Fuzzy Finite Element Method Applied To Euler-Bernoulli Beam Problem

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Abstract:

In this paper, the finite element analysis for static displacements of some complicated Euler-Bernoulli beam structure is considered in fuzzy environment, where the material and geometric properties are taken as crisp. The numerical examples deals with cantilever beam, a beam clamped at one end and supported by a linear elastic spring. Various loads such as constant distributed, linearly varying, and point loads are considered for the examples. Assembled system of the above structures converts into fuzzy system of linear equations by taking right hand side global force vector as fuzzy keeping coefficient matrix as crisp. The results obtained are represented in terms of plots.

Key words:

Euler-Bernoulli beam, Fuzzy finite element method, Triangular fuzzy number(TFN), Fuzzy system of linear equations.

I. INTRODUCTION:

Computational mechanics is a flourishing subject for applied science and engineering, in which physical mechanics problems are solved by cooperation of mechanics, computers and various numerical methods. At the same time, new theories and methods of computational mechanics itself are also being developed gradually. Finite Element Method is an important branch of computational mechanics. It is one kind of powerful numerical methods in which various mechanics problems are solved by dicretizing related continuums. In 1941 Finite element was introduced by Alexander Hrennikoff and Rechard Courant in 1942. For large deformation and in nonlinear problems the FEM began applied in 1970. Several important books [1]-[3] are written on FEM with engineering applications.

Several beam theories have been developed based on various assumption and lead to different level of accuracy. One of the simplest and most useful of these theories was first described by Euler and Bernoulli and is commonly called Euler-Bernoulli Beam theory. A fundamental assumption of this theory is that cross section of the beam is infinitely rigid in its own plane, i.e., no deformations occur in the plane of cross section. During deformation, the cross section is assumed to remain plane and normal to the deformed axis of the beam.

The structural parameters involved in finite element analysis, such as, mass, material properties, external loads or boundary conditions are considered as crisp. But the uncertainties may occur due to incomplete data, vaguely defined geometry, experimental error and different imposed conditions influenced by the systems which plays an important role in various fields of engineering and applied science. Different authors used some probabilistic and statistical approach to operate uncertainties [4,5]. The word fuzzy means "vagueness" and the concept of fuzzy number and its arithmetic operations was initiated by Zadeh [6], Dubois and Prade [7].

When FEM is considered in fuzzy environment then it is known as Fuzzy Finite Element Method(FFEM). Interval analysis provides a powerful set of tools which is directly applicable to
the problems are explained by [8]. Also [9] represents the modelling of uncertain structural systems using interval analysis. The numerical estimation of the static displacement bounds with uncertain parameters is studied by [10]. Uncertain boundary conditions and the effect of uncertain prescribed displacements of structural systems is discussed in [11]. An important book [12] is written on the theory of fuzzy arithmetic and its applications in engineering sciences. In [13] a unique approach of Fuzzy finite element method for the analysis of imprecisely defined structural systems is defined. Fuzzy finite element analysis of smart structures is discussed in [14]. A practical approach for analyzing the static response of structures with fuzzy parameters is investigated in [15]. The author of [16] presents the fuzzy finite element analysis for static displacements of structures with fuzzy nodal force. Fuzzy arithmetical approach to solve the finite element problems with uncertain parameters is used in [17]. Moreover structural analysis with fuzzy parameters are excellently studied by [18,19]. System of linear equations are important for studying and solving of real world applications in many branches of engineering and science. $n \times n$ fuzzy system of linear equations have been studied by many authors [20]-[23]. Friedman et al. [22] proposed a general model for solving such systems with coefficient matrix as crisp and the right hand column is an arbitrary fuzzy number vector.

In this paper we recall some fundamental results of fuzzy set theories to investigate the static responses of some Euler-Bernoulli’s beam problem. Finite Element method turns into a system of linear equations. The numerical examples when converts into such types of linear equations, the various distributed loads in terms of triangular fuzzy numbers are considered to discuss the fuzzy responses.

II. PRELIMINARIES:

Here some basic definitions and useful theories of fuzzy calculus are reviewed. The basic definition of a fuzzy number given in [24,25].

A. Definition (Fuzzy Number):

A fuzzy number is a fuzzy mapping defined as $\mu_{\tilde{A}}(x): R \rightarrow I = [0,1], \forall x \in R$, where $\mu_{\tilde{A}}(x)$ is membership function of fuzzy set which is piecewise continuous. Also an another definition of fuzzy number in parametric form is given by Kaleva and Ma [26,27]. The set of all fuzzy number is denoted by $E$.

B. Definition (Triangular Fuzzy Number):

A triangular fuzzy number is denoted by $\tilde{A} = (a_1, a_m, a_2)$ with membership function $\mu_{\tilde{A}}(x)$ is defined on $R$ as

$$
\mu_{\tilde{A}}(x) = \begin{cases} 
\frac{x-a_1}{a_m-a_1} & \text{when } a_1 \leq x \leq a_m \\
\frac{a_2-x}{a_2-a_m} & \text{when } a_m \leq x \leq a_2 \\
0 & \text{otherwise.}
\end{cases}
$$
When the point \( a_m \in (a_1, a_2) \) is located at the middle of the supporting interval i.e., \( a_m = \frac{a_1 + a_2}{2} \), then the fuzzy number \( \tilde{A} \) is called central triangular fuzzy number.

