A theorem concerning a product of polynomials and multivariable Gimel-function

Frédéric Ayant
Teacher in High School, France

ABSTRACT
The object of this paper is to establish a general theorem pertaining to a product of class of polynomials and multivariable Gimel-function. Certain integrals are also obtained by application of the theorem. The theorem is quite general nature and capable of yielding a number of new, interesting and useful integrals as its special cases.

KEYWORDS: Multivariable Gimel-function, multiple integral contours, class of polynomials, hypergeometric function.

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1. Introduction and preliminaries.

Throughout this paper, let \( \mathbb{C}, \mathbb{R} \) and \( \mathbb{N} \) be set of complex numbers, real numbers and positive integers respectively. Also \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \).

We define a generalized transcendental function of several complex variables noted \( \mathfrak{Z} \).

\[
\mathfrak{Z}(x_1, \ldots, x_r) = \sum_{n_0, n_1, \ldots, n_r = 0}^{n} \frac{x_{1}^{n_0} n_1! n_2! \cdots n_r!}{n_0! n_1! \cdots n_r!} G_{n_0}^{n_1} \ldots G_{n_r}^{n_r} \left( \tau_1, \tau_2, \ldots, \tau_r \right)
\]

\[
\tau_1, \tau_2, \ldots, \tau_r
\]

\[
\left[ (a_{2j}; \alpha_{2j}^{(2)}, A_{2j}) \right]_{n_2} \left[ \tau_{2j} \{ a_{2j}; \alpha_{2j}^{(2)}, A_{2j} \} \right]_{n_2+1, p_{2j}} \left[ (a_{3j}; \alpha_{3j}^{(3)}, A_{3j}) \right]_{n_3}
\]

\[
\left[ \tau_{3j} \{ b_{2j}; \beta_{2j}^{(2)}, B_{2j} \} \right]_{n_2, q_{2j}}
\]

\[
\tau_{1j} \{ a_{2j}; \alpha_{2j}^{(2)}, A_{2j} \} \tau_{3j} \{ b_{2j}; \beta_{2j}^{(2)}, B_{2j} \}
\]

\[
\left[ (a_{rj}; \alpha_{rj}^{(r)}, A_{rj}) \right]_{n_r+1, p_{rj}} \left[ \tau_{rj} \{ a_{rj}; \alpha_{rj}^{(r)}, A_{rj} \} \right]_{n_r}
\]

\[
\left[ \tau_{rj} \{ b_{rj}; \beta_{rj}^{(r)}, B_{rj} \} \right]_{n_r, q_{rj}}
\]

\[
\left[ (\epsilon_{j}^{(r)}; \gamma_{j}^{(r)}, C_{j}^{(r)}) \right]_{n_0+1, q_0}
\]

\[
\left[ \tau_{j} \{ \epsilon_{j}^{(r)}; \gamma_{j}^{(r)}, C_{j}^{(r)} \} \right]_{n_0, q_0+1}
\]

\[
\psi(s_1, \ldots, s_r) = \prod_{k=1}^{r} \theta_k(s_k) s_k^{\omega} \text{d}s_1 \cdots \text{d}s_r
\]

with \( \omega = \sqrt{-1} \)

\[
\psi(s_1, \ldots, s_r) = \frac{\prod_{j=1}^{n_0} \Gamma^{A_{2j}}(1 - a_{2j} + \sum_{k=1}^{a_{2j}} c_{2j}^{(k)} s_k)}{\sum_{j=1}^{n_2} \prod_{j=1}^{p_{2j}} \Gamma^{A_{2j}}(a_{2j} s_k) s_k} \frac{\prod_{j=1}^{n_3} \Gamma^{A_{3j}}(1 - a_{3j} + \sum_{k=1}^{a_{3j}} c_{3j}^{(k)} s_k)}{\sum_{j=1}^{n_3} \prod_{j=1}^{p_{3j}} \Gamma^{A_{3j}}(a_{3j} s_k) s_k}
\]

\[
\prod_{j=1}^{n_3} \Gamma^{A_{3j}}(1 - a_{3j} + \sum_{k=1}^{a_{3j}} c_{3j}^{(k)} s_k)
\]

