq \limsup_{r_n \to \infty} \frac{\log(M_g^{-1}(8\exp(\frac{\rho_h + \epsilon}{\rho_f - \epsilon} \log M_h(r_n))))}{\log r_n} \quad [\text{by equation (3.12)}]$$

$$\leq \limsup_{r_n \to \infty} \frac{\log(M_g^{-1}(8\exp(\frac{\rho_h + \epsilon}{\rho_f - \epsilon} \log(r_n)) \exp(\frac{\rho_h + \epsilon}{\rho_f - \epsilon} \log(r_n)) \log|a_m|(1 - \epsilon))}{\log r_n} \quad [\text{by equation (3.13)}]$$

$$= \limsup_{r_n \to \infty} \frac{\log \left(\frac{8r_n^{\rho_h + \epsilon}}{|a_m|(1 - \epsilon)}\right)^{\frac{1}{m}}}{\log r_n}$$

$$\leq \limsup_{r_n \to \infty} \frac{1}{m} \left\{ \log 8 + \frac{\rho_h + \epsilon}{\rho_f - \epsilon} \log r_n - \log |a_m|(1 - \epsilon) \right\}$$

$$= \frac{1}{m} \left( \frac{\rho_h + \epsilon}{\rho_f - \epsilon} \right)$$

Since $\epsilon$ is arbitrarily small,

$$\rho_{fog}(h) \leq \frac{\rho_h}{m \rho_f} \quad (3.22)$$

So, when $|g(0)| = 0$, from equation (3.20) and equation (3.22),
\[ \rho_{fog}(h) = \frac{\rho_h}{m_f} \]

**Theorem 3.2** Let \( f, h_1 \) be two entire functions of respective finite orders \( \rho_f \) and \( \rho_{h_1} \) such that \( \rho_f \neq 0 \) and \( g, h_2 \) be two polynomials of respective degree \( m_1 \) and \( m_2 \) such that \( |h_2(0)| = 0 \). The relative order of \( h_1 oh_2 \) with respect to \( fog \) satisfies the inequality: \( \rho_{fog}(h_1 oh_2) \geq \frac{m_2 h_{h_1}}{m_f} \).

The sign of equality occurs if \( |g(0)| = 0 \).

**Proof:**

Let \( g(z) = a_0 + a_1 z + \ldots + a_{m_1} z^{m_1} \) and \( h_2(z) = b_0 + b_1 z + \ldots + b_{m_2} z^{m_2} \) be two polynomials of degree \( m_1 \) and \( m_2 \) respectively. By definition of relative order of entire function with respect to another entire function we have

\[
\rho_{fog}(h_1 oh_2) = \lim_{r \to \infty} \frac{\log M_{fog}^{-1}(M_{h_1 oh_2}(r))}{\log r} \\
\geq \lim_{r \to \infty} \frac{\log M_g^{-1}(M_f^{-1}(M_{h_1 oh_2}(r)))}{\log r} \quad \text{[by equation (3.19)]} \\
\geq \lim_{r \to \infty} \frac{\log M_g^{-1}(M_{f}^{-1}(M_{h_1}^{-1}(\frac{1}{z} M_{h_2}(\frac{1}{z}))))}{\log r} \quad \text{[Since \( h_2(0) = 0 \) by assumption, using Lemma (2.3), [4] we get]} \\
\geq \lim_{r \to \infty} \frac{\log M_g^{-1}(\exp(\frac{1}{8} |b_{m_2}| (1 - \epsilon)(\frac{1}{8} M_{m_2})^{\rho_{h_1}}))}{\log r} \quad \text{by equation (3.14)} \\
\geq \lim_{u_n \to \infty} \frac{\log M_g^{-1}(\exp(\frac{1}{\rho_f} + \epsilon \log \log \exp(\frac{1}{8} |b_{m_2}| (1 - \epsilon)(\frac{1}{8} M_{m_2})^{\rho_{h_1}}))}{\log u_n} \quad \text{by equation (3.10)} \\
= \lim_{u_n \to \infty} \frac{\log M_g^{-1}(\exp(\frac{\rho_{h_1} - \epsilon}{\rho_f} \log \log \exp(\frac{1}{8} |b_{m_2}| (1 - \epsilon)(\frac{1}{8} M_{m_2})^{\rho_{h_1}}))}{\log u_n} \quad \text{[Since \( g \) is a polynomial of degree \( m_1 \), using equation (3.16) we get]}
\]
\[
\limsup_{u_n \to \infty} \frac{1}{m_1} \log \left( \frac{b_{m_2} |(1-\epsilon)(\frac{u_n}{m})^{m_2}|}{a_{m_1} |1+\epsilon|} \right) \geq \limsup_{u_n \to \infty} \frac{1}{m_1} \log u_n \geq \frac{\rho_h - \epsilon}{\rho_f + \epsilon} \log u_n \\
= \limsup_{u_n \to \infty} \frac{1}{m_1} \log \left( \frac{b_{m_2} |(1-\epsilon) + m_2 (\log u_n) - \log |a_{m_1}|(1+\epsilon)}{m_1} \right) = \left( \frac{\rho_h - \epsilon}{\rho_f + \epsilon} \right) \frac{m_2}{m_1} \\
\]