The TFN \( \tilde{A} = (a_1, a_m, a_2) \) may be represented into interval form through \( \alpha \)-cut approach as follows:

\[
\tilde{A} = (a_1, a_m, a_2) = [\alpha(\alpha), \bar{u}(\alpha)] = [a_1 + \alpha(a_m - a_1), a_2 - \alpha(a_2 - a_m)], \quad \alpha \in [0,1].
\]

C. Some Arithmetic Operations on TFN:

For arbitrary fuzzy numbers \( \tilde{u} = (u(\alpha), \bar{u}(\alpha)), \bar{v} = (v(\alpha), \bar{v}(\alpha)), \) for \( \alpha \in [0,1] \) and a real number \( k \), we define the addition and scalar multiplication of fuzzy numbers by using the extension principle [27] as

a) \( \tilde{u} = \tilde{v} \) if and only if \( u(\alpha) = v(\alpha) \) and \( \bar{u}(\alpha) = \bar{v}(\alpha) \).

b) \( \tilde{u} + \tilde{v} = (u(\alpha) + v(\alpha), \bar{u}(\alpha) + \bar{v}(\alpha)) \).

c) \( \tilde{u} - \tilde{v} = (u(\alpha) - v(\alpha), \bar{u}(\alpha) - \bar{v}(\alpha)) \).

d) \( k\tilde{u} = (ku(\alpha), k\bar{u}(\alpha)), k \geq 0 \)

\( = (k\tilde{u}(\alpha), k\tilde{u}(\alpha)), k < 0 \).

D. Definition (Fuzzy System of Linear Equations(FSLE)):

The \( n \times n \) linear systems

\[
\begin{align*}
 a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= y_1 \\
 a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= y_2 \\
 & \vdots \hfill \quad \vdots \hfill \\
 a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= y_n \\
\end{align*}
\]

where, the given coefficient matrix \( A = (a_{ij}), 1 \leq i \leq n \) and \( 1 \leq j \leq n \) is a crisp \( n \times n \) matrix, and \( y_i \in \mathbb{R}, 1 \leq i \leq n \), with the unknowns \( x_j \in \mathbb{R}, 1 \leq j \leq n \) is called fuzzy linear system of equations (FLSE) [22].

E. Definition: [22] A fuzzy number vector \( (x_1, x_2, \ldots, x_n)^T \), where \( x_j = (\underline{x}(\alpha), \bar{x}(\alpha)); j = 1, 2, \ldots, n \), for \( 0 \leq \alpha \leq 1 \), is called a solution of the FSLE (1) if

\[
\sum_{j=1}^{n} a_{ij} \underline{x}_j = \sum_{j=1}^{n} a_{ij} \bar{x}_j = y_i,
\]

if for a particular \( i, a_{ij} > 0 \) for all \( j \), we get

\[
\sum_{j=1}^{n} a_{ij} \underline{x}_j = y_i, \quad \sum_{j=1}^{n} a_{ij} \bar{x}_j = y_i.
\]

Thus in order to solve the systems (1) one must solve a \( 2n \times 2n \) crisp linear system:

\[
\begin{align*}
 s_{11} \underline{x}_1 + \ldots + s_{1n} \underline{x}_n + s_{1,n+1} \bar{x}_1 + \ldots + s_{1,2n} \bar{x}_n &= y_1 \\
 s_{n1} \underline{x}_1 + \ldots + s_{n,n} \underline{x}_n + s_{n,n+1} \bar{x}_1 + \ldots + s_{n,2n} \bar{x}_n &= y_n \\
 s_{n+1,1} \underline{x}_1 + \ldots + s_{n+1,n} \underline{x}_n + s_{n+1,n+1} \bar{x}_1 + \ldots + s_{n+1,2n} \bar{x}_n &= y_1 \\
 s_{2n,1} \underline{x}_1 + \ldots + s_{2n,n} \underline{x}_n + s_{2n,n+1} \bar{x}_1 + \ldots + s_{2n,2n} \bar{x}_n &= y_n
\end{align*}
\]
Where $s_{ij}$ are determined as follows,

\[ a_{ij} \geq 0 \Rightarrow s_{ij} = s_{i+j} = a_{ij} \]
\[ a_{ij} < 0 \Rightarrow s_{i+j} = s_{i+j} = a_{ij} \text{ for } 1 \leq i, j \leq n. \]

(3)

and any $s_{ij}$ which is not determined by (3) is zero. Using matrix notation (2) can be written as

\[ S \mathbf{x} = \mathbf{y}. \]

(4)

Where, $S = S_{ij}, 1 \leq i, j \leq 2n$, $X = (x_{1},...,x_{n},\bar{x}_{1},...,\bar{x}_{n})^{T}$ and $Y = (y_{1},...,y_{n},\bar{y}_{1},...,\bar{y}_{n})^{T}$.

