Supra – I– Compactness and Supra – I– Connectedness

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Abstract — In 2012 Sekar and Jayakumar introduced and investigated a new class of sets and functions between supra topological spaces called supra I–open sets and supra I–continuous functions respectively. In this paper we newly originate the notions of supra I–compact spaces, supra I–Lindelof spaces, countably supra I–compact spaces and supra I–connected spaces. We also interpret their several effects and characterizations.

Keywords — Supra Topological Space, Supra I– Open Set, Supra I– Compact Space, Supra I– Lindelof Space, Countably Supra I– Compact Space, Supra I– Connected Space.

2010 Mathematics Subject Classification: 54B05, 54D20, 54D30.

I. INTRODUCTION


II. PRELIMINARIES

DEFINITION 2.1. Let X be a nonempty set and let \( \tau^* \subseteq P(X) = \{ A : A \subseteq X \} \). Then \( \tau^* \) is called a supra topology on X if \( \phi \in \tau^* \), \( X \in \tau^* \) and for all \( \gamma \subseteq \tau^* \), it implies that \( \cup \gamma \in \tau^* \). The pair \( (X, \tau^*) \) is called a supra topological space. Each element \( A \in \tau^* \) is called a supra open set in \( (X, \tau^*) \) and the complement of \( A \) denoted by \( A^c = X - C \) is called a supra closed set in \( (X, \tau^*) \).

DEFINITION 2.2. Let \( (X, \tau^*) \) be a supra topological space. The supra closure of a set \( A \) is denoted by \( \text{Supra Cl}(A) \) and is defined by

\[
\text{Supra Cl}(A) = \{ B \subseteq X : B \text{ is a supra closed set in } X \text{ such that } A \subseteq B \}.
\]

The supra interior of a set \( A \) is denoted by \( \text{Supra Int}(A) \) and is defined by

\[
\text{Supra Int}(A) = \{ U \subseteq X : U \text{ is a supra open set in } X \text{ such that } U \subseteq A \}.
\]
DEFINITION 2.3. Let $(X, \tau)$ be a topological space and $\tau^*$ be a supra topology on $X$. We call $\tau^*$ a supra topology associated with $\tau$ if $\tau \subseteq \tau^*$.

DEFINITION 2.4. Let $(X, \tau^*)$ be a supra topological space. A subset $A$ of $X$ is called a supra $I-$ open set in $X$ if $A \subseteq \text{Supra } \text{Int} \left[ \text{Supra } \text{Cl}(A) \right]$. The complement of a supra $I-$ open set is called a supra $I-$ closed set.

DEFINITION 2.5. Let $(X, \tau^*)$ be a supra topological space. The supra $I-$ closure of a set $A$ is denoted by $\text{Supra } I- \text{Cl}(A)$, and is defined as given in the following:

$\text{Supra } I- \text{Cl}(A) = \{ B \subseteq X : B \text{ is supra } I- \text{closed set in } X \text{ such that } A \subseteq B \}$.

The supra $I-$ interior of a set $A$ is denoted by $\text{Supra } I- \text{Int}(A)$, and is defined by $\text{Supra } I- \text{Int}(A) = \{ U \subseteq X : U \text{ is supra } I- \text{open set in } X \text{ such that } U \subseteq A \}$. Clearly $\text{Supra } I- \text{Cl}(A)$ is a supra $I-$ closed set and $\text{Supra } I- \text{Int}(A)$ is a supra $I-$ open set.

Throughout this paper, $(X, \tau)$ and $(Y, \sigma)$ will denote topological spaces and we will denote by $\tau^*$ and $\sigma^*$ to be their associated supra topologies with $\tau$ and $\sigma$ respectively such that $\tau \subseteq \tau^*$ and $\sigma \subseteq \sigma^*$.

THEOREM 2.6. Let $(X, \tau^*)$ be a supra topological space. Then every supra open set in $X$ is supra $I-$ open set in $X$.

PROOF. Let $A$ be a supra open set in $X$. Then $A \subseteq \text{Supra } I- \text{Cl}(A)$, so $\text{Supra } I- \text{Int}(A) \subseteq \text{Supra } I- \text{Cl}(A)$. Since $A \in \tau^*$, so $\text{Supra } I- \text{Int}(A) = A$. Therefore $A \subseteq \text{Supra } I- \text{Int}(A)$, Hence it follows that $A$ is supra $I-$ open set in $X$.

The converse of the above theorem need not be true as shown by the following example.

EXAMPLE 2.7. Suppose $X = \{1, 2, 3, 4, 5\}$ and have the supra topology $\tau^* = \{ \phi, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, X \}$. The set $\{3\} \notin \tau^*$, so the set $\{3\}$ is not a supra open set in $(X, \tau^*)$.

Now since it clearly follows that $\text{Supra } I- \text{Int} (\text{Supra } I- \text{Cl}\{3\}) = \text{Supra } I- \text{Int}(X) = X$.

Therefore it follows that $\{3\}$ is a supra $I-$ open set in $(X, \tau^*)$.

DEFINITION 2.8. Let $(X, \tau^*)$ be a supra topological space. Then a subset $A$ of $X$ is called a supra semi–open set if $A \subseteq \text{Supra } I- \text{Cl}(A)$.

By the next two examples, we show that neither a supra $I-$ open set may be a supra semi–open set nor a semi–open set may be a supra $I-$ open set in a supra topological space.

EXAMPLE 2.9. Suppose $X = \{1, 2, 3, 4\}$ and have the supra topology as given by $\tau^* = \{ \phi, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, X \}$. Let $A = \{1, 2, 4\}$. Then $\text{Supra } I- \text{Cl}(A) = X$. Hence $A \subseteq \text{Supra } I- \text{Int}(\text{Supra } I- \text{Cl}(A)) = X$. It shows that $A$ is a supra $I-$ open set in $X$. Since $\text{Supra } I- \text{Cl}(A) = \phi$. It follows that $A$ is not a supra semi–open set.

EXAMPLE 2.10. Let $X = \{a, b, c\}$ and $\tau^* = \{ X, \phi, \{a\}, \{b\}, \{a, b\}\}$ be a supra topology on $X$.

Then clearly $\{b, c\}$ is a supra semi–open set, but not a supra $I-$ open set.

THEOREM 2.11. (i) Arbitrary union of Supra $I-$ open sets is always a supra $I-$ open set.