Since \( \epsilon \) is arbitrarily small, \( \rho_{fog}(h_1oh_2) \geq \frac{m_2 \rho_h}{m_1 \rho_f} \) (3.23)

On the other hand, using similar steps as done in Theorem (3.1), we can prove that

\( \rho_{fog}(h_1oh_2) \leq \frac{m_2 \rho_h}{m_1 \rho_f} \) (3.24)

Combining equation (3.23) and equation (3.24) we get \( \rho_{fog}(h_1oh_2) = \frac{m_2 \rho_h}{m_1 \rho_f} \).

**Theorem 3.3** Let \( f \) be an entire function of finite non-zero order \( \rho_f \), \( h \) be a meromorphic function of finite non-zero order \( \rho_h \) and \( g \) be a polynomial of degree \( m \). The relative order of \( h \) with respect to \( fog \) satisfies the inequality \( \rho_{fog}(h) \geq \frac{\rho_h}{m \rho_f} \). The sign of equality occurs if \( |g(0)| = 0 \).

**Proof:** From the definition of order of entire function, we have

\[
\limsup_{r \to \infty} \frac{\log \log M_f(r)}{\log r} = \limsup_{r \to \infty} \frac{\log T_f(r)}{\log r} = \rho_f
\]

Also for the meromorphic function \( h \) we have

\[
\limsup_{r \to \infty} \frac{\log T_h(r)}{\log r} = \rho_h
\]

So, for any \( \epsilon > 0 \) there exists \( r_0(\epsilon) > 0 \), \( r_1(\epsilon) > 0 \) such that

\[
T_f(r) < r^{\rho_f + \epsilon} \quad \text{for all } r > r_0(\epsilon) \quad (3.25)
\]

\[
T_h(r) < r^{\rho_h + \epsilon} \quad \text{for all } r > r_1(\epsilon) \quad (3.26)
\]

Let \( r^{\rho_f + \epsilon} = r_1 \). That implies \( \log r = \frac{1}{\rho_f + \epsilon} \log r_1 \) or \( r = r_1^{\frac{1}{\rho_f + \epsilon}} \)

\[
\text{Hence} \quad T_f^{-1}(r) > r^{\frac{1}{\rho_f + \epsilon}} \quad \text{for all } r > r_0(\epsilon) \quad (3.27)
\]
Also there exists sequence \{r_n\} and \{u_n\} strictly increasing and increases to \infty such that

\[ T_f(r_n) > r_n^{\rho_f - \varepsilon} \]  

and \[ T_h(u_n) > u_n^{\rho_h - \varepsilon} \]  \hspace{1cm} (3.28)

If \( f \) and \( g \) are entire functions, then we have from [8] for all large \( r \)

\[ T_{fog}(r) \leq 3T_f(2M_g(r)) \]  

This implies for all large \( r \)

\[ T_{fog}^{-1}(r) \geq M_g^{-1}(\frac{1}{2}T_f^{-1}(\frac{r}{3})) \]  