In this case the equation (4) is extended to the following crisp block form as

\[
\begin{pmatrix}
S_1 & S_2 \\
S_2 & S_1
\end{pmatrix}
\begin{pmatrix}
X_1 \\
X_2
\end{pmatrix}
=
\begin{pmatrix}
Y_1 \\
Y_2
\end{pmatrix}
\]

(5)

Where, $S_1$ and $S_2$ are $n \times n$ matrices with non-negative and non-positive elements respectively, and

\[
X_1 = \begin{pmatrix}
\tilde{x}_1(\alpha) \\
\cdot \\
\cdot \\
\tilde{x}_n(\alpha)
\end{pmatrix}, \quad X_2 = \begin{pmatrix}
\tilde{\bar{x}}_1(\alpha) \\
\cdot \\
\cdot \\
\tilde{\bar{x}}_n(\alpha)
\end{pmatrix}
\]

\[
Y_1 = \begin{pmatrix}
\tilde{y}_1(\alpha) \\
\cdot \\
\cdot \\
\tilde{y}_n(\alpha)
\end{pmatrix}, \quad Y_2 = \begin{pmatrix}
\tilde{\bar{y}}_1(\alpha) \\
\cdot \\
\cdot \\
\tilde{\bar{y}}_n(\alpha)
\end{pmatrix}
\]

III. FINITE ELEMENT FORMULATION OF EULER BERNOULLI’S BEAM ELEMENT:

Here the general form of the governing differential equation is

\[
\frac{d^2}{dx^2}(EI \frac{d^2w}{dx^2}) + c_f w = q(x), 0 < x < L
\]

(6)

Where, $E$ is the modulus of elasticity, $I$ denotes moment of inertia, $w$ the transverse deflection of the beam, $c_f$ is the elastic foundation modulus (if any), $q(x)$ is the transverse distributed load. The one dimensional finite element formulation for the above governing differential equation is well known but are given below for the purpose of completeness. When $c_f = 0$, then the suitable choice of approximation for $w$ over a typical element $(x_e, x_{e+1})$ of length $l_e$ gives the element equation in matrix form as

\[
[K^e] \{\Delta^e\} = \{F^e\}
\]

(7)

Here, $\{F^e\} = \{\mathbf{f}^e\} + \{\mathbf{Q}^e\}$ is the global load vector, $\{\mathbf{f}^e\} = [q^e, q^e_2, q^e_4]^T$ is the nodal force vector due to uniformly or varying load over the typical element and $\{\mathbf{Q}^e\} = [Q^e, Q^e_2, Q^e_3, Q^e_4]^T$ is the generalazied force vector, where, $Q^e_i (i = 1,3)$ and $Q^e_i (i = 2,4)$ denotes shear force and bending moment respectively. $\{\Delta\} = [\Delta_1, \Delta_2, \Delta_3, \Delta_4]^T$ is called the generalized displacement vector.
corresponding to the displacements and rotations at nodes.

\[
[K^e] = \frac{E_c I_c}{l_c^3} \begin{pmatrix}
12 & -6l_c & -12 & -6l_c \\
-6l_c & 4E_c l_c^2 & 6E_c l_c & 2I_c \\
-12 & 6E_c l_c^2 & 12 & 6E_c l_c \\
-6E_c l_c & 2I_c^2 & 6E_c l_c & 4I_c^2
\end{pmatrix}
\]  \hspace{1cm} (8)

is the stiffness matrix for a element with length, modulus of elasticity and moment of inertia are \( l_c, E_c \) and \( I_c \) respectively. If the element is subjected to uniformly distributed load of intensity \( q_e \) then

\[
\{q^e\} = \frac{qE}{12}[6, -l_c, 6l_c]^T
\]

When the distributed load is a function of \( x \), say \( q(x) \), then the components \( q^e_i \) of \( \{q^e\} \) obtained as

\[
q^e_i = \int_{x_i}^{x_{i+1}} \phi_i^e(x) q(x) \, dx.
\]  \hspace{1cm} (10)

Where, \( \phi_i^e(x) \) are the Hermite cubic interpolation function.

Depending upon the geometry, the domain of the problem in this method is discretized into a collection of finite elements. Each element gives a stiffness matrix of the form (8). To get the assembled coefficient matrix of the complete domain we need to combine all the stiffness matrices. When we discretized a beam with \( n \) – elements then the final stiffness matrix in global system \([K]\) looks as

\[
\begin{pmatrix}
12E_1 I_1 & -6E_1 I_1 & -12E_1 I_1 & -6E_1 I_1 & 0 & 0 & 0 \\
-6E_1 I_1 & 4E_1 I_1^2 & 6E_1 I_1 & 2E_1 I_1 & 0 & 0 & 0 \\
-12E_1 I_1 & 6E_1 I_1 & 12E_1 I_1 & 6E_1 I_2 & 0 & 0 & 0 \\
6E_1 I_1 & -6E_1 I_2 & -6E_1 I_2 & 6E_1 I_3 & 0 & 0 & 0 \\
6E_1 I_1 & -2E_1 I_3 & -6E_1 I_3 & 6E_1 I_4 & \ddots & \ddots & \vdots \\
-l_2^3 & l_2^2 & -l_2^1 & l_2 & 6E_n I_n & 2E_n I_n & 0 \\
0 & 0 & 0 & \vdots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & -6E_{n-1} I_{n-1} & 6E_{n-1} I_{n-1} & 12E_{n-1} I_{n-1} & 6E_{n-1} I_{n-1} \\
0 & 0 & 0 & -6E_{n-1} I_{n-1} & 2E_{n-1} I_{n-1} & 6E_{n-1} I_{n-1} & 6E_{n-1} I_{n-1} \\
0 & 0 & 0 & -6E_{n-1} I_{n-1} & 6E_{n-1} I_{n-1} & 4E_{n-1} I_{n-1} & + 4E_{n-1} I_{n-1}
\end{pmatrix}
\]