(ii) Finite intersection of supra $I-$ open sets may fail to be a supra $I-$ open set.
PROOF. (i) Let \((X, \tau')\) be a supra topological space. Let \(\mathcal{I} = \{S_i : i \in I\}\) be a family of supra \(I\)-open sets in \(X\). Let \(S = \bigcup_{i \in I} S_i\). Since for each \(i \in I\), \(S_i\) is supra \(I\)-open set. Hence it follows that \(S \subseteq \text{Supra } \text{Int} \left[\text{Supra } \text{Cl} (S_i)\right] \subseteq \text{Supra } \text{Int} \left[\text{Supra } \text{Cl} (S)\right]\), for all \(i \in I\). So \(S_i \subseteq \text{Supra } \text{Int} \left[\text{Supra } \text{Cl} (S)\right]\), for all \(i \in I\). Therefore clearly it follows that 
\[
S = \bigcup_{i \in I} S_i \subseteq \text{Supra } \text{Int} \left[\text{Supra } \text{Cl} (S)\right].
\]
Thus we conclude that \(S\) is supra \(I\)-open set.

(ii) Let \(X = \{a, b, c\}\) and \(\tau' = \{X, \emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}, X\}\) be a supra topology on \(X\). Then \(\{a, b\}\) and \(\{b, c\}\) are supra \(I\)-open sets but their intersection \(\{b\}\) is not a supra \(I\)-open set.

THEOREM 2.12. (i) The arbitrary intersection of supra \(I\)-closed sets is always supra \(I\)-closed.

(ii) A finite union of supra \(I\)-closed sets may fail to be a supra \(I\)-closed set.

PROOF. (i) Follows from Theorem 2.11 (i).

(ii) Let \(X = \{1, 2, 3, 4, 5\}\) and \(\tau' = \{X, \emptyset, \{1, 2\}, \{1, 2, 3\}, \{4\}, \{1, 2, 4\}, \{3, 4\}, \{1, 2, 3, 4\}\}\) be a supra topology on \(X\). Then \(\{4, 5\}\) and \(\{1, 2, 5\}\) are supra \(I\)-closed sets but their union \(\{1, 2, 4, 5\}\) is a not a supra \(I\)-closed set as its complement \(\{3\}\) is not supra \(I\)-open set.

THEOREM 2.13. Let \((X, \tau')\) be a supra topological space. Let \(A\) be a subset of \(X\). Then the following statements are true.

(a) \(\text{Supra } I - \text{Int} (X - A) = X - \left[\text{Supra } I - \text{Cl} (A)\right]\).

(b) \(\text{Supra } I - \text{Cl} (X - A) = X - \left[\text{Supra } I - \text{Int} (A)\right]\).

(c) \(\text{Supra } I - \text{Int} (A)\) is supra \(I\)-open.

(d) \(\text{Supra } I - \text{Cl} (A)\) is supra \(I\)-closed.

(e) \(\text{Supra } I - \text{Int} (A) = \emptyset\) if and only if \(A\) is a supra \(I\)-open set.

(f) \(\text{Supra } I - \text{Cl} (A) = \emptyset\) if and only if \(A\) is a supra \(I\)-closed set.

(g) \(\text{Supra } I - \text{Int} (A) = \{x \in X : \text{There exists a supra } I - \text{open set } U \text{ such that } x \in U \subseteq A\}\).

(h) \(\text{Supra } I - \text{Cl} (A) = \{x \in X : \text{for every supra } I - \text{open subset } U \text{ containing } x, U \cap A \neq \emptyset\}\).

DEFINITION 2.14. A function \(f : (X, \tau') \rightarrow (Y, \sigma')\) is called a supra \(I\)-continuous functions if the inverse image of each supra open set in \(Y\) is a supra \(I\)-open set in \(X\).

DEFINITION 2.15. A function \(f : (X, \tau') \rightarrow (Y, \sigma')\) is called a supra \(I\)-irresolute function if \(f^{-1}(V)\) is supra \(I\)-closed set in \((X, \tau')\), for every supra \(I\)-closed set \(V\) in \((Y, \sigma')\).

DEFINITION 2.16. A function \(f : (X, \tau') \rightarrow (Y, \sigma')\) is called strongly supra \(I\)-continuous if the inverse image \(f^{-1}(V)\) of every supra \(I\)-closed set \(V\) in \(Y\) is supra closed in \(X\).

DEFINITION 2.17. A function \(f : (X, \tau') \rightarrow (Y, \sigma')\) is called perfectly supra \(I\)-continuous if the inverse image \(f^{-1}(V)\) of every supra \(I\)-closed set \(V\) in \(Y\) is both supra closed and supra open in \(X\).

THEOREM 2.18. Every continuous function is supra \(I\)-continuous functions.
PROOF. Let \((X, \tau)\) and \((Y, \sigma)\) be two topological spaces and let \(\tau^*\) and \(\sigma^*\) be associated supra topologies with \(\tau\) and \(\sigma\) respectively. Let \(f : X \longrightarrow Y\) be a continuous function. Therefore \(f^{-1}(A)\) is an open set in \(X\) for each open set \(A\) in \(Y\). But, \(\tau^*\) is associated with \(\tau\). That is \(\tau \subseteq \tau^*\). This implies that \(f^{-1}(A)\) is a supra open set in \(X\). Since every supra open set is supra \(I\)–open set, this implies \(f^{-1}(A)\) is supra \(I\)–open in \(X\). Hence \(f\) is a supra \(I\)–continuous function. The converse of the above theorem is not true as shown in the following example.

EXAMPLE 2.19. Let \(X = \{a, b, c\}\) and \(\tau = \{X, \{a\}, \{b\}, \{a, b\}\}\) be a topology on \(X\). The supra topology \(\tau^*\) is defined as follows. \(\tau^* = \{X, \{a\}, \{c\}, \{a, b\}\}\). Suppose that \(f : X \longrightarrow X\) is a function defined as follows: \(f(a) = b, f(b) = c, f(c) = a\). The inverse image of the open set \(\{a, b\}\) is \(\{a, c\}\) which is not an open set but it is supra \(I\)–open. Also \(f^{-1}(\{c\}) = \{b\}\) is a supra \(I\)–open set in \(X\). Then \(f\) is supra \(I\)–continuous but it is not continuous.

III. SUPRA \(I\)–COMPACTNESS

DEFINITION 3.1. A collection \(\{A_i : i \in I\}\) of supra \(I\)–open sets in a supra topological space \((X, \tau^*)\) is called a supra \(I\)–open cover of a subset \(B\) of \(X\) if \(B \subseteq \bigcup\{A_i : i \in I\}\) holds.

DEFINITION 3.2. A supra topological space \((X, \tau^*)\) is called supra \(I\)–compact if every supra \(I\)–open cover of \(X\) has a finite subcover.

DEFINITION 3.3. A subset \(B\) of a supra topological space \((X, \tau^*)\) is said to be supra \(I\)–compact relative to \((X, \tau^*)\) if, for every collection \(\{A_i : i \in I\}\) of supra \(I\)–open subsets of \(X\) such that \(B \subseteq \bigcup\{A_i : i \in I\}\) there exists a finite subset \(I_0\) of \(I\) such that \(B \subseteq \bigcup\{A_i : i \in I_0\}\).