Let \( g(z) = a_0 + a_1 + \ldots + a_mz^m \) be a polynomial of degree \( m \). By definition of relative order of meromorphic function with respect to entire function [6],

\[
\rho_{fog}(h) = \limsup_{r \to \infty} \frac{\log T_{fog}^{-1}(T_h(r))}{\log r} \\
\geq \limsup_{r \to \infty} \frac{\log M_g^{-1}(\frac{1}{2}T_f^{-1}(T_h(r)))}{\log r} \hspace{1cm} \text{by equation (3.31)} \\
\geq \limsup_{u_n \to \infty} \frac{\log M_g^{-1}(\frac{1}{2}T_f^{-1}(\frac{\rho_h - \varepsilon}{3}))}{\log u_n} \hspace{1cm} \text{by equation (3.29)} \\
\geq \limsup_{u_n \to \infty} \frac{\log M_g^{-1}(\frac{1}{2}(\frac{\rho_h - \varepsilon}{3})^\frac{1}{\rho_f + \varepsilon})}{\log \rho_f} \hspace{1cm} \text{by equation (3.27)} \\
\text{[Since } g \text{ is a polynomial of order } m, \text{ by equation(3.16)]} \\
\geq \limsup_{u_n \to \infty} \frac{1}{m} \left\{ \log \left( \frac{1}{2} \left( \frac{\rho_h - \varepsilon}{3} \right)^\frac{1}{\rho_f + \varepsilon} \right) - \log |a_m|(1 + \varepsilon) \right\} \\
= \limsup_{u_n \to \infty} \left[ \frac{1}{m} \left( \frac{\rho_h - \varepsilon}{\rho_f + \varepsilon} \right) \log u_n - \frac{1}{m} \log |a_m|(1 + \varepsilon) \right] \\
= \frac{1}{m} \left( \frac{\rho_h - \varepsilon}{\rho_f + \varepsilon} \right) \\
\text{[Since } \varepsilon > 0 \text{ is arbitrarily small,} \\
\rho_{fog}(h) \geq \frac{\rho_h}{m\rho_f} \]  

Since \( f \) is entire function and \( g \) is a polynomial, by Lemma (2.3,[4]) and Lemma (2.1), if \(|g(0)| = 0\), for all sufficiently large values of \( r \),
\[ M_f\left(\frac{1}{8} M_g \frac{r}{2}\right) \leq M_{fog}(r) \]

that implies \( \log(M_f(\frac{1}{8} M_g \frac{r}{2})) \leq \log(M_{fog}(r)) \leq 3T_fog(2r) \)

or \( T_{fog}^{-1}(\frac{1}{3} \log(M_f(\frac{1}{8} M_g \frac{r}{2}))) \leq 2r \)

Let \( r_1 = \frac{1}{3} \log(M_f(\frac{1}{8} M_g \frac{r}{2})) \)

or \( M_f(\frac{1}{8} M_g \frac{r}{2}) = \exp(3r_1) \)

or \( \frac{1}{8} M_g \frac{r}{2} = M_f^{-1}(\exp(3r_1)) \)

or \( \frac{r}{2} = M^{-1}_g(8M_f^{-1}(\exp(3r_1))) \)

or \( r = 2M^{-1}_g(8M_f^{-1}(\exp(3r_1))) \)

Therefore for all large \( r \),

\[ T_{fog}^{-1}(r_1) \leq 4M^{-1}_g(8M_f^{-1}(\exp(3r_1))) \] (3.33)

On the other hand

\[ \rho_{fog}(h) = \limsup_{r \to \infty} \frac{\log(T_{fog}^{-1}(T_h(r)))}{\log r} \]

\[ \leq \limsup_{r \to \infty} \frac{\log(4M^{-1}_g(8M_f^{-1}(\exp(3T_h(r)))))}{\log r} \]

[by equation(3.33)]

\[ \leq \limsup_{r \to \infty} \log\left(\frac{M^{-1}_g(8(\log(\exp(3T_h(r))))^{\frac{1}{1-\epsilon}})}{\log r_n}\right) \]

[by equation(3.12)]

\[ = \limsup_{r \to \infty} \log\left(\frac{M^{-1}_g(8(3T_h(r_n))^{\frac{1}{1-\epsilon}})}{\log r_n}\right) \]

[by equation(3.26)]

\[ \leq \limsup_{r \to \infty} \log\left(\frac{M^{-1}_g(8(3r_n)^{\frac{\rho_h+\epsilon}{\rho_f+\epsilon}})^{\frac{1}{1-\epsilon}}}{\log r_n}\right) \]