which is a symmetric matrix of order \( 2(n+1) \). The right hand side global load vector for \( n \) elements as
If the global displacement vector is given by the following equation

\[ \{F\} = \{q\} + \{Q\} = \begin{bmatrix} q_1^1 \\ q_2^1 \\ q_3^1 + q_1^1 \\ q_4^1 + q_2^1 \\ q_5^2 + q_1^2 \\ \vdots \\ \vdots \\ q_3^n \\ q_4^n \\ \end{bmatrix} + \begin{bmatrix} Q_1^1 \\ Q_2^1 \\ Q_1^2 + Q_2^2 \\ Q_4^2 + Q_3^2 \\ Q_5^3 \\ \vdots \\ \vdots \\ Q_3^n \\ Q_4^n \\ \end{bmatrix} \]

If the global displacement vector is given by the following equation

\[ U = [U_1, U_2, U_3, \ldots, U_{2(n+1)}]^T \]

then the assembled equations becomes

\[ [K] \{U\} = \{F\}. \quad (1) \]

In the assembly procedure, when we select three finite elements, there are four global nodes and eight global generalized displacements and eight generalized forces. For each node there are two degrees of freedom. At the node \( i \) degrees of freedom for \( U_{2i-1} \) is the transverse displacement and degrees of freedom for \( U_{2i} \) is a rotation.

![Finite element mesh](image1)

**Fig.1:** (a) Finite element mesh of 3 elements with global displacement vector. (b) Generalized forces on a typical element

**IV. FINITE ELEMENT FORMULATION WITH FUZZY INPUT LOAD:**

It is easy to handle the assembled equations when the coefficient matrix and the right hand side force vector are crisp. But when the information about the input loads involved in the problems are imprecise in nature then to deal with the corresponding equation needing much effort. Due to uncertain distributed load the stiffness matrix \( S \) of the global system is obtained from equations (3) and the right hand side fuzzy load vector can be written as
In this study, TFN is used through $\alpha-cut$ for fuzzy input loads. The displacements and rotations in this approach would also give intervals in results. Thus, if the solution vector for fuzzy loads is given by the equation

$$U^* = [U_1, U_2, ..., U_x(n+1), \bar{U}_1, \bar{U}_2, ..., \bar{U}_x(n+1)]^T$$

then the assembled equations becomes

$$[S] \{U^*\} = \{\tilde{F}\}.$$  

V. Numerical Examples and Results: Example 1:

Here we consider a indeterminate beam of length 48$\text{in.}$ with nonuniform mesh of three elements and subjected to the combination of linearly varying, constant distributed load and point load as shown fig.3.[2]. Only vertical displacements $U_i (i = 1, 3, 5, 7)$ and angle of rotations $U_i (i = 2, 4, 6, 8)$ of nodes are considered here. The physical and material properties such as Young’s modulus ($E = 30 \times 10^6 \text{ psi}$) and moment of inertia ($I = 4.5 \text{in.}^4$) same for each elements and considered as crisp variables. Two different types of distributed loads $q^{(1)}(x)$ and $q^{(2)}(x)$ acting on elements 1 and 2 respectively and the point load $F_0$ at node 4 are considered both as crisp and TFN to compute the static response of the beam. Discretization of the domain of this problem is the same as fig. 1(a)
Case I (Crisp Load):

We consider here the distributed and point loads as crisp such as,

\[ q^{(1)}(x) = \begin{cases} 30 \text{ lb/in.}, & x = 0 \\ 20 \text{ lb/in.}, & x = 16 \end{cases} \]

\[ q^{(2)}(x) = 20 \text{ lb/in.}, \quad 16 \leq x \leq 36 \]

Here \( q^{(1)}(x) \) is linearly varying distributed load in \([0, 16]\). If we take \( Bx = \frac{30}{20}x \), then the boundary conditions on it gives \( A = 30 \) and \( B = -10/16 \). Now we evaluate the contribution of \( q^{(1)}(x) \) to the element load vector from the equation (10).