DEFINITION 3.4. A subset \(B\) of a supra topological space \((X, \tau^*)\) is said to be supra \(I\)–compact if \(B\) is supra \(I\)–compact as a subspace of \(X\).

THEOREM 3.5. Every supra \(I\)–compact space \((X, \tau^*)\) is supra compact.

PROOF. Let \(\{A_i : i \in I\}\) be a supra open cover of \(X\). Since every supra open set in \(X\) has a finite subcover.

Therefore the supra \(I\)–open cover \(\{A_i : i \in I\}\) of \((X, \tau^*)\) has a finite subcover say \(\{A_i : i = 1, 2, \ldots, n\}\) for \(X\). Hence \((X, \tau^*)\) is a supra compact space.

THEOREM 3.6. Every supra \(I\)–closed subset of a supra \(I\)–compact space is supra \(I\)–compact relative to \(X\).

PROOF. Let \(A\) be a supra \(I\)–closed subset of a supra topological space \((X, \tau^*)\). Then \(A^C = X - A\) is supra \(I\)–open in \((X, \tau^*)\). Let \(\gamma = \{A_i : i \in I\}\) be a supra \(I\)–open cover of \(A\) by supra \(I\)–open subsets in \((X, \tau^*)\). Let \(\gamma^* = \{A_i : i \in I\} \cup \{A^C\}\) be a supra \(I\)–open cover of \((X, \tau^*)\). That is \(X = U \gamma^* = (U\{A_i : i \in I\}) \cup A^C\). By hypothesis \((X, \tau^*)\) is supra \(I\)–compact and hence \(\gamma^*\) is reducible to a finite subcover of \((X, \tau^*)\) say \(X = A_1 \cup A_2 \cup \ldots \cup A_n \cup A^C\); \(A_k \in \gamma\) for \(k = 1, 2, \ldots, n\). But \(A\) and \(A^C\) are disjoint. Hence \(A \subseteq A_1 \cup A_2 \cup \ldots \cup A_n;\)
\(A_k \in \gamma \) for \(k = 1, 2, \ldots, n\). Thus a supra I- open cover \(\gamma\) of \(A\) contains a finite subcover. Hence \(A\) is supra I- compact relative to \((X, \tau')\).

**THEOREM 3.7** A supra I- continuous image of a supra I- compact space is supra compact.

**PROOF.** Let \(f : (X, \tau') \longrightarrow (Y, \sigma')\) be a supra I- continuous map from a supra I- compact space \(X\) onto a supra topological space \(Y\). Let \(\{A_i : i \in I\}\) be a supra open cover of \(Y\). Then \(\{f^{-1}(A_i) : i \in I\}\) is a supra I- open cover of \(X\), as \(f\) is supra I- continuous. Since \(X\) is supra I- compact, the supra I- open cover of \(X\), \(\{f^{-1}(A_i) : i \in I\}\) has a finite subcover say \(\{f^{-1}(A_i) : i = 1, 2, \ldots, n\}\). Therefore \(X = \bigcup \{f^{-1}(A_i) : i = 1, 2, \ldots, n\}\), which implies \(f(X) = \bigcup \{A_i : i = 1, 2, \ldots, n\}\). That is \(\{A_i : i = 1, 2, \ldots, n\}\) is a finite subcover of \(\{A_i : i \in I\}\) for \(Y\). Hence \(Y\) is supra compact.

**THEOREM 3.8.** Suppose that a map \(f : (X, \tau') \longrightarrow (Y, \sigma')\) is supra I- irresolute and a subset \(S\) of \(X\) is supra I- compact relative to \((X, \tau')\), then the image \(f(S)\) is supra I- compact relative to \((Y, \sigma')\).

**PROOF.** Let \(\{A_i : i \in I\}\) be a collection of supra I- open cover of \((Y, \sigma')\), such that \(f(S) \subseteq \bigcup \{A_i : i \in I\}\). Since \(f\) is supra I- irresolute. Therefore \(S \subseteq \bigcup \{f^{-1}(A_i) : i \in I\}\), where \(\{f^{-1}(A_i) : i \in I\}\) is a family of supra I- open sets in \(X\). Since \(S\) is supra I- compact relative to \((X, \tau')\), so there exists a finite subcollection \(\{A_1, A_2, \ldots, A_n\}\) such that \(S \subseteq \bigcup \{f^{-1}(A_i) : i = 1, 2, \ldots, n\}\). That is \(f(S) \subseteq \bigcup \{A_1, A_2, \ldots, A_n\}\). Hence \(f(S)\) is supra I- compact relative to \((Y, \sigma')\).

**THEOREM 3.9.** Suppose that a map \(f : (X, \tau') \longrightarrow (Y, \sigma')\) is strongly supra I- continuous map from a supra compact space \((X, \tau')\) onto a supra topological space \((Y, \sigma')\), then \((Y, \sigma')\) is supra compact.

**PROOF.** Let \(\{A_i : i \in I\}\) be a supra open cover of \((Y, \sigma')\). Since \(f\) is strongly supra I- continuous, \(\{f^{-1}(A_i) : i \in I\}\) is a supra I- open cover of \((X, \tau')\). Again, since \((X, \tau')\) is supra I- compact, the supra I- open cover \(\{f^{-1}(A_i) : i \in I\}\) of \((X, \tau')\) has a finite subcover say \(\{f^{-1}(A_i) : i = 1, 2, \ldots, n\}\). Therefore \(X = \bigcup \{f^{-1}(A_i) : i = 1, 2, \ldots, n\}\), which implies \(f(X) = \bigcup \{A_i : i = 1, 2, \ldots, n\}\), so that \(Y = \bigcup \{A_i : i = 1, 2, \ldots, n\}\). That is \(\{A_1, A_2, \ldots, A_n\}\) is a finite subcover of \(\{A_i : i \in I\}\) for \((Y, \sigma')\). Hence \((Y, \sigma')\) is supra compact.

**THEOREM 3.10.** Suppose that a map \(f : (X, \tau') \longrightarrow (Y, \sigma')\) is perfectly supra I- continuous map from a supra compact space \((X, \tau')\) onto a supra topological space \((Y, \sigma')\). Then \((Y, \sigma')\) is supra compact.

**PROOF.** Let \(\{A_i : i \in I\}\) be a supra I- open cover of \((Y, \sigma')\). Since \(f\) is perfectly supra I- continuous, \(\{f^{-1}(A_i) : i \in I\}\) is a supra open cover of \((X, \tau')\). Again, since \((X, \tau')\) is supra compact, the supra open cover \(\{f^{-1}(A_i) : i \in I\}\) of \((X, \tau')\) has a finite subcover say \(\{f^{-1}(A_i) : i = 1, 2, \ldots, n\}\).
Therefore $X = \bigcup \{ f^{-1}(A_i) : i = 1, 2, \ldots, n \}$, which implies $f(X) = \bigcup \{ A_i : i = 1, 2, \ldots, n \}$, so that $Y = \bigcup \{ A_i : i = 1, 2, \ldots, n \}$. That is, $\{ A_1, A_2, \ldots, A_n \}$ is a finite subcover of $\{ A_i : i \in I \}$ for $(Y, \sigma^*)$. Hence $(Y, \sigma^*)$ is supra compact.