[by equation(3.17)]

\[ = \limsup_{r \to \infty} \frac{1}{m} \left[ \log(8.3^{\frac{1}{1-\epsilon}}) + \log\left(\frac{\rho_h+\epsilon}{\rho_f+\epsilon}\right) \right] \]

\[ \log r_n \]
\[
\begin{align*}
&= \limsup_{r_n \to \infty} \frac{1}{m} \left[ \log \left( \frac{\rho_h + \epsilon}{\rho_f - \epsilon} \right) \right] \\
&= \frac{1}{m} \left( \frac{\rho_h + \epsilon}{\rho_f - \epsilon} \right)
\end{align*}
\]

Since \( \epsilon \) is arbitrarily small,

\[\rho_{fog}(h) \leq \frac{\rho_h}{m \rho_f} \quad (3.34)\]

Combining equation (3.32) and equation (3.34) we get

\[\rho_{fog}(h) = \frac{\rho_h}{m \rho_f}\]

**Theorem 3.4** Let \( f, h \) be two meromorphic functions of finite non-zero orders \( \rho_f, \rho_h \) such that \( \rho_f \neq 0 \) and \( g \) be a polynomial of degree \( m \). The relative order of \( h \) with respect to \( fog \) satisfies the inequality \( \rho_{fog}(h) \geq \frac{\rho_h}{m \rho_f} \).

The sign of equality occurs if \( |g(0)| = 0 \).

**Proof:** Let \( g(z) = a_0 + a_1 + \ldots + a_m z^m \) be a polynomial of degree \( m \). We know by ([1], [2])

\[
\rho_{fog}(h) = \limsup_{r \to \infty} \frac{\log T_h(r)}{\log T_{fog}(r)}
\]

\[
\geq \limsup_{r \to \infty} \frac{\log T_h(r)}{\log \log M_{fog}(r)} \quad [\text{by Lemma (2.1)}]
\]

\[
\geq \limsup_{r \to \infty} \frac{\log T_h(r)}{\log \log M_f(M_g r^m)} \quad [\text{by Lemma (2.3)}]
\]

\[
\geq \limsup_{r \to \infty} \frac{\log T_h(r)}{\log (3T_f(2M_g r^m))} \quad [\text{by Lemma (2.1)}]
\]

\[
\geq \limsup_{r \to \infty} \frac{\log T_h(r)}{\log 3 + \log(T_f(2M_g r^m))}
\]

\[
= \limsup_{r \to \infty} \frac{\log T_h(r)}{\log (T_f(2M_g r^m))}
\]

\[
\geq \limsup_{r \to \infty} \frac{\log T_h(r)}{\log \left( T_f(2|a_m|(1 + \epsilon) r^m) \right)} \quad [\text{by equation (3.15)}]
\]

\[
\geq \limsup_{u_n \to \infty} \frac{(\rho_h - \epsilon) \log u_n}{\log(T_f(2|a_m|(1 + \epsilon) u_m^n))} \quad [\text{by equation (3.29)}]
\]

\[
\geq \limsup_{u_n \to \infty} \frac{(\rho_h - \epsilon) \log u_n}{(\rho_f + \epsilon) \log(2|a_m|(1 + \epsilon) u_m^n)} \quad [\text{by equation (3.26)}]
\]
\[
\limsup_{u_n \to \infty} \frac{(\rho_h - \epsilon) \log u_n}{(\rho_f + \epsilon) \log (2|a_m|(1 + \epsilon)) + (\rho_f + \epsilon)m \log u_n} = \limsup_{u_n \to \infty} \frac{(\rho_h - \epsilon) \log u_n}{(\rho_f + \epsilon)m \log u_n}
\]

Since \(\epsilon > 0\) is arbitrarily small,

\[
\rho_{foh}(h) \geq \frac{\rho_h}{m \rho_f} \quad (3.35)
\]

\[
\rho_{foh}(h) = \limsup_{r \to \infty} \frac{\log T_h(r)}{\log T_{foh}(r)}
\]

\[
\leq \limsup_{r \to \infty} \frac{\log T_h(r)}{\log(\frac{1}{\frac{1}{8} \log M_{foh}(\frac{x}{T})})} \quad [\text{by Lemma}(2.1)]
\]

\[
= \limsup_{r \to \infty} \frac{\log T_h(r)}{\log \log M_{foh}(\frac{x}{T})}
\]

\[
\leq \limsup_{r \to \infty} \frac{\log T_h(r)}{\log \log M_{f}(\frac{1}{8} M_g(\frac{r}{T}))} \quad [\text{by Lemma}(2.3), \text{since } |g(0)| = 0]
\]