\[
q_1^1 = \int_0^1 \phi_1^1(x) q^{(1)}(x) \, dx = \int_0^1 \left\{ 1 - 3 \left( \frac{x}{l_1} \right)^2 + 2 \left( \frac{x}{l_1} \right)^3 \right\} (A + Bx) \, dx = 216.
\]

\[
q_2^1 = \int_0^1 \phi_2^1(x) q^{(1)}(x) \, dx = -\int_0^1 x \left( 1 - \frac{x}{l_1} \right)^2 (A + Bx) \, dx = -554.6667.
\]

\[
q_3^1 = \int_0^1 \phi_3^1(x) q^{(1)}(x) \, dx = \int_0^1 \left\{ 3 \left( \frac{x}{l_1} \right)^2 - 2 \left( \frac{x}{l_1} \right)^3 \right\} (A + Bx) \, dx = 184.
\]

\[
q_4^1 = \int_0^1 \phi_4^1(x) q^{(1)}(x) \, dx = -\int_0^1 \left( \frac{x}{l_1} - \frac{x}{l_1} \right) (A + Bx) \, dx = 512. \text{ where, } l_1 = 16 \text{ in.}
\]

Therefore, \( \{q^1\} = [q_1^1, q_2^1, q_3^1, q_4^1]^T = [216, -554.6667, 184, 512]^T \).

Again, \( q^{(2)}(x) \) is constant distributed load in \([16, 36]\).

\[
\text{So, } \{q^2\} = [q_1^2, q_2^2, q_3^2, q_4^2]^T = \frac{q^{(2)}(x) l_2}{12} \{6, -l_2, 6, l_2\}^T = [200, -666.6667, 200, 666.6667]^T.
\]

where, \( l_2 = 20 \text{ in.} \) and, \( \{q^3\} = [q_1^3, q_2^3, q_3^3, q_4^3]^T = [0, 0, 0, 0]^T \).

The boundary and balanced conditions for this problem are

\[
U_1 = U_2 = U_5 = 0.
\]

\[
Q^1_3 + Q^2_3 = Q^1_4 + Q^2_4 = Q^3_3 + Q^3_4 = 0, \quad Q^1_1 = 500 \text{ lb.}, \quad Q^3_1 = 0.
\]

The homogeneous boundary conditions on the primary variables are imposed by
elimination method which gives the permission to delete the rows and columns 1, 2 and 5 of the assembled coefficient matrix to get the condensed equations. Thus, the condensed equations are

\[
\begin{pmatrix}
0.598 & 1.1394 & 2.025 & 0 & U_3 \\
1.1391 & 60.75 & 13.5 & 0 & U_4 \\
10^6 & -2.025 & 13.5 & 72 & 5.625 & 2 U_5 \\
& 0 & 0 & 5.625 & 0.9375 & 5.625 & 45 U_6 \\
& 0 & 0 & 22.5 & 5.625 & 45 U_8 & 0
\end{pmatrix} = \begin{pmatrix}
384 \\
-154.6667 \\
666.6667 \\
500 \\
0
\end{pmatrix}
\]

Solving the above systems we get the static responses as

\[
\begin{align*}
U_3 &= -0.322 \text{ in.} \\
U_4 &= 0.0594 \text{ rad.} \\
U_5 &= 10^{-3} -0.2514 \text{ rad.} \\
U_6 &= 5.1497 \text{ in.} \\
U_7 &= 0.2514 \text{ rad.} \\
U_8 &= -0.518 \text{ rad.}
\end{align*}
\]

The results obtained above for crisp parameters are found to be similar with the crisp solution of [2].

**Case II (Fuzzy Load):**

Here, the distributed and point loads are considered as triangular fuzzy number that is

\[
\tilde{q}^{(1)}(x) = \begin{cases}
(28, 30, 32) \text{ lb/in.}, x = 0 \\
(18, 20, 22) \text{ lb/in.}, x = 16
\end{cases}
\]

\[
\tilde{q}^{(2)}(x) = (18, 20, 22) \text{ lb/in. at } x = 16 \leq x \leq 36, \quad \tilde{q}^{(3)}(x) = 0 \text{ at } 36 \leq x \leq 48,
\]
and \(\bar{F}_0 = (495, 500, 505)\) lb.

The corresponding interval forms in terms of \(\alpha\)-cut of the triangular fuzzy loads are

\[
\tilde{q}^{(1)}(x) = [q^{(1)}(x)] = \begin{cases}
28 + 2\alpha, 32 - 2\alpha & \text{lb/in.} \\
18 + 2\alpha, 22 - 2\alpha & \text{lb/in.}
\end{cases}
\]

\[
\tilde{q}^{(2)}(x) = [q^{(2)}(x), q^{(2)}(x)] = [18 + 2\alpha, 22 - 2\alpha] \text{ lb/in. at } x \in [16, 36]
\]

\[
\tilde{q}^{(3)}(x) = [q^{(3)}(x), q^{(3)}(x)] = [0, 0] \text{ at } x \in [36, 48]
\]

and, \(\bar{F}_0 = [E_0, \bar{F}_0] = [495 + 5\alpha, 505 - 5\alpha]\), where, \(\alpha \in [0, 1]\).