**THEOREM 3.11.** Suppose that a function $f : (X, \tau^*) \rightarrow (Y, \sigma^*)$ is supra $I-$ irresolute map from a supra $I-$ compact space $(X, \tau^*)$ onto a supra topological space $(Y, \sigma^*)$. Then $(Y, \sigma^*)$ is supra $I-$ compact.

**PROOF.** Let $\{ A_i : i \in I \}$ be a supra $I-$ open cover of $(Y, \sigma^*)$. Then $\{ f^{-1}(A_i) : i \in I \}$ is a supra $I-$ open cover of $(X, \tau^*)$, since $f$ is supra $I-$ irresolute. As $(X, \tau^*)$ is supra $I-$ compact, the supra $I-$ open cover $\{ f^{-1}(A_i) : i \in I \}$ of $(X, \tau^*)$ has a finite subcover $\{ f^{-1}(A_i) : i = 1, 2, \ldots, n \}$. Therefore $X = \bigcup \{ f^{-1}(A_i) : i = 1, 2, \ldots, n \}$, which implies $f(X) = \bigcup \{ A_i : i = 1, 2, \ldots, n \}$, so that $Y = \bigcup \{ A_i : i = 1, 2, \ldots, n \}$. That is, $\{ A_1, A_2, \ldots, A_n \}$ is a finite subcover of $\{ A_i : i \in I \}$ for $(Y, \sigma^*)$. Hence $(Y, \sigma^*)$ is supra $I-$ compact.

**THEOREM 3.12.** If $(X, \tau^*)$ is supra compact and every supra $I-$ closed set in $X$ is also supra closed in $X$, then $(X, \tau^*)$ is supra $I-$ compact.

**PROOF.** Let $\{ A_i : i \in I \}$ be a supra $I-$ open cover of $X$. Since every supra $I-$ closed set in $X$ is also supra closed in $X$, Thus $\{ X - A_i : i \in I \}$ is a supra closed cover of $X$ and hence $\{ A_i : i \in I \}$ is a supra open cover of $X$. Since $(X, \tau^*)$ is supra compact. So there exists a finite subcover $\{ A_i : i = 1, 2, \ldots, n \}$ of $\{ A_i : i \in I \}$ such that $X = \bigcup \{ A_i : i = 1, 2, \ldots, n \}$. Hence $(X, \tau^*)$ is a supra $I-$ compact space.

**THEOREM 3.13.** A supra topological space $(X, \tau^*)$ is supra $I-$ compact if and only if every family of supra $I-$ closed sets of $(X, \tau^*)$ having finite intersection property has a nonempty intersection.

**PROOF.** Suppose $(X, \tau^*)$ is supra $I-$ compact, Let $\{ A_i : i \in I \}$ be a family of supra $I-$ closed sets with finite intersection property. Suppose $\bigcap_{i \in I} A_i = \phi$, then $X - I \{ \{ A_i : i \in I \} \} = X$. This implies $\bigcup \{ (X - A_i) : i \in I \} = X$. Thus $\{ (X - A_i) : i \in I \}$ is a supra $I-$ open cover of $(X, \tau^*)$. Then as $(X, \tau^*)$ is supra $I-$ compact, the supra $I-$ open cover $\{ (X - A_i) : i \in I \}$ of $X$ has a finite subcover say $\{ (X - A_i) : i = 1, 2, \ldots, n \}$. This implies that $X = \bigcup \{ (X - A_i) : i = 1, 2, \ldots, n \}$, which implies $X - X = I \{ A_i : i = 1, 2, \ldots, n \}$, and which implies $\phi = I \{ A_i : i = 1, 2, \ldots, n \}$. This disproves the assumption. Hence $I \{ A_i : i \in I \} \neq \phi$.

Conversely, suppose $(X, \tau^*)$ is not supra $I-$ compact. Then there exits a supra $I-$ open cover of $(X, \tau^*)$ say $\{ G_i : i \in I \}$ having no finite subcover. This implies that for any finite subfamily $\{ G_i : i = 1, 2, \ldots, n \}$ of $\{ G_i : i \in I \}$, we have $\bigcup \{ G_i : i = 1, 2, \ldots, n \} \neq X$, which implies $X - (\bigcup \{ G_i : i = 1, 2, \ldots, n \}) \neq X - X$, hence $I \{ X - G_i : i = 1, 2, \ldots, n \} \neq \phi$. Therefore the family $\{ X - G_i : i \in I \}$ of supra $I-$ closed sets has a finite intersection property. Then by assumption
\(I \{X - G_i : i \in I\} \neq \emptyset\) which implies \(X - \left(\bigcup \{G_i : i \in I\}\right) \neq \emptyset\), so that \(\bigcup \{G_i : i \in I\} \neq X\). This implies that \(\{G_i : i \in I\}\) is not a cover of \((X, \tau^*)\). This disproves the fact that \(\{G_i : i \in I\}\) is a cover for \((X, \tau^*)\). Therefore any supra \(I-\) open cover \(\{G_i : i \in I\}\) of \((X, \tau^*)\) has a finite subcover \(\{G_i : i = 1, 2, \ldots, n\}\). Hence \((X, \tau^*)\) is supra \(I-\) compact.

**THEOREM 3.14.** Let \(A\) be a supra \(I-\) compact set relative to a supra topological space \(X\) and \(B\) be a supra \(I-\) closed subset of \(X\). Then \(AI \ B\) is supra \(I-\) compact relative to \(X\).

**PROOF.** Let \(A\) be supra \(I-\) compact relative to \(X\). Let \(\{A_i : i \in I\}\) be a cover of \(AI \ B\) by supra \(I-\) open sets in \(X\). Then \(\{A_i : i \in I\}\) is a cover of \(A\) by supra \(I-\) open sets in \(X\), but \(A\) is supra \(I-\) compact relative to \(X\), so there exist \(i_1, i_2, \ldots, i_n \in I\) such that \(A \subseteq \left(\bigcup \{A_{i_k} : k = 1, 2, \ldots, n\}\right) \cup B^C\). Then \(AI \ B \subseteq \bigcup \{A_{i_k} B : k = 1, 2, \ldots, n\} \subseteq \bigcup \{A_{i_k} : k = 1, 2, \ldots, n\}\). Hence \(AI \ B\) is supra \(I-\) compact relative to \(X\).