\[
\leq \limsup_{r \to \infty} \frac{\log T_h(r)}{\log T_{f}(\frac{1}{8} |a_m|(1 + \epsilon)4^{-m} u_n^m)} \quad [\text{by Lemma}(2.1)]
\]

\[
\leq \limsup_{r \to \infty} \frac{\log T_h(r)}{\log(\frac{1}{\frac{1}{8} |a_m|(1 + \epsilon)4^{-m} u_n^m})} \quad [\text{by equation}(3.15)]
\]

\[
\leq \limsup_{r \to \infty} \frac{(\rho_h + \epsilon) \log r}{\log(\frac{1}{\frac{1}{8} |a_m|(1 + \epsilon)4^{-m} u_n^m})} \quad [\text{by equation}(3.26)]
\]

\[
\leq \limsup_{u_n \to \infty} \frac{(\rho_h + \epsilon) \log u_n}{(\rho_f - \epsilon) \log (\frac{1}{8} |a_m|(1 + \epsilon)4^{-m} u_n^m)} \quad [\text{by equation}(3.29)]
\]

\[
= \limsup_{u_n \to \infty} \frac{(\rho_f - \epsilon)m \log u_n + (\rho_f - \epsilon) \log (\frac{1}{8} |a_m|(1 + \epsilon)4^{-m})}{(\rho_h + \epsilon) \log u_n}
\]

\[
= \frac{\rho_h + \epsilon}{m(\rho_f - \epsilon)}
\]

Since \(\epsilon > 0\) is arbitrarily small,

\[
\rho_{foh}(h) \leq \frac{\rho_h}{m \rho_f} \quad (3.36)
\]

Combining equation (3.35) and equation (3.36) we get
\[ \rho_{fog}(h) = \frac{\rho_h}{m \rho_f} \]

**Theorem 3.5** Let \( f, h \) be meromorphic functions and \( g \) be entire such that \( f \circ g \) is meromorphic and

1. \( \liminf_{r \to \infty} \frac{\log r}{(\log T_h(r))^\alpha} = A \)
2. \( \liminf_{r \to \infty} \frac{\log T_f(\exp r^\mu)}{(\log r)^{\beta+1}} = B \)

where \( A \) and \( B \) are positive real numbers and \( \alpha, \beta, \mu \) are any arbitrary real numbers satisfying \( 0 < \alpha < 1, \beta > 0, \alpha(\beta + 1) > 1 \) and \( 0 < \mu < \rho_g \leq \infty \) then \( \rho_h(f \circ g) = \infty \)

**Proof:** By (i) we have for any arbitrary \( \epsilon > 0 \) there exists \( r_0(\epsilon) > 0 \) such that

\[ \log r \geq (A - \epsilon)(\log T_h(r))^\alpha \quad \text{for all} \quad r > r_0(\epsilon) \quad (3.37) \]

By (ii) we have for any arbitrary \( \epsilon > 0 \) there exists \( r_1(\epsilon) > 0 \) such that

\[ \log T_f(\exp r^\mu) \geq (B - \epsilon)(\log r)^{\beta+1} \quad \text{for all} \quad r > r_1(\epsilon) \quad (3.38) \]

By definition of relative order of meromorphic function with respect to another meromorphic function given by D. Banerjee ([1], [2]) we have

\[
\rho_h(f \circ g) = \limsup_{r \to \infty} \frac{\log T_{f \circ g}(r)}{\log T_h(r)} \\
\geq \limsup_{r \to \infty} \frac{\log T_f(\exp r^\mu)}{\log T_h(r)} \quad [\text{by Lemma (2.6)}] \\
\geq \liminf_{r \to \infty} \frac{(B - \epsilon)(\log r)^{\beta+1}}{\log T_h(r)} \quad [\text{by equation (3.38)}] \\
\geq \liminf_{r \to \infty} (B - \epsilon)(A - \epsilon)^{\beta+1}(\log T_h(r))^{\alpha(\beta+1)} \quad [\text{by equation (3.37)}]
\]

We know by Hayman [5] that \( T_h(r) \) is a convex increasing function of \( \log r \). Since \( \alpha(\beta + 1) > 1 \), by the above inequality we get that for any arbitrarily small \( \epsilon > 0, A, B \) constants

\[ \rho_h(f \circ g) = \infty \]