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\[
\Pi_{j=1}^{n_3} \Gamma^{A_{3j}}_3 \left(1 - a_{3j} + \sum_{k=1}^{3} \alpha_{ijj}^{(k)} s_k \right) \\
\sum_{i=1}^{R_3} \tau_{i_3} \prod_{j=1}^{n_3+1} \Gamma^{A_{3j} + 3} \left(a_{3ji} - \sum_{k=1}^{3} \alpha_{ijj}^{(k)} s_k \right) \prod_{j=1}^{n_3+1} \Gamma^{B_{3j} + 3} \left(1 - b_{3ji} + \sum_{k=1}^{3} \beta_{ijj}^{(k)} s_k \right)
\]

(1.2)

and

\[
\theta_k(s_k) = \frac{\prod_{i=1}^{m^{(k)}} \Gamma^{D^{(k)}}(d^{(k)} - \delta^{(k)}_j s_k) \prod_{i=1}^{n^{(k)}} \Gamma^{C^{(k)}}(1 - c^{(k)}_k + \gamma^{(k)}_k s_k)}{\sum_{i=1}^{\bar{R}^{(k)}} \tau_{i_3} \prod_{j=m^{(k)}+1}^{n^{(k)+1}} \Gamma^{D^{(k)}_j}(1 - d^{(k)}_j s_k) + \delta^{(k)}_j s_k) \prod_{j=m^{(k)}+1}^{n^{(k)+1}} \Gamma^{E^{(k)}_j}(c^{(k)}_{jkm} - \gamma^{(k)}_j s_k)}
\]

(1.3)

1) \([c^{(1)}_j, \tau^{(1)}_j]_{1,n_1} \) stands for \((c^{(1)}_1, \tau^{(1)}_1), \ldots, (c^{(1)}_{n_1}, \tau^{(1)}_{n_1})\).

2) \(n_2, \ldots, n_r, m^{(1)}, n^{(1)}, \ldots, m^{(r)}, n^{(r)}, p_{i_3}, q_{i_3}, n^{(r)}, p_{i_3}, q_{i_3}, R_{i_3}, \tau_{i_3}, \cdots, p_{i_3}, q_{i_3}, R_{i_3}, \tau_{i_3}, p_{i_3}, q_{i_3}, R_{i_3}, \tau_{i_3}, R^{(r)} \in \mathbb{N}\) and verify:

0 \(\leq m_2, 0 \leq n_2 \leq p_{i_3}, \ldots, 0 \leq m_r, 0 \leq n_r \leq p_{i_3}, 0 \leq m^{(1)} \leq q_{i_3}, \ldots, 0 \leq m^{(r)} \leq q_{i_3}.

3) \(\tau_{i_3}(i_2 = 1, \ldots, R_{i_3}) \in \mathbb{R}^+; \tau_{i_3} \in \mathbb{R}^+; \tau_{i_3} \in \mathbb{R}^+; \tau_{i_3} \in \mathbb{R}^+(i = 1, \ldots, R^{(k)}), (k = 1, \ldots, r).

4) \(c^{(k)}_j, \tau^{(k)}_j \in \mathbb{R}^+; (j = 1, \ldots, n_k); (k = 1, \ldots, r); d^{(k)}_j, D^{(k)}_j \in \mathbb{R}^+; (j = 1, \ldots, m_k); (k = 1, \ldots, r).

5) \(a^{(k)}_{jkm}, B_{jkm} \in \mathbb{R}^+; (j = 1, \ldots, n_k); (k = 1, \ldots, r); k = 1, \ldots, k.

6) \(b^{(k)}_{jkm} \in \mathbb{C}; (j = 1, \ldots, n_k); (k = 1, \ldots, r); d^{(k)}_j \in \mathbb{C}; (j = 1, \ldots, m_k); (k = 1, \ldots, r).

