Mean Edge-Antimagic Vertex Labeling of Graphs

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Abstract - A connected graph G(p,q) is said to be Mean Edge-Antimagic vertex labeling if there exists a bijection f : V→{0,1,2,...q} such that the induced mapping g_f : E→{1,2,...q} defined by

\[ g_f(e = uv) = \begin{cases} \frac{f(u) + f(v)}{2} & \text{if } f(u) + f(v) \text{ is even} \\ \frac{f(u) + f(v) + 1}{2} & \text{if } f(u) + f(v) \text{ is odd} \end{cases} \]

In this article we mainly investigate mean edge-antimagic vertex labeling of paths, spider, \(K_{1,n}\) and odd and even mean antimagic labeling of paths, dragon, \(K_{1,n}\) and cycle graphs.

Keywords - Mean Edge-Antimagic vertex Labeling, odd and even Mean Antimagic Labeling, Path, Spider, \(K_{1,n}\), Cycles, Dragon graphs.

AMS Subject Classification (2010): 05C78.

1. Introduction

In this paper we mean a graph G by finite, connected, undirected graph G = (V, E) without any loops and multiple edges with \(|V| = p\) vertices and \(|E| = q\) edges. Terms not defined here are used in the sense of Harary [1].

The concept of graph labeling was introduced by Rosa [4] in 1967. A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. The process of vertex labeling is label the vertices with integers. Under this vertex labeling, the edge weight of an edge e = uv is defined as W(e) = W(uv) = f(u) + f(v).

In 1994, N.Hartsfield and Ringel [2] introduced the concept of antimagic graph. Each vertex labeling f of a graph G = (p,q) from\{0,1,2,...q\} induces a edge labeling g_f where g_f(e) is sum the labels of end vertices of an edge e. Labeling f is called antimagic if and only if all the edge labelings are pair wisely distinct.

By an edge antimagic vertex labeling we mean a one-to-one mapping V(G) into \{0,1,2,...q\} such that the set of edge weights of all edges in G is \{1,2,...q\}.


We follow the notation and terminology of [7]. Different kinds of antimagic graphs were studied by T.Nicholas, S.Somasundaram and V.Vilfred [3].

In this paper we prove that Mean Edge-Antimagic vertex labeling of Paths, Spider and Cycles.

Definition 1.1

Let G(V, E) be a graph of order \(|V| \geq 2\) with \(|V| = p\) vertices and \(|E| = q\) edges. A bijective mapping f : V → \{0,1,...q\} with the induced mapping g_f : E → \{1,2,...q\} defined by
\[ g_f(e = uv) = \begin{cases} \frac{f(u) + f(v)}{2} & \text{if } f(u) + f(v) \text{ is even} \\ \frac{f(u) + f(v) + 1}{2} & \text{if } f(u) + f(v) \text{ is odd} \end{cases} \]

is called *Mean Edge-Antimagic vertex labeling*. A graph that admits a mean edge-antimagic vertex labeling is called a *mean antimagic graph*.

**Observation 1.2**

Every mean edge-antimagic vertex labeling is also antimagic.

**Definition 1.3**

A *Caterpillar* is a tree which has path \( P_n = a_1, a_2, \ldots, a_n \) of order \( n \) and is obtained by attaching \( X_i \) (possibly zero) end vertices at the vertex \( a_i \) of \( P_n \), \( i = 1, 2, \ldots, n \) by end edges. It is denoted as \( T = S(X_1, X_2, \ldots, X_n) \). The order of \( T \) is \( n + \sum X_i \).

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**Example 1.4**

![Caterpillar diagram](image1)

**Definition 1.5**

The Complete graph \( K_{1,n} \) is called a *Star graph*.

**Example 1.6**

The following graph is the example for the Spider \( SP(P_{4,2}) \)

![Spider diagram](image2)
Example 1.6

The above figure represents $K_{1,3}$

Fig (1.3)

2. Main Results

In this section we prove the Mean – Edge Antimagic vertex labeling of some special graphs.

Theorem 2.1

Every path $P_n$, $n \geq 2$ admits Mean – Edge Antimagic vertex labeling.

Proof:

Let us define a vertex labeling $f: V \rightarrow \{0,1,2,\ldots,q\}$ by $f(v_i) = i - 1$, $i = 1,2,\ldots,n$.

Then the induced edge labels are $g_f(e = uv) = f^*(v_{i+1} - v_i) = i$, $i = 1,2,\ldots,n$.

The above defined function $f$ provides mean – edge antimagic vertex labeling of path $P_n$.

Definition 2.2

The starting and end vertices of the 1–sided infinite path $P_1$ is of degree one and all other vertices are of degree two.

Example 2.3

Theorem 2.4

The 1–sided infinite path $P_1$ is $(1,1)$ edge antimagic vertex labeling using Partition technique.