**THEOREM 3.15.** Suppose that a function \(f : (X, \tau^*) \longrightarrow (Y, \sigma^*)\) is supra \(I-\) irresolute and a subset \(B\) of \(X\) is supra \(I-\) compact relative to \(X\). Then \(f(B)\) is supra \(I-\) compact relative to \(Y\).

**PROOF.** Let \(\{A_i : i \in I\}\) be a cover of \(f(B)\) by supra \(I-\) open subsets of \(Y\). Since \(f\) is supra \(I-\) irresolute. Then \(\left\{f^{-1}(A_i) : i \in I\right\}\) is a cover of \(B\) by supra \(I-\) open subsets of \(X\). Since \(B\) is supra \(I-\) compact relative to \(X\), so \(\left\{f^{-1}(A_i) : i \in I\right\}\) has a finite subcover say \(\left\{f^{-1}(A_1), f^{-1}(A_2), \ldots, f^{-1}(A_n)\right\}\) for \(B\). Then it implies that \(\{A_i : i = 1, 2, \ldots, n\}\) is a finite subcover of \(\{A_i : i \in I\}\) for \(f(B)\). So \(f(B)\) is supra \(I-\) compact relative to \(Y\).

**IV. COUNTABLY SUPRA \(I-\) COMPACTNESS**

In this section, we present the concept of countably supra \(I-\) compactness and its properties.

**DEFINITION 4.1.** A supra topological space \((X, \tau^*)\) is said to be countably supra \(I-\) compact if every countable supra \(I-\) open cover of \(X\) has a finite subcover.

**THEOREM 4.2.** If \((X, \tau^*)\) is a countably supra \(I-\) compact space, then \((X, \tau^*)\) is countably supra compact.

**PROOF.** Let \((X, \tau^*)\) be a countably supra \(I-\) compact space. Let \(\{A_i : i \in I\}\) be a countable supra open cover of \((X, \tau^*)\). Since every supra open set in \(X\) is always supra \(I-\) open set in \(X\). So \(\{A_i : i \in I\}\) is a countable supra \(I-\) open cover of \((X, \tau^*)\). Since \((X, \tau^*)\) is countably supra \(I-\) compact, so the countable supra \(I-\) open cover \(\{A_i : i \in I\}\) of \((X, \tau^*)\) has a finite subcover say \(\{A_i : i = 1, 2, \ldots, n\}\) for \(X\). Hence \((X, \tau^*)\) is a countably supra compact space.

**THEOREM 4.3.** If \((X, \tau^*)\) is countably supra compact and every supra \(I-\) closed subset of \(X\) is supra closed in \(X\), then \((X, \tau^*)\) is countably supra \(I-\) compact.

**PROOF.** Let \((X, \tau^*)\) be a countably supra compact space. Let \(\{A_i : i \in I\}\) be a countable supra \(I-\) open cover of \((X, \tau^*)\). Since every supra \(I-\) closed subset of \(X\) is supra closed in \(X\). Thus every supra
I—open set in X is supra open in X. Therefore \( \{ A_i : i \in I \} \) is a countable supra open cover of \( (X, \tau^* ) \).
Since \( (X, \tau^* ) \) is countably supra compact, so the countable supra open cover \( \{ A_i : i \in I \} \) of \( (X, \tau^* ) \) has a finite subcover say \( \{ A_i : i = 1, 2, \ldots, n \} \) for X. Hence \( (X, \tau^* ) \) is a countably supra I—compact space.

THEOREM 4.4. Every supra I—compact space is countably supra I—compact.

PROOF. Let \( (X, \tau^* ) \) be a supra I—compact space. Let \( \{ A_i : i \in I \} \) be a countable supra I—open cover of \( (X, \tau^* ) \). Since \( (X, \tau^* ) \) is supra I—compact, so the supra I—open cover \( \{ A_i : i \in I \} \) of \( (X, \tau^* ) \) has a finite subcover say \( \{ A_i : i = 1, 2, \ldots, n \} \) for \( (X, \tau^* ) \). Hence \( (X, \tau^* ) \) is countably supra I—compact space.

THEOREM 4.5. Let \( f : (X, \tau^* ) \to (Y, \sigma^* ) \) be a supra I—continuous onto mapping. If X is countably supra I—compact space, then \( (Y, \sigma^* ) \) is countably supra I—compact.

PROOF. Let \( f : (X, \tau^* ) \to (Y, \sigma^* ) \) be a supra I—continuous map from a countably supra I—compact space \( (X, \tau^* ) \) onto a supra topological space \( (Y, \sigma^* ) \). Let \( \{ A_i : i \in I \} \) be a countable supra I—open cover of \( Y \). Then \( \{ f^{-1}(A_i) : i \in I \} \) is a countable supra I—open cover of \( X \), as \( f \) is supra I—continuous. Since \( X \) is countably supra I—compact. So the countable supra I—open cover \( \{ f^{-1}(A_i) : i \in I \} \) of \( X \) has a finite subcover say \( \{ f^{-1}(A_i) : i = 1, 2, \ldots, n \} \). Therefore \( X = \bigcup \{ f^{-1}(A_i) : i = 1, 2, \ldots, n \} \), which implies \( Y = f(X) = \bigcup \{ A_i : i = 1, 2, \ldots, n \} \). That is \( \{ A_i : i = 1, 2, \ldots, n \} \) is a finite subcover of \( \{ A_i : i \in I \} \) for \( Y \). Hence \( Y \) is countably supra I—compact.

THEOREM 4.6. Suppose that a map \( f : (X, \tau^* ) \to (Y, \sigma^* ) \) is perfectly supra I—continuous map from a countably supra compact space \( (X, \tau^* ) \) onto a supra topological space \( (Y, \sigma^* ) \). Then \( (Y, \sigma^* ) \) is countably supra I—compact.

PROOF. Let \( \{ A_i : i \in I \} \) be a countable supra I—open cover of \( (Y, \sigma^* ) \). Since \( f \) is perfectly supra I—continuous. So \( \{ f^{-1}(A_i) : i \in I \} \) is a countable supra open cover of \( (X, \tau^* ) \). Again, since \( (X, \tau^* ) \) is countably supra compact. Hence the countable supra open cover \( \{ f^{-1}(A_i) : i \in I \} \) of \( (X, \tau^* ) \) has a finite subcover say \( \{ f^{-1}(A_i) : i = 1, 2, \ldots, n \} \). Therefore \( X = \bigcup \{ f^{-1}(A_i) : i = 1, 2, \ldots, n \} \), which implies \( f(X) = \bigcup \{ A_i : i = 1, 2, \ldots, n \} \). That is \( \{ A_1, A_2, \ldots, A_n \} \) is a finite subcover of \( \{ A_i : i \in I \} \) for \( (Y, \sigma^* ) \). Hence \( (Y, \sigma^* ) \) is countably supra I—compact.