Therefore, \(\tilde{q}^{(1)}(x) = \tilde{A} + \tilde{B} x\) gives,

\[
\tilde{A} = [28 + 2\alpha, 32 - 2\alpha], \text{ and } \tilde{B} = [\frac{\alpha}{4}, \frac{7}{8}, \frac{\alpha}{4}, \frac{3}{8}].
\]

Equation (13) says that, when we evaluate the contribution of \(\tilde{q}^{(1)}(x)\) to the element load vector, we only calculate the components \(\tilde{q}^{1}_3\) and \(\tilde{q}^{1}_4\) of \(\tilde{q}^{1}\)

\[
\tilde{q}_3 = \int_0^1 \phi_3(x) \tilde{q}^{(1)}(x) dx = \int_0^1 \left(3 \left(\frac{x}{l_1}\right)^2 - 2 \left(\frac{x}{l_1}\right)^3\right) (\tilde{A} + \tilde{B} x) dx = \frac{728}{5} + \frac{192\alpha}{5}, \quad \frac{1112}{5} - \frac{192\alpha}{5}.
\]
\[
\tilde{q}_i = \int_0^l \phi_i(x) \tilde{q}^{(1)}(x) \, dx = -\int_0^l x \left( \frac{x}{l} \right)^2 \left( \frac{x}{l} \right) (\tilde{A} + \tilde{B}x) \, dx = \left[ \frac{6272}{15}, \frac{1408\alpha}{15}, \frac{9088}{15}, -\frac{1408\alpha}{15} \right].
\]

Thus,
\[
\left\{ \tilde{q}^{(3)} \right\} = [\tilde{q}_1, \tilde{q}_2, \tilde{q}_3, \tilde{q}_4]^T = [[*,*],[*,*],[*,*],[*,*]].
\]

The balanced conditions for triangular fuzzy loads are:
\[
\tilde{Q}_1 + \tilde{Q}_2 = \tilde{Q}_3 + \tilde{Q}_4 = [0,0] \quad \tilde{Q}_3 = [495+5\alpha, 505-5\alpha] \quad \text{and} \quad \tilde{Q}_4 = [0].
\]

Therefore, the condensed equations (13) for uncertain distributed loads becomes

\[
\begin{bmatrix}
0.598 & 1.1391 & -2.025 & 0 & 0 \\
1.1391 & 60.75 & 13.5 & 0 & 0 \\
\frac{10^6}{-2.025} & 13.5 & 72 & 5.625 & 22.5 \\
0 & 0 & 5.625 & 0.9375 & 5.625 \\
0 & 0 & 22.5 & 5.625 & 45
\end{bmatrix}
\begin{bmatrix}
U_1, U_2, U_3, U_4, U_5, U_6, U_7, U_8
\end{bmatrix}^T =
\begin{bmatrix}
\frac{1628}{5} + \frac{292\alpha}{5}, \frac{2212}{5} - \frac{292\alpha}{5} \\
-\frac{4728}{15} + \frac{2408\alpha}{15}, \frac{88}{15} - \frac{2408\alpha}{15} \\
\frac{600}{3} + \frac{200\alpha}{3}, \frac{2200}{3} - \frac{200\alpha}{3} \\
[495+5\alpha, 505-5\alpha] \quad \text{and} \quad [0]
\end{bmatrix}.
\]

Solving the above fuzzy system of linear equations we get the lower and upper bounds of fuzzy static responses for triangular fuzzy loads and the obtained results are given in the table 1.

Table 1: Lower and upper bounds of fuzzy static responses for triangular fuzzy load for Example 1:
\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
\(\alpha\) & 0 & 0.2 & 0.4 & 0.6 & 0.8 & 1 \\
\hline
\(\frac{U_3}{\overline{U}_3}\) & -0.4512e-3 & -0.4254e-3 & -0.3995e-3 & -0.3737e-3 & -0.3479e-3 & -0.3221e-3 \\
& -0.1931e-3 & -0.2189e-3 & -0.2447e-3 & -0.2705e-3 & -0.2963e-3 & -0.3221e-3 \\
\hline
\(\frac{U_4}{\overline{U}_4}\) & 0.0567e-3 & 0.0573e-3 & 0.0578e-3 & 0.0583e-3 & 0.0588e-3 & 0.0594e-3 \\
& 0.0620e-3 & 0.0614e-3 & 0.0609e-3 & 0.0604e-3 & 0.0599e-3 & 0.0594e-3 \\
\hline
\(\frac{U_6}{\overline{U}_6}\) & -0.2406e-3 & -0.2408e-3 & -0.2449e-3 & -0.2471e-3 & -0.2492e-3 & -0.2514e-3 \\
& -0.2621e-3 & -0.2600e-3 & -0.2578e-3 & -0.2557e-3 & -0.2535e-3 & -0.2514e-3 \\
\hline
\(\frac{U_7}{\overline{U}_7}\) & 4.9996e-3 & 5.0296e-3 & 5.0596e-3 & 5.0897e-3 & 5.1197e-3 & 5.1497e-3 \\
& 5.2999e-3 & 5.2699e-3 & 5.2398e-3 & 5.2098e-3 & 5.1798e-3 & 5.1497e-3 \\
\hline
\(\frac{U_8}{\overline{U}_8}\) & -0.5046e-3 & -0.5073e-3 & -0.5100e-3 & -0.5127e-3 & -0.5154e-3 & -0.5180e-3 \\
& -0.5314e-3 & -0.5288e-3 & -0.5261e-3 & -0.5234e-3 & -0.5207e-3 & -0.5180e-3 \\
\hline
\end{tabular}
\end{table}
Figure 3. (a) and (d) represents the minimum and maximum bounds of transverse displacements at nodes 2 and 4 respectively. (b),(c) and (e) represents the minimum and maximum bounds of angle of rotations at nodes 2, 3 and 4 respectively of problem 1.