Proof:

The vertex labels of $P_1$ are $0,1,2,\ldots,q$.

The induced edge labels are $1,2,\ldots,q$.

A partition scheme of numbers is

$0 + 1 = \frac{0+1+1}{2}$, $0 + 1$ is odd

$1 + 2 = \frac{1+2+1}{2}$, $1 + 2$ is odd

$2 + 3 = \frac{2+3+1}{2}$, $2 + 3$ is odd and so on.

The last number of the $(a,d) = (1,1)$ mean – antimagic label is $a + (q-1)d = 1 + (q-1)1 = q$.

Thus the 1-sided infinite path $P_1$ is $(1,1)$ mean – antimagic.
Theorem 2.5
The Spider SP \((P_{n,2})\), where \(n \geq 2\) is mean – edge antimagic vertex labeling.

Proof:
Here \(|V(\text{SP}(P_{n,2}))| = n + \sum X_i = n + 2\)
\(|E(\text{SP}(P_{n,2}))| = n + 1\)
Define \(f\) on \(V(\text{SP}(P_{n,2}))\) by \(f(a_i) = n + 2 - i, \ i = 1,2,\ldots,n, \ f(u_1) = 0, \ f(u_2) = 1.\)
Therefore the induced mapping \(g_f : E \to \{ 1,2,\ldots,q\}\) defined by
\[f^*(a_i a_{i+1}) = n + 2 - i, \ i = 1,2,\ldots,n,
\]
\[f^*(a_n u_1) = 1\]
\[f^*(a_n u_2) = 2.\]
Thus the condition is satisfied.

Definition 2.6
A regular Spider SP \((k,n)\) is a tree obtained by identifying one end vertex of \(k\) number of paths each of length \(n\).

Observation 2.7
A regular Spiders SP \((2, 2), SP (3, 3), SP (3, 2), SP (4,2)\) are mean – edge antimagic vertex labeling.

Theorem 2.8
The Complete graph \(K_{1,n}\) is mean – edge antimagic, if \(n \leq 3\).

Definition 3.1
A connected graph \(G\) with \(|V| = p\) vertices and \(|E| = q\) edges is said to be odd – mean edge antimagic vertex labeling if there is a bijection \(f : V \to \{1,3,\ldots,2q+1\}\) such that the mapping \(f^* : E \to \{2,4,\ldots,2q\}\) defined by
\[f^*(v_i v_{i+1}) = \begin{cases} \frac{f(u) + f(v)}{2} & \text{if } f(u) + f(v) \text{ is even} \\ \frac{f(u) + f(v) + 1}{2} & \text{if } f(u) + f(v) \text{ is odd} \end{cases} \]
then the resulting edges are distinct.

Theorem 3.2
The path \(P_n\) \((n \geq 2)\) admits on odd – mean edge antimagic vertex labeling.

Proof:
The assignment of vertex labels is as follows.
Define \(f : V \to \{1,3,\ldots,2q+1\}\) by \(f(v_i) = 2i - 1, \ i = 1,2,\ldots,n.\) Then the induced edge labels are defined as follows.
\[f^*(v_i v_{i+1}) = \begin{cases} \frac{f(u) + f(v)}{2} & \text{if } f(u) + f(v) \text{ is even} \\ \frac{f(u) + f(v) + 1}{2} & \text{if } f(u) + f(v) \text{ is odd} \end{cases} \]
The resulting distinct edge labels are \(f^*(v_i v_{i+1}) = 2i, \ i = 1,2,\ldots,n.\) Thus proved.
Definition 3.3

The graph G = C_n @ P_m is called dragon consists of a cycle C_n together with a path P_m one end vertex a_1 of P_m is joined with a node u_n of C_n. That is V(G) = { V_1 ∪ V_2 } where V_1 = { u_1, u_2, ...., u_n } of vertices of cycle C_n and V_2 = { a_1, a_2, ...., a_m } of vertices of Path P_m. Therefore V(G) = { v_1, v_2, ...., v_{n+m} } and E(G) = E(C_n) ∪ E(P_m) ∪ {u_n, a_1}.

Hence C_n @ P_m contains m + n vertices and equal number of edges.

Example 3.4

The below diagram shows the C_5 @ P_4

![Diagram](image_url)

Fig (3.1)

Theorem 3.4

The graph G = C_n @ P_m where n ≥ 3 and n is odd admits on odd mean - edge antimagic vertex labeling.

Proof:

We define a vertex labeling of G = C_n @ P_m as follows.

f : V(G) → { 1, 3, ..., 2(n+m) + 1 } by f(v_i) = 2i - 1, i = 1, 2, ..., n+m. The induced edge labels are defined by A = f*(v_i v_{i+1}) = 2i, i = 1, 2, ..., n+m-1, B = f*(v_n v_1) = n. All these edge labelings are distinct.