THEOREM 4.7. Suppose that a map \( f : (X, \tau^* ) \to (Y, \sigma^* ) \) is strongly supra I—continuous map from a countably supra compact space \( (X, \tau^* ) \) onto a supra topological space \( (Y, \sigma^* ) \). Then \( (Y, \sigma^* ) \) is countably supra I—compact.

PROOF. Let \( \{ A_i : i \in I \} \) be a countable supra I—open cover of \( (Y, \sigma^* ) \). Since \( f \) is strongly supra I—continuous, so \( \{ f^{-1}(A_i) : i \in I \} \) is a countable supra open cover of \( (X, \tau^* ) \). Again, since \( (X, \tau^* ) \) is countably supra compact, so the countable supra open cover \( \{ f^{-1}(A_i) : i \in I \} \) of \( (X, \tau^* ) \) has a finite
subcover say \( \{ f^{-1}(A_i) : i = 1, 2, ..., n \} \). Therefore \( X = \bigcup \{ f^{-1}(A_i) : i = 1, 2, ..., n \} \), which implies \( f(X) = \bigcup A_i \) so that \( Y = \bigcup \{ A_i : i = 1, 2, ..., n \} \). That is \( \{ A_1, A_2, ..., A_n \} \) is a finite subcover of \( \{ A_i : i \in I \} \) for \( (Y, \sigma') \). Hence \( (Y, \sigma') \) is countably supra \( I \)–compact.

**THEOREM 4.8.** The image of a countably supra \( I \)–compact space under a supra \( I \)–irresolute map is countably supra \( I \)–compact.

**PROOF.** Suppose that a map \( f : (X, \tau^*) \rightarrow (Y, \sigma^*) \) is a supra \( I \)–irresolute map from a countably \( I \)–compact space \( (X, \tau^*) \) onto a supra topological space \( (Y, \sigma^*) \). Let \( \{ A_i : i \in I \} \) be a countable supra \( I \)–open cover of \( (Y, \sigma^*) \). Then \( \{ f^{-1}(A_i) : i \in I \} \) is a countable supra \( I \)–open cover of \( (X, \tau^*) \), since \( f \) is supra \( I \)–irresolute. As \( (X, \tau^*) \) is countably supra \( I \)–compact, so the countable supra \( I \)–open cover \( \{ f^{-1}(A_i) : i \in I \} \) of \( (X, \tau^*) \) has a finite subcover say \( \{ f^{-1}(A_i) : i = 1, 2, ..., n \} \). Therefore \( X = \bigcup \{ f^{-1}(A_i) : i = 1, 2, ..., n \} \), which implies \( f(X) = \bigcup A_i \) so that \( Y = \bigcup \{ A_i : i = 1, 2, ..., n \} \). That is \( \{ A_1, A_2, ..., A_n \} \) is a finite subcover of \( \{ A_i : i \in I \} \) for \( (Y, \sigma^*) \). Hence \( (Y, \sigma^*) \) is countably supra \( I \)–compact.

**V. SUPRA \( I \)–LINDELOF SPACE**

In this section, we concentrate on the concept of supra \( I \)–Lindelof space and its properties.

**DEFINITION 5.1.** A supra topological space \( (X, \tau^*) \) is said to be supra \( I \)–Lindelof space if every supra \( I \)–open cover of \( X \) has a countable subcover.

**THEOREM 5.2.** Every supra \( I \)–Lindelof space \( (X, \tau^*) \) is supra Lindelof space.

**PROOF.** Let \( (X, \tau^*) \) be a supra \( I \)–Lindelof space. Let \( \{ A_i : i \in I \} \) be a supra open cover of \( (X, \tau^*) \). Since every supra open set in \( X \) is always supra \( I \)–open set in \( X \). Therefore \( \{ A_i : i \in I \} \) is a supra \( I \)–open cover of \( (X, \tau^*) \). Since \( (X, \tau^*) \) is supra \( I \)–Lindelof space, so the supra \( I \)–open cover \( \{ A_i : i \in I \} \) of \( (X, \tau^*) \) has a countable subcover say \( \{ A_i : i = 1, 2, ..., n \} \) for \( X \). Hence \( (X, \tau^*) \) is a supra Lindelof space.

**THEOREM 5.3.** If \( (X, \tau^*) \) is supra \( I \)–Lindelof space, then \( (X, \tau) \) is Lindelof space.

**PROOF.** Let \( \{ A_i : i \in I \} \) be an open cover of \( X \). Since every open set in \( X \) being a supra open set in \( X \) is also supra \( I \)–open set in \( X \). Therefore \( \{ A_i : i \in I \} \) is a supra \( I \)–open cover of \( (X, \tau^*) \). Since \( (X, \tau^*) \) is supra \( I \)–Lindelof, so the supra \( I \)–open cover \( \{ A_i : i \in I \} \) of \( (X, \tau^*) \) has a countable subcover say \( \{ A_i : i = 1, 2, ..., n \} \) for \( X \). Hence \( (X, \tau) \) is a Lindelof space.

**THEOREM 5.4.** Every supra \( I \)–compact space is supra \( I \)–Lindelof space.

**PROOF.** Let \( (X, \tau^*) \) be a supra \( I \)–compact space. Let \( \{ A_i : i \in I \} \) be a supra \( I \)–open cover of \( (X, \tau^*) \). Since \( (X, \tau^*) \) is supra \( I \)–compact space. Then \( \{ A_i : i \in I \} \) has a finite subcover say \( \{ A_i : i = 1, 2, ..., n \} \). Since every finite subcover is always countable subcover and therefore \( \{ A_i : i = 1, 2, ..., n \} \) is a countable subcover of \( \{ A_i : i \in I \} \). Hence \( (X, \tau^*) \) is a supra \( I \)–Lindelof space.
THEOREM 5.5. A supra $I-$ continuous image of a supra $I-$ Lindelof space is supra Lindelof space.

PROOF. Let $f: (X, \tau^*) \longrightarrow (Y, \sigma^*)$ be a supra $I-$ continuous map from a supra $I-$ Lindelof space $X$ onto a supra topological space $Y$. Let $\{A_i: i \in I\}$ be a supra open cover of $Y$. Then $\{f^{-1}(A_i): i \in I\}$ is a supra $I-$ open cover of $X$, as $f$ is supra $I-$ continuous. Since $X$ is supra $I-$ Lindelof space, so the supra $I-$ open cover $\{f^{-1}(A_i): i \in I\}$ of $X$ has a countable subcover say $\{f^{-1}(A_i): i \in I_0\}$ for some countable set $I_0 \subseteq I$. Therefore $X = \bigcup \{f^{-1}(A_i): i \in I_0\}$, which implies $f(X) = \bigcup \{A_i: i \in I_0\}$, then $Y = \bigcup \{A_i: i \in I_0\}$. That is $\{A_i: i \in I_0\}$ is a countable subcover of $\{A_i: i \in I\}$ for $Y$. Hence $Y$ is a supra Lindelof space.