**Theorem 3.6** Let \( f, h \) be meromorphic functions and \( g \) be entire such that \( f \circ g \) is meromorphic and
\( \lim \inf_{r \to \infty} \frac{r^\mu}{\left( \log \log T_h(r) \right)^\alpha} = A \)

\( \lim \inf_{r \to \infty} \frac{\log T_f(\exp r^\mu)}{(r)^{\mu\beta}} = B \)

where \( A \) and \( B \) are positive real numbers and \( \alpha, \beta, \mu \) are any arbitrary real numbers satisfying \( 0 < \beta < 1, \alpha > 1, \mu \beta > 1 \) and \( 0 < \mu < \rho_g \leq \infty \) then \( \rho_h(fog) = \infty \)

**Proof:** From (i) we have for any arbitrary \( \epsilon > 0 \) there exists \( r_0(\epsilon) > 0 \) such that

\[ r^\mu \geq (A - \epsilon) \left( \log \log T_h(r) \right)^\alpha \]  
for all \( r > r_0(\epsilon) \) \quad (3.39)

From (ii) we have for any arbitrary \( \epsilon > 0 \) there exists \( r_1(\epsilon) > 0 \) such that

\[ \log \left[ \frac{\log T_f(\exp r^\mu)}{r^\mu} \right] \geq (B - \epsilon)(r)^{\mu\beta} \]  
for all \( r > r_1(\epsilon) \)

That implies \[ \log T_f(\exp r^\mu) \geq \exp \left( (B - \epsilon)(r)^{\mu\beta} \right) \] \quad (3.40)

By definition of relative order of meromorphic function with respect to another meromorphic function given by D. Banerjee (\cite{1}, \cite{2}) we have

\[ \rho_h(fog) = \lim \sup_{r \to \infty} \frac{\log T_{fog}(r)}{\log T_h(r)} \]
\[ \geq \lim \sup_{r \to \infty} \frac{\log T_f(\exp r^\mu)}{\log T_h(r)} \quad \text{[by Lemma(2.6)]} \]
\[ = \lim \sup_{r \to \infty} \frac{\log T_f(\exp r^\mu)}{r^\mu \log T_h(r)} \]
\[ \geq \lim \inf_{r \to \infty} \exp \left( (B - \epsilon)(r)^{\mu\beta} \right) \cdot \frac{(A - \epsilon) \left( \log \log T_h(r) \right)^\alpha}{\log T_h(r)} \]

[By equation (3.39) and (3.40)]

We know by Hayman \cite{5} that \( T_h(r) \) is a convex increasing function of \( \log r \). Since \( \mu \beta, \alpha > 1 \), by the above inequality we get that for any arbitrarily small \( \epsilon > 0 \), \( A, B \) constants

\[ \rho_h(fog) = \infty \]

**Theorem 3.7** Let \( f \) and \( h \) be two meromorphic functions and \( g \) be an entire functions such that \( fog \) is meromorphic, \( 0 < \rho_g \leq \infty \) and \( \lambda_h(f) > 0 \). Then \( \rho_h(fog) = \infty \)
**Proof:** By definition of relative order of meromorphic function with respect to another meromorphic function given by D. Banerjee ([1], [2]) we have

\[ \rho_h(f) = \limsup_{r \to \infty} \frac{\log T_f(r)}{\log T_h(r)} \]

Therefore the lower order

\[ \lambda_h(f) = \liminf_{r \to \infty} \frac{\log T_f(r)}{\log T_h(r)} \]

That implies for all arbitrary \( \epsilon > 0 \) there exists \( r_0(\epsilon) > 0 \) such that

\[ \frac{\log T_f(r)}{\log T_h(r)} > \lambda_h(f) - \epsilon \quad \text{for all } r > r_0(\epsilon) \]

Hence

\[ T_f(r) > \left( T_h(r) \right)^{\lambda_h(f) - \epsilon} \]

(3.41)

By Lemma (2.6) for a sequence of values of \( r \) tending to infinity

\[ T_{fof}(r) \geq T_f(\exp r^\mu) \geq \left( T_h(\exp r^\mu) \right)^{\lambda_h(f) - \epsilon} \quad \text{[by equation(3.41)]} \]

That implies, for a sequence of values of \( r \) tending to infinity

\[ \log T_{fof}(r) \geq (\lambda_h(f) - \epsilon) \log \left( T_h(\exp r^\mu) \right) \]