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The contour \( L_k \) is in the \( s_k(k = 1, \cdots, r) \)-plane and run from \( \sigma - i\infty \) to \( \sigma + i\infty \) where \( \sigma \) is a real number with loop, if necessary to ensure that the poles of \( \Gamma^{A_{sj}}\left(1 - a_{2j} + \sum_{k=1}^{2} \alpha_{sj}^{(k)}s_k\right) \) (\( j = 1, \cdots, n_2 \)), \( \Gamma^{A_{j}}\left(1 - a_{sj} + \sum_{k=1}^{1} \alpha_{sj}^{(k)}s_k\right) \) (\( j = 1, \cdots, n_3 \)) and \( \Gamma^{A_{j}}\left(1 - a_{sj} + \sum_{k=1}^{1} \alpha_{sj}^{(k)}s_k\right) \) (\( j = 1, \cdots, n_4 \)) to the right of the contour \( L_k \) and the poles of \( \Gamma^{D_{j}}\left(\delta^{(k)}_{j} - \delta^{(k)}_{j} s_k\right) \) (\( j = 1, \cdots, m^{(k)}(k = 1, \cdots, r) \)) lie to the left of the contour \( L_k \). The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as:

\[
|\text{arg}(z_k)| < \frac{1}{2} A_{i}^{(k)} \pi \quad \text{where} \quad A_{i}^{(k)} = \sum_{j=1}^{m^{(k)}} D_{j}^{(k)} \delta_{j}^{(k)} + \sum_{j=1}^{r^{(k)}} C_{j}^{(k)} \gamma_{j}^{(k)} - \tau_{i}^{(k)} \quad \text{and} \quad \tau_{i}^{(k)} = \sum_{j=n_{2}+1}^{p_{1}} \sum_{j=1}^{q_{1}} A_{2j_{1}s_{2j_{2}}}^{(k)} + \sum_{j=1}^{p_{2}} \sum_{j=1}^{q_{2}} B_{2j_{1}s_{2j_{2}}}^{(k)} - \cdots - \tau_{r}^{(k)} \quad \text{and} \quad \tau_{r}^{(k)} = \sum_{j=n_{3}+1}^{p_{3}} \sum_{j=1}^{q_{3}} A_{rj_{1}s_{rj_{2}}}^{(k)} + \sum_{j=1}^{p_{4}} \sum_{j=1}^{q_{4}} B_{rj_{1}s_{rj_{2}}}^{(k)}
\]

Following the lines of Braaksma ([2] p. 278), we may establish the asymptotic expansion in the following convenient form:

\[
N(z_{1}, \cdots, z_{r}) = 0 \quad (|z_{1}|^{\alpha_{1}}, \cdots, |z_{r}|^{\alpha_{r}}), \quad \text{max}(|z_{1}|, \cdots, |z_{r}|) \rightarrow 0
\]

\[
N(z_{1}, \cdots, z_{r}) = 0 \quad (|z_{1}|^{\beta_{1}}, \cdots, |z_{r}|^{\beta_{r}}), \quad \text{min}(|z_{1}|, \cdots, |z_{r}|) \rightarrow \infty \quad \text{where} \quad i = 1, \cdots, r:
\]

\[
\alpha_{i} = \min_{1 \leq j \leq m^{(i)}} \text{Re} \left[ D_{j}^{(i)} \left( \frac{d^{(i)}_{j}}{\delta_{j}^{(i)}} \right) \right] \quad \text{and} \quad \beta_{i} = \max_{1 \leq j \leq m^{(i)}} \text{Re} \left[ C_{j}^{(i)} \left( \frac{\gamma_{j}^{(i)} - 1}{\gamma_{j}^{(i)}} \right) \right]
\]

**Remark 1.**

If \( n_{2} = \cdots = n_{r-1} = p_{1} = q_{1} = \cdots = p_{r-1} = q_{r-1} = 0 \) and \( A_{2j_{1}s_{2j_{2}}} = B_{2j_{1}s_{2j_{2}}} = \cdots = A_{rj_{1}s_{rj_{2}}} = B_{rj_{1}s_{rj_{2}}} = 1 \) and \( A_{3j} = A_{rj_{1}s_{rj_{2}}} = 1 \), then the multivariable Gimel-function reduces in the multivariable Aleph-function defined by Ayant [1].