Example 2:
Here we consider a two stepped indeterminant beam clamped at left end and whose right end is linear elastic spring supported with spring constant $k$. A rigid loading frame is placed at the middle of the beam which is subjected to a point load $F_0$ as shown in figure 5. The beam is discretized into two elements of equal length. On the first element an uniformly distributed load $q_0$ is acting. The material and geometric properties are considered crisp as

$$EI = 50 \times 10^6 \text{ N} \cdot \text{m}^2; \; h = 4 \text{ m}; \; k = 10^6 \text{ N/m} \; \text{and} \; \ell = 0.5 \text{ m}.$$ 

The elastic spring acting here as another finite element with element equation as

$$k \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} u_i^x \\ u_i^y \end{pmatrix} = \begin{pmatrix} Q_i^x \\ Q_i^y \end{pmatrix}$$ \hspace{1cm} (14)

Where $(Q_i^x, Q_i^y)$ and $(u_i^x, u_i^y)$ are end forces and end displacements respectively of the spring element with spring constant $k$. We assemble the beam element with spring element in such a manner that the vertical displacement of beam is the same as axial displacement of the spring. Therefore, three elements (e.g., two beam elements and one spring element) are used to discuss the problem. The transverse displacements $U_i (i = 1, 3, 5, 7)$ and angle of rotations $U_i (i = 2, 4, 6)$ of nodes are considered here.
The assembled equations are

\[
\begin{bmatrix}
12 & -6h & -12 & -6h & 0 & 0 & 0 \\
-6h & 4h^2 & 6h & 2h^2 & 0 & 0 & 0 \\
-12 & 6h & 18 & 3h & -6 & -3h & 0 \\
-6h & 2h^2 & 3h & 6h^2 & 3h & h^2 & 0 \\
0 & 0 & -6 & 3h & 6+a & 3h & -a \\
0 & 0 & -3h & h^2 & 3h & 2h^2 & 0 \\
0 & 0 & 0 & 0 & -a & 0 & a
\end{bmatrix}
\begin{bmatrix}
U_1 \\
U_2 \\
U_3 \\
U_4 \\
U_5 \\
U_6 \\
U_7
\end{bmatrix}
= \begin{bmatrix}
q_1^1 \\
q_2^1 \\
q_3^1 + q_1^2 \\
q_4^1 + q_2^2 \\
q_3^2 + q_1^3 \\
q_4^2 \\
q_3^3
\end{bmatrix}
+ \begin{bmatrix}
Q_1^1 \\
Q_2^1 \\
Q_3^1 + Q_2^2 \\
Q_4^1 + Q_2^2 \\
Q_3^2 + Q_1^3 \\
Q_2^3 \\
Q_3^3
\end{bmatrix}
\] (15)

where \( k h^3 \frac{2E}{h^3} \)

With the given geometric and material properties the distributed load \( q_0 \) and the concentrated load \( F_0 \) are considered both as crisp and triangular fuzzy number to compute the static responses of the beam.

**Case I (Crisp Load):**

Let us consider the distributed load \( q_0 \) and the point load \( F_0 \) as crisp, where 

\[ q_0 = 10^3 \text{N/m}; F_0 = 5000 \text{N}. \]

The contribution of \( q_0 \) to the element load vector is given by

\[
\{q^i\} = \begin{bmatrix} q_1^i, q_2^i, q_3^i, q_4^i \end{bmatrix} = \frac{q_0 h}{12} \begin{bmatrix} 6, -h, 6, h \end{bmatrix} = \begin{bmatrix} 2000, -\frac{4000}{3}, 2000, \frac{4000}{3} \end{bmatrix}^T.
\]

Since, there are no distributed loads on the other elements, the components of load vector \( \{q^i\} \) for \( (i = 2, 3) \) are zero. The global node 2 have a downward load of \( F_0 = 5000 \text{N} \) and bending moment of \( -d.F_0 = -2500 \text{N.m} \). The specified global displacements, forces and balanced equilibrium conditions are

\[
U_1 = U_2 = U_7 = 0
\]

\[
Q_1^1 + Q_2^1 = F_0 = 5000 \text{N}, Q_4^1 + Q_2^2 = -d.F_0 = -2500 \text{N.m}, Q_3^2 + Q_1^3 = Q_4^2 = 0.
\]
The condensed equations for the unknown global displacements are given by deleting the rows and columns corresponding to the specified global displacements. Thus, by deleting rows and columns 1,2 and 7, one may obtain the \(4 \times 4\) matrix equations as

\[
\begin{pmatrix}
28.125 & 18.75 & -9.375 & -18.75 \\
18.75 & 150 & 18.75 & 25 \\
-9.375 & 18.75 & 10.375 & 18.75 \\
-18.75 & 25 & 18.75 & 50
\end{pmatrix}
\begin{pmatrix}
U_3 \\
U_4 \\
U_5 \\
U_6
\end{pmatrix}
= \begin{pmatrix}
7000 \\
3500 \\
3 \\
0
\end{pmatrix}.
\tag{16}
\]