Therefore

\[ \rho_h(fof) = \limsup_{r \to \infty} \frac{\log T_{fof}(r)}{\log T_h(r)} \geq \limsup_{r \to \infty} \frac{\log T_f(\exp r^\mu)}{\log T_h(r)} \quad \text{[by Lemma(2.6)]} \]

\[ \geq \liminf_{r \to \infty} \frac{(\lambda_h(f) - \epsilon) \log \left( T_h(\exp r^\mu) \right)}{\log T_h(r)} \quad \text{[by equation(3.41)]} \]

Since \( T_h(r) \) is convex increasing function of \( \log r \),

\[ \liminf_{r \to \infty} \frac{\log \left( T_h(\exp r^\mu) \right)}{\log T_h(r)} \to \infty \]

Hence \( \rho_h(fof) = \infty \)
Theorem 3.8 Let \( f \) and \( h \) be two meromorphic functions and \( g \) be an entire function such that \( 0 < \lambda_h(g) \leq \rho_h(g) < \infty \). If for any real positive constant \( k \)

\[
\limsup_{r \to \infty} \frac{\log T_f(\alpha M_g(r))}{\log T_g(r)} = k < \infty
\]

then \( \lambda_h(fog) \leq k\lambda_h(g) \leq \rho_h(fog) \leq \rho_h(g) \). Where \( \alpha \) is any real positive constant.

Proof: By definition of relative order of meromorphic function with respect to another meromorphic function given by D. Banerjee ( [1], [2] ) we have

\[
\rho_h(fog) = \limsup_{r \to \infty} \frac{\log T_{fog}(r)}{\log T_h(r)}
\]

Therefore the lower order

\[
\lambda_h(fog) = \liminf_{r \to \infty} \frac{\log T_{fog}(r)}{\log T_h(r)} \leq \liminf_{r \to \infty} \frac{\log 3T_f(2M_g(r))}{\log T_h(r)} \quad [8]
\]

\[
\leq \limsup_{r \to \infty} \frac{\log T_f(2M_g(r))}{\log T_h(r)} \cdot \liminf_{r \to \infty} \frac{\log T_g(r)}{\log T_h(r)}
\]

\[
= k \cdot \lambda_h(g)
\]

Therefore

\[
\lambda_h(fog) \leq k \lambda_h(g) \quad (3.42)
\]

Again

\[
\rho_h(fog) = \limsup_{r \to \infty} \frac{\log T_{fog}(r)}{\log T_h(r)} \geq \limsup_{r \to \infty} \frac{\log \left( \frac{1}{3} \log M_f(\frac{1}{3}M_g(\frac{r}{3}) + o(1)) \right)}{\log T_h(r)} \quad [\text{by Lemma (2.2)}]
\]

\[
= \limsup_{r \to \infty} \frac{\log \left( \frac{1}{3} \log M_f(\frac{1}{3}M_g(\frac{r}{3})) \right)}{\log T_h(r)}
\]

\[
\geq \limsup_{r \to \infty} \frac{\log \left( \frac{1}{3} T_f(\frac{1}{3}M_g(\frac{r}{3})) \right)}{\log T_h(r)} \quad [\text{by Lemma (2.1)}]
\]

\[
\geq \limsup_{r \to \infty} \frac{\log T_f(\frac{1}{3}M_g(\frac{r}{3}))}{\log T_g(r)} \cdot \liminf_{r \to \infty} \frac{\log T_g(r)}{\log T_h(r)}
\]

\[
= k \cdot \lambda_h(g)
\]

Therefore
\( k\lambda_h(g) \leq \rho_h(fog) \) \hspace{3cm} (3.43)

Finally

\[
\rho_h(fog) = \limsup_{r \to \infty} \frac{\log T_{fog}(r)}{\log T_h(r)} \\
\leq \limsup_{r \to \infty} \frac{\log 3T_f(2M_g(r))}{\log T_h(r)} \quad \text{[by [8]]} \\
\leq \limsup_{r \to \infty} \frac{\log T_f(2M_g(r))}{\log T_g(r)} \cdot \limsup_{r \to \infty} \frac{\log T_g(r)}{\log T_h(r)} \\
= k\rho_h(g)
\]

Hence

\[
\rho_h(fog) \leq k\rho_h(g) \quad (3.44)
\]

From equation (3.42), equation (3.43), and (3.44) the theorem follows.

References