**Remark 2.**

If \( n_{2} = \cdots = n_{r-1} = p_{1} = q_{1} = \cdots = p_{r-1} = q_{r-1} = 0 \) and \( A_{3j_{1}} = A_{rj_{1}s_{rj_{2}}} = \cdots = A_{rj_{1}s_{rj_{2}}} = 1 \) and \( \tau_{2} = \cdots = \tau_{r} = \tau_{i}^{(1)} = \cdots = \tau_{i}^{(r)} = R_{2} = \cdots = R_{r} = R^{(1)} = \cdots = R^{(r-1)} = 1 \), then the multivariable Gimel-function reduces in a multivariable I-function defined by Prathima et al. [4].

**Remark 3.**

If \( A_{2j_{1}s_{2j_{2}}} = B_{2j_{1}s_{2j_{2}}} = \cdots = A_{3j_{1}} = A_{rj_{1}s_{rj_{2}}} = 1 \) and \( \tau_{3} = \cdots = \tau_{r} = \tau_{i}^{(1)} = \cdots = \tau_{i}^{(r)} = R_{2} = \cdots = R_{r} = R^{(1)} = \cdots = R^{(r-1)} = 1 \), then the generalized multivariable Gimel-function reduces in a multivariable I-function defined by Prasad [3].

**Remark 4.**

If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the multivariable H-function defined by Srivastava and Panda [7,8].

In your investigation, we shall use the following notations:

\[
A = \{(a_{2j}; \alpha_{2j}^{(1)}; \alpha_{2j}^{(2)}; A_{2j})\}_{1, n_{2}}, \quad \tau_{2}^{(1)}(a_{2j_{1}s_{2j_{2}}}; \alpha_{2j_{1}}^{(1)}; \alpha_{2j_{2}}^{(2)}; A_{2j_{1}s_{2j_{2}}})_{n_{2}+1, p_{2}+1, q_{2}}, \quad \{(a_{3j}; \alpha_{3j}^{(1)}; \alpha_{3j}^{(2)}; \alpha_{3j}^{(3)}; A_{3j})\}_{1, n_{3}}, \\
\tau_{3}^{(1)}(a_{2j_{1}s_{2j_{2}}}; \alpha_{2j_{1}}^{(1)}; \alpha_{2j_{1}}^{(2)}; A_{2j_{1}s_{2j_{2}}})_{n_{3}+1, p_{3}+1, q_{3}}, \cdots \quad \{(a_{r-1,j_{1}}; \alpha_{r-1,j_{1}}^{(1)}; \alpha_{r-1,j_{1}}^{(2)}; \cdots; \alpha_{r-1,j_{1}}^{(r-1)}; A_{r-1,j_{1}})\}_{1, n_{r-1}},
\]

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Srivastava ([6], p. 1, Eq. (1)) have introduced the general class of polynomials:

\[
\tau_{r-1}(a_{(r-1)i_{j},-1}; \alpha^{(1)}_{(r-1)i_{j},-1}, \cdots, \alpha^{(r-1)}_{(r-1)i_{j},-1}; A_{(r-1)i_{j},-1}, p_{i_{j},-1})
\]

(1.5)

\[
A = \left[ (a_{rj}; \alpha^{(1)}_{rj}, \cdots, \alpha^{(r)}_{rj}; A_{rj})_{1, n_{r}} \bigg| \tau_{r}(a_{rj}, \alpha^{(1)}_{rj}, \cdots, \alpha^{(r)}_{rj}; A_{rj})_{n+1, p_{r}} \right]
\]

(1.6)

\[
A = \left[ (c^{(1)}_{j}, \gamma^{(1)}_{j}; C^{(1)}_{j})_{1, n_{j}} \bigg| \tau_{j}(c^{(1)}_{j}, \gamma^{(1)}_{j}; C^{(1)}_{j})_{n_{j}+1, p_{j}} \right] ; \cdots ; \left[ (c^{(r)}_{j}, \gamma^{(r)}_{j}; C^{(r)}_{j})_{1, n_{j}} \bigg| \tau_{j}(c^{(r)}_{j}, \gamma^{(r)}_{j}; C^{(r)}_{j})_{n_{j}+1, p_{j}} \right] ; \cdots
\]