Hence the theorem follows.

Theorem 3.5

For n ≥ 1, the Cycle C_{2n+1} is odd – mean edge antimagic vertex labeling.

Proof:

The assignment of vertex labels are f : V → { 1, 3, ..., 4n+1 } by f(v_i) = 2i - 1, i = 1, 2, ..., 2n+1.

Then the induced edge labels are defined by

f*(v_i v_{i+1}) = \begin{cases} \frac{f(u) + f(v)}{2} & \text{if } f(u) + f(v) \text{ is even} \\ \frac{f(u) + f(v) + 1}{2} & \text{if } f(u) + f(v) \text{ is odd} \end{cases}

and the distinct edge labels are A = f*(v_i v_{i+1}) = 2i, i = 1, 2, ..., 2n.

B = f*(v_n v_1) = 2n+1.

This completes the proof.

Theorem 3.6
The Complete graph $K_{1,n}$ is odd – mean edge antimagic vertex labeling.

**Proof:**
Define $f : V \rightarrow \{1,3,\ldots,2q+1\}$ by $f(v_i) = 2i - 1$, $i = 1,2,\ldots,n + 1$. Then the induced edge labels are defined as follows.

$$f^*(v_i v_i) = i, i = 2,3,\ldots,n.$$ All these edge labelings are distinct. Hence the condition of antimagic is satisfied.

4. Even – Mean edge Antimagic vertex labeling

Here we present the even mean – edge antimagic vertex labeling and even mean – edge antimagic vertex labeling of different kinds of graphs.

**Definition 4.1**
A connected graph $G$ with $|V| = p$ vertices and $|E| = q$ edges is said to be even – mean edge antimagic vertex labeling if there is a bijection $f : V \rightarrow \{2,4,\ldots,2q+2\}$ such that the mapping $f^* : E \rightarrow \{3,5,\ldots,2q+1\}$ defined by

$$f^*(v_i v_i) = \begin{cases} \frac{f(u) + f(v)}{2} & \text{if } f(u) + f(v) \text{ is even} \\ \frac{f(u) + f(v) + 1}{2} & \text{if } f(u) + f(v) \text{ is odd} \end{cases}$$

then the resulting edges are distinct.

*Theorem 4.2*

The path $P_n$ ($n \geq 2$) admits on even – mean edge antimagic vertex labeling.

**Proof:**
The assignments of vertex labels are as follows.
Define $f : V \rightarrow \{2,4,\ldots,2q+2\}$ by $f(v_i) = 2i$, $i = 1,2,\ldots,n$. Then the induced edge labels are defined as follows.

$$f^*(v_i v_i) = \begin{cases} \frac{f(u) + f(v)}{2} & \text{if } f(u) + f(v) \text{ is even} \\ \frac{f(u) + f(v) + 1}{2} & \text{if } f(u) + f(v) \text{ is odd} \end{cases}$$

The resulting distinct edge labels are $f^*(v_i v_{i+1}) = 2i + 1$, $i = 1,2,\ldots,n$. Thus proved.

*Theorem 4.3*

For $n \geq 2$, the Cycle $C_{2n}$ is even – mean edge antimagic vertex labeling.

**Proof:**
The assignments of vertex labels are as follows.
Define $f : V \rightarrow \{2,4,\ldots,4n\}$ by $f(v_i) = 2i$, $i = 1,2,\ldots,n$. Then the induced edge labels are defined as follows.

$$f^*(v_i v_i) = \begin{cases} \frac{f(u) + f(v)}{2} & \text{if } f(u) + f(v) \text{ is even} \\ \frac{f(u) + f(v) + 1}{2} & \text{if } f(u) + f(v) \text{ is odd} \end{cases}$$

The resulting distinct edge labels are $f^*(v_i v_{i+1}) = 2i + 1$, $i = 1,2,\ldots,n$. Hence proved.

*Theorem 4.4*

The Complete graph $K_{1,n}$ is even – mean edge antimagic vertex labeling.

**Proof:**
Define $f : V \rightarrow \{2,4,\ldots,2q+2\}$ such that $f(v_i) = 2i$, $i = 1,2,\ldots,1+n$. 

ISSN: 2231-5373  http://www.ijmttjournal.org  Page 305
The edge labels are defined by $f^*(v_i) = i$, $i = 2, 3, \ldots, n$ and all are distinct.

Hence $K_{1,n}$ is even – mean edge antimagic vertex labeling.

5. Conclusion

In this article we have investigated Mean – Edge Antimagic vertex labeling of Paths, Spider, and odd and even Mean – Edge Antimagic vertex labeling of Paths, Cycles and dragon graphs. Analogous work can be carried out for other families and in the context of different types of graph labeling techniques.

References


