A New Closure and Interior Operators via V-Closed Sets and V-Open Sets

S. Saranya A, Dr. K. Bageerathi
Assistant professors of mathematics
Aditanar college of Arts and Science, Tiruchendur,
Tamil Nadu-628215, INDIA.

Abstract
The purpose of this paper is to introduce the some operators via $v$-open sets and $v$-closed sets in topological spaces and obtain some of interesting properties of these operators.

AMS Subject Classification (2010): 54A05

Keywords: $v$-open, $v$-closed, $v$-interior, $v$-closure.

1. INTRODUCTION

2. PRELIMINARIES
Throughout this paper, spaces $(X, \tau)$ (or simply $X$) always mean non empty topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset $A$ of a space $(X, \tau)$, $cl(A), int(A), cl^*(A), int^*(A)$ and $X/A$ denote the closure of $A$, the interior of $A$, $g$-closure of $A$, $g$-interior of $A$ and the complement of $A$ respectively. The following definitions and results are very useful in the subsequent sections

Definition 2.1. A subset $A$ of a topological space $(X, \tau)$ is said to be a $v$-open set if $A \subseteq int^*(cl(A)) \cup cl^*(int(A))$.

3. $v$-INTERIOR OPERATOR

Definition 3.1. Let $A$ be a subset of a topological space $(X, \tau)$. Then the union of all $v$-open sets contained in $A$ is called the $v$-interior of $A$ and it is denoted by $vint(A)$. That is, $vint(A) = \{V: V \subseteq A \text{ and } V \in v - O(X)\}$.

Remark 3.2. Since the union of $v$-open subsets of $X$ is $v$-open in $X$, then $vint(A)$ is $v$-open in $X$.

Definition 3.3. Let $A$ be a subset of a topological space $X$. A point $x \in X$ is called a $v$-interior point of $A$ if there exists a $v$-open set $G$ such that $x \in G \subseteq A$.

Theorem 3.4. Let $A$ be a subset of a topological space $(X, \tau)$. Then

(i) $vint(A)$ is the largest $v$-open set contained in $A$.
(ii) $A$ is $v$-open if and only if $vint(A)=A$.
(iii) $vint(A)$ is the set of all $v$-interior points of $A$.
(iv) $A$ is $v$-open if and only if every point of $A$ is a $v$-interior point of $A$.

Proof:
(i) Being the union of all \( v \)-open sets, \( \text{vint}(A) \) is \( v \)-open and contains every \( v \)-open subset of \( A \). Hence \( \text{vint}(A) \) is the largest \( v \)-open set contained in \( A \).

(ii) Necessity: Suppose \( A \) is \( v \)-open. Then by Definition 3.1, \( A \subseteq \text{vint}(A) \). But \( \text{vint}(A) \subseteq A \) and therefore, \( \text{vint}(A) = A \). Sufficiency: suppose \( \text{vint}(A) = A \). Then by Remark 3.2, \( \text{vint}(A) \) is \( v \)-open set. Hence \( A \) is \( v \)-open.

(iii) Let \( x \in \text{vint}(A) \) \( \iff \) \( x \in \bigcup \{ V \cdot V \subseteq A \text{ and } V \in v - O(X) \} \)

\[ \iff \text{there exists a } v \text{-open set } G \text{ such that } x \in G \subseteq A. \]

\[ \iff A \text{ is a } v \text{-nbhd of } x. \]

\[ \iff x \text{ is a } v \text{-interior point of } A. \]

Hence \( \text{vint}(A) \) is the set of all \( v \)-interior points of \( A \).

(iv) Suppose \( A \) is \( v \)-open. Then by part (ii) and (iii), we have every point of \( A \) is the \( v \)-interior point of \( A \).

**Theorem 3.5.** Let \( A \) and \( B \) be subsets of \((X, \tau)\). Then the following results hold.

(i) \( \text{vint}(\phi) = \phi \) and \( \text{vint}(X) = X \).

(ii) If \( B \) is any \( v \)-open set contained in \( A \), then \( B \subseteq \text{vint}(A) \).

(iii) If \( A \subseteq B \), then \( \text{vint}(A) \subseteq \text{vint}(B) \).

(iv) \( \text{int}(A) \subseteq \text{s'\text{int}}(A) \subseteq \text{vint}(A) \subseteq A \).

(v) \( \text{vint}(\text{vint}(A)) = \text{vint}(A) \).

**Proof:**

(i) Since \( \phi \) is the only \( v \)-open set contained in \( \phi \), then \( v\text{cl}(\phi) = \phi \). Since \( X \) is \( v \)-open and \( \text{vint}(X) \) is the union of all \( v \)-open sets contained in \( X \), \( \text{vint}(X) = X \).

(ii) Suppose \( B \) is \( v \)-open set contained in \( A \). Since \( \text{vint}(A) \) is the union of all \( v \)-open set contained in \( A \), then we have \( B \subseteq \text{vint}(A) \).