THEOREM 5.6. The image of a supra $I-$ Lindelof space under a supra $I-$ irresolute map is supra $I-$ Lindelof space.

PROOF. Suppose that a map $f: (X, \tau^*) \longrightarrow (Y, \sigma^*)$ is supra $I-$ irresolute map from a supra $I-$ Lindelof space $\left( X, \tau^* \right)$ onto a supra topological space $\left( Y, \sigma^* \right)$. Let $\{A_i: i \in I\}$ be a supra $I-$ open cover of $\left( Y, \sigma^* \right)$. Since $f$ is supra $I-$ irresolute. Therefore $\{f^{-1}(A_i): i \in I\}$ is a supra $I-$ open cover of $\left( X, \tau^* \right)$. As $\left( X, \tau^* \right)$ is supra $I-$ Lindelof space, so the supra $I-$ open cover $\{f^{-1}(A_i): i \in I\}$ of $\left( X, \tau^* \right)$ has a countable subcover say $\{f^{-1}(A_i): i \in I_0\}$ for some countable set $I_0 \subseteq I$. Therefore $X = \bigcup \{f^{-1}(A_i): i \in I_0\}$, which implies $f(X) = \bigcup \{A_i: i \in I_0\}$, so that $Y = \bigcup \{A_i: i \in I_0\}$. That is $\{A_i: i \in I_0\}$ is a countable subcover of $\{A_i: i \in I\}$ for $Y$. Hence $\left( Y, \sigma^* \right)$ is a supra $I-$ Lindelof space.

THEOREM 5.7. If $\left( X, \tau^* \right)$ is supra $I-$ Lindelof space and countably supra $I-$ compact space, then $\left( X, \tau^* \right)$ is supra $I-$ compact space.

PROOF. Suppose $\left( X, \tau^* \right)$ is supra $I-$ Lindelof space and countably supra $I-$ compact space. Let $\{A_i: i \in I\}$ be a supra $I-$ open cover of $\left( X, \tau^* \right)$. Since $\left( X, \tau^* \right)$ is supra $I-$ Lindelof space, so $\{A_i: i \in I\}$ has a countable subcover say $\{A_i: i \in I_0\}$ for some countable set $I_0 \subseteq I$. Therefore $\{A_i: i \in I_0\}$ is a countable supra $I-$ open cover of $\left( X, \tau^* \right)$. Again, since $\left( X, \tau^* \right)$ is countably supra $I-$ compact space, so $\{A_i: i \in I_0\}$ has a finite subcover and say $\{A_i: i = 1, 2, \ldots, n\}$. Therefore $\{A_i: i = 1, 2, \ldots, n\}$ is a finite subcover of $\{A_i: i \in I\}$ for $\left( X, \tau^* \right)$. Hence $\left( X, \tau^* \right)$ is a supra $I-$ compact space.

THEOREM 5.8. If a function $f: (X, \tau^*) \longrightarrow (Y, \sigma^*)$ is supra $I-$ irresolute and a subset $B$ of $X$ is supra $I-$ Lindelof relative to $X$, then $f(B)$ is supra $I-$ Lindelof relative to $Y$.

PROOF. Let $\{A_i: i \in I\}$ be a cover of $f(B)$ by supra $I-$ open subsets of $Y$. By hypothesis $f$ is supra $I-$ irresolute and so $\{f^{-1}(A_i): i \in I\}$ is a cover of $B$ by supra $I-$ open subsets of $X$. Since $B$ is supra $I-$ Lindelof relative to $X$, $\{f^{-1}(A_i): i \in I\}$ has a countable subcover say $\{f^{-1}(A_i): i \in I_0\}$ for $B$, where $I_0$ is a countable subset of $I$. Now $\{A_i: i \in I_0\}$ is a countable subcover of $\{A_i: i \in I\}$ for $f(B)$. So $f(B)$ is supra $I-$ Lindelof relative to $Y$. 
VI. SUPRA $I-$CONNECTEDNESS

DEFINITION 6.1. A supra topological space $(X, \tau^*)$ is said to be supra connected if $X$ cannot be written as a disjoint union of two nonempty supra open subsets of $X$. A subset of $(X, \tau^*)$ is supra connected if it is supra connected as a subspace.

DEFINITION 6.2. A supra topological space $(X, \tau^*)$ is said to be supra $I-$connected if $X$ cannot be written as a disjoint union of two nonempty supra $I-$open sets. A subset of $(X, \tau^*)$ is supra $I-$connected if it is supra $I-$connected as a subspace.

THEOREM 6.3. Every supra $I-$connected space $(X, \tau^*)$ is supra connected.

PROOF. Let $A$ and $B$ be two nonempty disjoint proper supra open sets in $X$. Since every supra open set is supra $I-$open set. Therefore $A$ and $B$ are nonempty disjoint proper supra $I-$open sets in $X$. By hypothesis $X$ is supra $I-$connected space. Hence $X \neq A \cup B$. Therefore $X$ is supra $I-$connected.

The converse of the above theorem need not be true in general, which follows from the following example.

EXAMPLE 6.4. Let $X = \{1, 2, 3, 4\}$ and $\tau^* = \{\emptyset, \{1, 2\}, \{1, 2, 3\}, X\}$. Then $(X, \tau^*)$ is a supra topological space. Since $X$ cannot be written as a disjoint union of any two nonempty supra open sets. Therefore $(X, \tau^*)$ is a supra connected topological space. We notice that both $\{1\}$ and $\{2, 3, 4\}$ are supra $I-$open sets in $(X, \tau^*)$ because

$\{1\} \subseteq \text{Supra-Int}\left[\text{Supra-Closure}\left(\{1\}\right)\right] = \text{Supra-Closure}(X) = X$ and $\{2, 3, 4\} \subseteq \text{Supra-Int}\left[\text{Supra-Closure}\left(\{2, 3, 4\}\right)\right] = \text{Supra-Closure}(X) = X$. Therefore $\{1\}$ and $\{2, 3, 4\}$ are nonempty disjoint supra $I-$open sets such that $X = \{1\} \cup \{2, 3, 4\}$. Hence $(X, \tau^*)$ is not a supra $I-$connected space.

THEOREM 6.5. Let $(X, \tau^*)$ be a supra topological space. Then the following statements are equivalent

(i) $(X, \tau^*)$ is supra $I-$connected.

(ii) The only subsets of $(X, \tau^*)$ which are both supra $I-$open and supra $I-$closed are the empty sets $\emptyset$ and $X$.

(iii) Each supra $I-$continuous map of $(X, \tau^*)$ into a discrete space $(Y, \sigma^*)$ with at least two points is a constant map.