(1.7)

\[
B = \left[ \tau_{2j_{j},(b^{(1)}_{2j_{j}}, \beta^{(1)}_{2j_{j}}, B_{2j_{j}})_{1, q_{j}} \bigg| \tau_{2j_{j}}(b^{(1)}_{2j_{j}}, \beta^{(1)}_{2j_{j}}, B_{2j_{j}})_{1, q_{j}} \right] ; \cdots ; \left[ \tau_{2j_{j},(b^{(1)}_{2j_{j}}, \beta^{(1)}_{2j_{j}}, \beta^{(2)}_{2j_{j}}, B_{2j_{j}})_{1, q_{j}} \bigg| \tau_{2j_{j}}(b^{(1)}_{2j_{j}}, \beta^{(1)}_{2j_{j}}, \beta^{(2)}_{2j_{j}}, B_{2j_{j}})_{1, q_{j}} \right] ; \cdots ; \left[ \tau_{2j_{j},(b^{(1)}_{2j_{j}}, \beta^{(1)}_{2j_{j}}, B_{2j_{j}})_{1, q_{j}} \bigg| \tau_{2j_{j}}(b^{(1)}_{2j_{j}}, \beta^{(1)}_{2j_{j}}, B_{2j_{j}})_{1, q_{j}} \right] ; \cdots
\]

(1.8)

\[
B = \left[ \tau_{r}(b_{rj}, \beta^{(1)}_{rj}, \cdots, \beta^{(r-1)}_{rj}; B_{rj})_{1, q_{j}} \bigg| \tau_{r}(b_{rj}, \beta^{(1)}_{rj}, \cdots, \beta^{(r-1)}_{rj}; B_{rj})_{1, q_{j}} \right]
\]

(1.9)

\[
B = \left[ (d^{(1)}_{j}, \delta^{(1)}_{j}; D^{(1)}_{j})_{1, m_{j}} \bigg| \tau_{j}(d^{(1)}_{j}, \delta^{(1)}_{j}; D^{(1)}_{j})_{m_{j}+1, q_{j}} \right] ; \cdots ; \left[ (d^{(r)}_{j}, \delta^{(r)}_{j}; D^{(r)}_{j})_{1, m_{j}} \bigg| \tau_{j}(d^{(r)}_{j}, \delta^{(r)}_{j}; D^{(r)}_{j})_{m_{j}+1, q_{j}} \right] ; \cdots
\]

(1.10)

\[
U = 0, n_{2}; 0, n_{3}; \cdots ; 0, n_{r-1}; V = m_{(1)}, n_{(1)}; m_{(2)}, n_{(2)}; \cdots ; m_{(r)}, n_{(r)}
\]

(1.11)

\[
X = p_{1}, q_{1}; \tau_{1}; R_{1}; \cdots ; p_{r-1}, q_{r-1}, \tau_{r-1}; R_{r-1}; Y = p_{1}, q_{1}; \tau_{1}; R_{1}; \cdots ; p_{r-1}, q_{r-1}, \tau_{r-1}; R_{r-1}; R^{(1)}; \cdots ; p_{r}, q_{r}; \tau_{r}; R^{(r)};
\]

(1.12)

Srivastava ([6], p. 1, Eq. (1)) have introduced the general class of polynomials:

\[
S_{N}^{M}(z) = \sum_{K=0}^{N/M} \frac{(-N)_{MK}}{K!} A_{N,K} z^{K}
\]

(1.13)

where \( M \) is an arbitrary positive integer and the coefficients \( A_{N,K} \) are arbitrary constants real or complex. On specializing these coefficients \( A_{N,K}, S_{N}^{M}[] \) yields a number of known polynomials as special cases. These include, among others, Hermite polynomials, Jacobi polynomials, Laguerre polynomials, bessel polynomials and several others([10], p. 158-161).

We shall note

\[
a_{NK} = \frac{(-N)_{MK}}{K!} A_{N,K}
\]

(1.14)

2. Main formula.

Theorem.