(iii) Suppose \( A \subseteq B \). Let \( x \in \text{vint}(A) \). Then \( x \) is a \( v \)-interior point of \( A \) and hence there exists a \( v \)-open set \( G \) such that \( x \in G \subseteq A \). Since \( A \subseteq B \), then \( x \in G \subseteq B \). Therefore \( x \) is a \( v \)-interior point of \( A \). Hence \( x \in \text{vint}(B) \). This proves (iii).

(iv) Since \( \text{semi}^* \text{-open set is } v \)-open, \( \text{s'\text{int}}(A) \subseteq \text{vint}(A) \). Every open set is semi* open, \( \text{int}(A) \subseteq \text{s'\text{int}}(A) \). Therefore \( \text{int}(A) \subseteq \text{s'\text{int}}(A) \subseteq \text{vint}(A) \subseteq A \). This proves (iv).

(v) By Remark 3.2, \( \text{vint}(A) \) is \( v \)-open and by Theorem 3.4, \( \text{vint}(\text{vint}(A)) = \text{vint}(A) \). This proves (v).

**Theorem 3.6.** Let \( A \) and \( B \) are the subsets of a topological space \( X \). Then,

(i) \( \text{vint}(A) \cup \text{vint}(B) \subseteq \text{vint}(A \cup B) \).

(ii) \( \text{vint}(A \cap B) \subseteq \text{vint}(A) \cap \text{vint}(B) \).

**Proof:**

(i) Let \( A \) and \( B \) be subsets of \( X \). We have \( A \subseteq A \cup B \) and \( B \subseteq A \cup B \). By Theorem 3.5(iii), \( \text{vint}(A) \subseteq \text{vint}(A \cup B) \) and \( \text{vint}(B) \subseteq \text{vint}(A \cup B) \) which implies that, \( \text{vint}(A) \cup \text{vint}(B) \subseteq \text{vint}(A \cup B) \). This proves (i).

(ii) We have \( A \cap B \subseteq A \) and \( A \cap B \subseteq B \). Then by Theorem 3.5(iii), \( \text{vint}(A \cap B) \subseteq \text{vint}(A) \) and \( \text{vint}(A \cap B) \subseteq \text{vint}(B) \) which implies, \( \text{vint}(A \cap B) \subseteq \text{vint}(A) \cap \text{vint}(B) \). This proves (ii).

**Theorem 3.7.** For any subset \( A \) of \( X \),

(i) \( \text{int}(\text{vint}(A)) = \text{int}(A) \)

(ii) \( \text{vint}(\text{int}(A)) = \text{int}(A) \).
Proof (i) Since \( \text{vint}(A) \subseteq A \), then \( \text{int}(\text{vint}(A)) \subseteq \text{int}(A) \). By Theorem 3.5(iv), \( \text{int}(A) \subseteq (\text{vint}(A)) \), we have \( \text{int}(A) = \text{int}(\text{int}(A)) \subseteq \text{int}(\text{vint}(A)) \). Hence \( \text{int}(\text{vint}(A)) = \text{int}(A) \).

(ii) Since \( \text{int}(A) \) is open and hence \( v \)-open, by Theorem 3.3, \( \text{vint}(\text{int}(A)) = \text{int}(A) \).

4. \( v \)-CLOSURE OPERATOR

**Definition 4.1.** Let \( A \) be a subset of a topological space \( (X, \tau) \). Then the intersection of all \( v \)-closed sets in \( X \) containing \( A \) is called the \( v \)-closure of \( A \) and it is denoted by \( \text{vcl}(A) \). That is, \( \text{vcl}(A) = \text{int} \{ F : A \subseteq F \text{ and } F \in v - C(X) \} \).

**Remark 4.2.** Since the intersection of \( v \)-closed set is \( v \)-closed, then \( \text{vcl}(A) \) is \( v \)-closed.

**Theorem 4.3.** Let \( A \) be a subset of a topological space \( (X, \tau) \). Then

(i) \( \text{vcl}(A) \) is the smallest \( v \)-closed set containing \( A \).

(ii) \( A \) is \( v \)-closed if and only if \( \text{vcl}(A) = A \).

Proof:

(i) Being the intersection of all \( v \)-closed sets, \( \text{vcl}(A) \) is \( v \)-closed and contained in every \( v \)-closed set containing \( A \). Hence \( \text{vcl}(A) \) is the smallest \( v \)-closed set containing \( A \).

(ii) Necessity: Suppose \( A \) is \( v \)-closed. Then by Definition 4.1, \( \text{vcl}(A) \subseteq A \). But \( A \subseteq \text{vcl}(A) \) and therefore \( \text{vcl}(A) = A \). Sufficiency: Suppose \( \text{vcl}(A) = A \). Then by Remark, \( \text{vcl}(A) \) is \( v \)-closed set. Hence \( A \) is \( v \)-closed.

**Theorem 4.4.** Let \( A \) and \( B \) be a two subsets of a topological space \( (X, \tau) \). Then the following results hold.

(i) \( \text{vcl}(\phi) = \phi \) and \( \text{vcl}(X) = X \).

(ii) If \( B \) is any \( v \)-closed set containing \( A \), then \( \text{vcl}(A) \subseteq B \).