PROOF: (i) $\Rightarrow$ (ii): Let $G$ be a nonempty proper supra $I-$open and supra $I-$closed subset of $(X, \tau^*)$. Then $X - G$ is also both supra $I-$open and supra $I-$closed set. Then $X = G \cup (X - G)$ is a disjoint union of two nonempty supra $I-$open sets, which contradicts the fact that $(X, \tau^*)$ is supra $I-$connected. Hence $G = \emptyset$ or $G = X$.

(ii) $\Rightarrow$ (i): Suppose that $X = A \cup B$ where $A$ and $B$ are disjoint nonempty supra $I-$open subsets of $(X, \tau^*)$. Since $A = X - B$, then $A$ is both supra $I-$open and supra $I-$closed set. By assumption $A = \emptyset$ or $A = X$, which is a contradiction. Hence $(X, \tau^*)$ is supra $I-$connected.

(iii) $\Rightarrow$ (ii): Let $f : (X, \tau^*) \longrightarrow (Y, \sigma^*)$ be a supra $I-$continuous map, where $(Y, \sigma^*)$ is discrete space with at least two points. Then $f^{-1}(y)$ is supra $I-$closed and supra $I-$open for each...
\( y \in Y \). Thus \((X, \tau^\ast)\) is covered by supra \(-\) closed and supra \(-\) opening \(\{f^{-1}(y) : y \in Y\}\).

By assumption, \(f^{-1}(y) = \emptyset\) or \(f^{-1}(y) = X\) for each \(y \in Y\). If \(f^{-1}(y) = \emptyset\) for each \(y \in Y\), then \(f\) fails to be a map. Therefore their exists at least one point say \(y^* \in Y\) such that \(f^{-1}(y^*) = X\). This shows that \(f\) is a constant map.

\((iii) \Rightarrow (ii)\): Let \(G\) be both supra \(-\) open and supra \(-\) closed nonempty set in \((X, \tau^\ast)\). Suppose \(G \neq X\). Then \(f : (X, \tau^\ast) \rightarrow (Y, \sigma^\ast)\) is a nonconstant supra \(-\) continuous map defined by \(f(G) = \{a\}\) and \(f(X - G) = \{b\}\) where \(a \neq b\) and \(a, b \in Y\). By assumption, \(f\) is constant so we conclude that \(G = X\).

**THEOREM 6.6.** If \(f : (X, \tau^\ast) \rightarrow (Y, \sigma^\ast)\) is a supra \(-\) continuous surjection and \((X, \tau^\ast)\) is supra \(-\) connected, then \((Y, \sigma^\ast)\) is supra connected.

**PROOF.** Suppose \((Y, \sigma^\ast)\) is not supra connected. Let \(Y = A \cup B\), where \(A\) and \(B\) are disjoint nonempty supra open subsets of \((Y, \sigma^\ast)\). Since \(f\) is supra \(-\) continuous and onto, so \(X = f^{-1}(A) \cup f^{-1}(B)\), where \(f^{-1}(A)\) and \(f^{-1}(B)\) are disjoint nonempty supra \(-\) open subsets of \((X, \tau^\ast)\). This disproves the fact that \((X, \tau^\ast)\) is supra \(-\) connected. Hence \((Y, \sigma^\ast)\) is supra connected.

**THEOREM 6.7.** If \(f : (X, \tau^\ast) \rightarrow (Y, \sigma^\ast)\) is a supra \(-\) irresolute surjection and \(X\) is supra \(-\) connected, then \((Y, \sigma^\ast)\) is supra \(-\) connected.

**PROOF.** Suppose that \(Y\) is not supra \(-\) connected. Let \(Y = A \cup B\), where \(A\) and \(B\) are disjoint nonempty supra \(-\) open sets in \(Y\). Since \(f\) is supra \(-\) irresolute and onto, so \(X = f^{-1}(A) \cup f^{-1}(B)\), where \(f^{-1}(A)\) and \(f^{-1}(B)\) are disjoint nonempty supra \(-\) open sets in \((X, \tau^\ast)\). This contradicts the fact that \((X, \tau^\ast)\) is supra \(-\) connected. Hence \((Y, \sigma^\ast)\) is supra \(-\) connected.

**THEOREM 6.8.** If every supra \(-\) closed set in \(X\) is supra closed in \(X\) and \(X\) is supra \(-\) connected, then \(X\) is supra \(-\) connected.

**PROOF.** Suppose that \(X\) is supra connected. Then \(X\) cannot be expressed as a disjoint union of two nonempty proper supra open subset of \(X\). Suppose \(X\) is not supra \(-\) connected space. Let \(A\) and \(B\) be any two nonempty supra \(-\) open subsets of \(X\) such that \(X = A \cup B\), where \(A \cap B = \emptyset\). Since every supra \(-\) closed set in \(X\) is supra closed in \(X\). Therefore every supra \(-\) open set in \(X\) is supra open in \(X\). Hence \(A\) and \(B\) are supra open subsets of \(X\), which contradicts the fact that \(X\) is supra connected. Therefore \(X\) is supra \(-\) connected.

**THEOREM 6.9.** If two supra \(-\) open sets \(C\) and \(D\) form a separation of \(X\) and if \(Y\) is supra \(-\) connected subspace of \(X\), then \(Y\) lies entirely within \(C\) or \(D\).

**PROOF.** By hypothesis \(C\) and \(D\) are both supra \(-\) open sets in \(X\). The sets \(C \cap Y\) and \(D \cap Y\) are supra \(-\) open in \(Y\). These two sets are disjoint and their union is \(Y\). If they were both nonempty, then they would constitute a separation of \(Y\). Therefore, one of them is empty. Hence \(Y\) must lie entirely in \(C\) or \(D\).

**THEOREM 6.10.** Let \(A\) be a supra \(-\) connected subspace of \(X\). If \(A \subseteq B \subseteq \text{Supra-Cl}(A)\), then \(B\) is also supra \(-\) connected.

**PROOF.** Let \(A\) be supra \(-\) connected. Let \(A \subseteq B \subseteq \text{Supra-Cl}(A)\). Suppose that \(B = C \cup D\) is a separation of \(B\) by supra \(-\) open sets. Thus by the previous theorem \(A\) must lie entirely in \(C\) or \(D\). Suppose that \(A \subseteq C\), then it implies that \(\text{Supra-Cl}(A) \subseteq \text{Supra-Cl}(C)\). Since
Supra \( I - \text{Cl}(C) \) and \( D \) are disjoint, \( B \) cannot intersect \( D \). This disproves the fact that \( D \) is nonempty subset of \( B \). So \( D = \emptyset \) which implies \( B \) is supra \( I - \) connected.

ACKNOWLEDGEMENT

The author is highly and gratefully indebted to the Prince Mohammad Bin Fahd University, Saudi Arabia, for providing research facilities during the preparation of this research paper.

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