If \( _{2}F_{1}(a, b; c; x) = \sum_{l=0}^{\infty} C_{l} x^{l} \)

(2.1)

then

\[
\int_{0}^{1} dF_{3} \left[ \frac{a, b, c+d, c+d-1}{a+b, c, d} \right] 4z(1-z) S_{N}^{M}(z)_{j} (a_{1} z_{1}^{h_{1}}, \cdots, a_{r} z_{r}^{h_{r}}) dz = \sum_{l=0}^{N/M} \sum_{K=0}^{[N/M]} a_{NK} \frac{(c + d - 1)_{l} C_{l}}{(a+b)_{l}^{m}}
\]

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provided

\[ h_i > 0 (i = 1, \ldots, r), \quad 1 + \sum_{i=1}^{r} h_i \min_{1 \leq j \leq m(t)} \text{Re} \left( D_j^{(i)} \frac{d^{(i)}}{d^{(i)}} \right) > 0 \]

\[ |\text{arg}(a_i z^{h_i})| < \frac{1}{2} A_i^{(h)} \pi \] where \( A_i^{(h)} \) is defined by (1.4) and the series on the right-hand side of (2.2) is absolutely convergent.

**Proof**

To prove the above theorem, we use the following formula due to Slater ([5], p. 79, Eq. (2.5.27))

\[ \sum_{l=0}^{\infty} \frac{(c + d - 1)_l}{(a)_l} C_l z^l \] (2.3)

where \( C_l \) is given by (2.1). Multiplying the both-sides of (2.3) by \( S^M_N (z^h) \J (a_1 z^{h_1}, \ldots, a_r z^{h_r}) \) and integrating with respect to \( z \) between 0 to 1, we get

\[ \int_0^1 \sum_{l=0}^{\infty} \frac{(c + d - 1)_l}{(a)_l} C_l z^l \] (2.4)

Substituting the expression of the multivariable Gimel-function in terms of Mellin-Barnes multiple integrals contour with the help of (1.1) and \( S^M_N (z^h) \J (a_1 z^{h_1}, \ldots, a_r z^{h_r}) \) with the help of (1.13), interchanging the order of integrations and summation (which is permissible under the conditions mentioned in (2.2)), now, evaluating the inner \( z \)-integral and interpreting the Mellin-Barnes multiple integrals contour in terms of the multivariable Gimel-function, we get the desired result (2.2).

3. Particular cases.

Taking \( b = c = d \) in the theorem, we get the following interesting integral:

**Corollary 1.**

\[ \int_0^1 2F_1 \left[ \begin{array}{c} a, \frac{c + d}{2} \cr a + c \end{array} \right] \text{d}(1-z) \] \[ \sum_{l=0}^{\infty} \frac{(c + d - 1)_l}{(a)_l} C_l z^l \] (2.5)

under the same existence conditions mentioned in (2.2).

Taking \( a = -c \) in the above corollary, it reduces to the interesting integral

**Corollary 2.**

\[ \sum_{l=0}^{\infty} \frac{(c + d - 1)_l}{(a)_l} C_l z^l \] (2.6)
\[ \int_0^1 \sum \binom{n}{r} \left[ \binom{\frac{1}{4}}{r} \right] 4z(1-z) \sum_{l=0}^{N/M} \sum_{K=0}^{2 \pi} \frac{(2\pi - 1)(-2\pi)^l}{(\pi - l)l!} \right] \\
\sum_{Y \in \mathbb{N}, X \in \mathbb{N}^+} \left( \begin{array}{c} a_1 \\ A_1 \end{array} \right) \left( \begin{array}{c} a_r \\ a_r \end{array} \right) \sum_{Y \in \mathbb{N}, X \in \mathbb{N}^+} \left( \begin{array}{c} A \right) \left( \begin{array}{c} B \end{array} \right) \\
\text{under the same existence conditions mentioned in (2.2).} \\

4. Conclusion \\
The main integral (2.2) established here are unified and act as key formulae. Thus the multivariable Gimel-function occurring in these integrals can be suitably specialized to a remarkably wide variety of special functions (or product of several such special functions) which are expressible in terms of E, F, G, H, I, Aleph-functions of one or more variables. Again the class of polynomials involved in the integral (2.2) reduces to a large number of polynomials listed by Srivastava and Singh ([9], p.158-161), therefore, from the integral (2.2) we can further obtain various integrals involving a number of simpler polynomials. \\

REFERENCES. \\