(iii) If \( A \subseteq B \), then \( \text{vcl}(A) \subseteq \text{vcl}(B) \).

(iv) \( A \subseteq \text{vcl}(A) \subseteq \text{sc}(A) \subseteq \text{cl}(A) \).

(v) \( \text{vcl}((\text{vcl}(A))) = \text{vcl}(A) \).

Proof:

(i) Since \( \phi \) is \( v \)-closed and \( \text{vcl}(\phi) \) is the intersection of all \( v \)-closed sets containing \( \phi \), \( \text{vcl}(\phi) = \phi \). since \( X \) is the only \( v \)-closed set containing \( X \), then \( \text{vcl}(X) = X \).

(ii) Suppose \( B \) is \( v \)-closed set containing \( A \). Since \( \text{vcl}(A) \) is the intersection of all \( v \)-closed set containing \( A \), then we have \( \text{vcl}(A) \subseteq B \).

(iii) Suppose \( A \subseteq B \). Let \( F \) be any \( v \)-closed set containing \( B \). Since \( A \subseteq B \), then \( A \subseteq F \) and hence by part (ii), \( \text{vcl}(A) \subseteq F \). Therefore \( \text{vcl}(A) \subseteq \{ F / B \subseteq F \text{ and } F \text{ is } v \text{-closed} \} = \text{vcl}(B) \). This proves (iii).

(iv) Since \( \text{semi}^{*} \)-closed set is \( v \)-closed, \( \text{vcl}(A) \subseteq \text{semi}^{*}\text{cl}(A) \) and every closed set is \( v \)-closed, \( \text{vcl}(A) \subseteq \text{cl}(A) \). Therefore \( A \subseteq \text{vcl}(A) \subseteq \text{semi}^{*}\text{cl}(A) \subseteq \text{cl}(A) \). This proves (iv).

(v) By Remark 4.2, \( \text{vcl}(A) \) is \( v \)-closed and by Theorem 4.3, \( \text{vcl}((\text{vcl}(A))) = \text{vcl}(A) \). This proves (v).

**Theorem 4.5.** Let \( A \) and \( B \) be subsets of a topological space \( (X, \tau) \). Then,

(i) \( \text{vcl}(A) \cup \text{vcl}(B) \subseteq \text{vcl}(A \cup B) \).

(ii) \( \text{vcl}(A \cap B) \subseteq \text{vcl}(A) \cap \text{vcl}(B) \).

Proof: (i) Let \( A \) and \( B \) be subsets of \( X \). We have \( A \subseteq A \cup B \) and \( B \subseteq A \cup B \). By Theorem 4.4 (iii), \( \text{vcl}(A) \subseteq \text{vcl}(A \cup B) \) and \( \text{vcl}(B) \subseteq \text{vcl}(A \cup B) \) which implies that, \( \text{vcl}(A) \cup \text{vcl}(B) \subseteq \text{vcl}(A \cup B) \). This proves (i). (ii) We have \( A \cap B \subseteq A \) and \( A \cap B \subseteq B \). Then by Theorem 4.4 (iii), \( \text{vcl}(A \cap B) \subseteq \text{vcl}(A) \) and \( \text{vcl}(A \cap B) \subseteq \text{vcl}(B) \) which implies, \( \text{vcl}(A \cap B) \subseteq \text{vcl}(A) \cap \text{vcl}(B) \). This proves (ii).
Theorem 4.6. For a subset $A$ of $X$ and $x \in X$, $x \in vcl(A)$ if and only if $V \cap A \neq \phi$ for every $v$-open set $V$ containing $x$.

Proof: Necessity: Let $x \in vcl(A)$. Suppose there is a $v$-open set $V$ containing $x$ such that $V \cap A = \phi$. Then $A \subseteq X \setminus V$ and $X \setminus V$ is $v$-closed and hence $vcl(A) \subseteq X \setminus V$. Since $x \in vcl(A)$, then $x \in X \setminus V$ which contradicts to $x \in V$.

Sufficiency: Assume that $V \cap A \neq \phi$ for every $v$-open set $V$ containing $x$. Suppose $x \notin vcl(A)$. Then there exists a $v$-closed set $F$ such that $A \subseteq F$ and $x \notin F$. Therefore $x \in X \setminus F$, $A \cap (X \setminus F) = \phi$ and $X \setminus F$ is $v$-open. This is a contradiction to our assumption. Hence $x \in vcl(A)$.

Theorem 4.7. For any subset $A$ of $X$,

(i) $cl(vcl(A)) = cl(A)$
(ii) $vcl(cl(A)) = cl(A)$.

Proof: (i) Since $A \subseteq vcl(A)$, then $cl(A) \subseteq cl(vcl(A))$. By Theorem 4.4(iv), $vcl(A) \subseteq cl(A)$, we have $cl(vcl(A)) \subseteq cl(cl(A))$ = $cl(A)$. Hence $cl(vcl(A)) = cl(A)$. (ii) Since $cl(A)$ is closed and hence $v$-closed, by Theorem 4.3, $vcl(cl(A)) = cl(A)$.

REFERENCES