Solving the above systems we get the static responses as

\[
\begin{pmatrix}
U_3 \\
U_4 \\
U_5 \\
U_6
\end{pmatrix}
= 10^{-3}
\begin{pmatrix}
0.8536377 \text{ m.} \\
-0.2768036 \text{ rad.} \\
1.3744292 \text{ m.} \\
-0.0568949 \text{ rad.}
\end{pmatrix}.
\]

**Case II (Fuzzy Load):**

Next, we consider the distributed load \(q_0\) and the concentrated load \(F_0\) as triangular fuzzy number, where

\[
\tilde{q}_0 = (0.95 \times 10^3, 10^3, 1.05 \times 10^3) \text{N/m}; \tilde{F}_0 = (4900, 5000, 5100) \text{N}.
\]

The corresponding interval forms in terms of \(\alpha \text{-cut}\) of triangular fuzzy loads are given by

\[
\tilde{q}_0 = [q_0, -q_0] = [950 + 50\alpha, 1050 - 50\alpha] \text{ and } \tilde{F}_0 = [F_0, -F_0] = [4900 + 100\alpha, 5100 - 100\alpha], \text{where, } \alpha \in [0, 1].
\]

The element load vector due to \(\tilde{q}_0\) is

\[
\begin{pmatrix}
\tilde{q}_1^1 \\
\tilde{q}_1^2 \\
\tilde{q}_3^1 \\
\tilde{q}_4^1
\end{pmatrix}
= \frac{3}{12} \begin{pmatrix}
4200 \text{ and } 200\alpha \\
300 \text{ and } 200\alpha \\
300 \text{ and } 300\alpha \\
300 \text{ and } 300\alpha
\end{pmatrix}
\]

and the other two load vectors \(\begin{pmatrix}
\tilde{q}_1^i \\
\tilde{q}_2^i \\
\tilde{q}_3^i \\
\tilde{q}_4^i
\end{pmatrix}\) (for \(i = 2,3\)) are zero. The specified conditions of the internal forces for triangular fuzzy loads are

\[
\tilde{Q}_3^1 + \tilde{Q}_4^1 = \tilde{F}_0 = [4900 + 100\alpha, 5100 - 100\alpha] \text{N}, \tilde{Q}_3^2 + \tilde{Q}_4^2 = -d \tilde{F}_0 = -0.5[4900 + 100\alpha, 5100 - 100\alpha]
\]

\[
= [-2500 + 50\alpha, 2450 - 50\alpha] \text{N/m}.
\]

So, for uncertain distributed loads, the condensed equations (16) takes the form
Solving the above systems we get the lower and upper bounds of fuzzy static responses and that are given in table 2.

Table 2: Lower and upper bounds of fuzzy static responses for triangular fuzzy load for example 2:

<table>
<thead>
<tr>
<th>α</th>
<th>U₃</th>
<th>U₄</th>
<th>U₅</th>
<th>U₆</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.8435e-3</td>
<td>-0.2776e-3</td>
<td>1.3838e-3</td>
<td>-0.0562e-3</td>
</tr>
<tr>
<td>0.2</td>
<td>0.8455e-3</td>
<td>-0.2774e-3</td>
<td>1.3819e-3</td>
<td>-0.0563e-3</td>
</tr>
<tr>
<td>0.4</td>
<td>0.8475e-3</td>
<td>-0.2773e-3</td>
<td>1.3800e-3</td>
<td>-0.0565e-3</td>
</tr>
<tr>
<td>0.6</td>
<td>0.8496e-3</td>
<td>-0.2771e-3</td>
<td>1.3782e-3</td>
<td>-0.0566e-3</td>
</tr>
<tr>
<td>0.8</td>
<td>0.8516e-3</td>
<td>-0.2770e-3</td>
<td>1.3763e-3</td>
<td>-0.0568e-3</td>
</tr>
<tr>
<td>1</td>
<td>0.8536e-3</td>
<td>-0.2768e-3</td>
<td>1.3744e-3</td>
<td>-0.0569e-3</td>
</tr>
</tbody>
</table>

\[
\begin{pmatrix}
28.125 & 18.75 & -9.375 & -18.75 \\
18.75 & 150 & 18.75 & 25 \\
-9.375 & 18.75 & 10.375 & 18.75 \\
-18.75 & 25 & 18.75 & 50
\end{pmatrix}
\begin{pmatrix}
U_3 \\
U_4 \\
U_5 \\
U_6
\end{pmatrix}
= \frac{1}{10^6}
\begin{pmatrix}
[6800 + 200\alpha, 7200 - 200\alpha] \\
3850 + 350\alpha, 3150 - 350\alpha \\
0, 0 \\
0, 0
\end{pmatrix}
\]
Figure 5. (f) and (h) represents the minimum and maximum bounds of transverse displacements at nodes 2 and 3 respectively. (g) and (i) represents the minimum and maximum bounds of angle of rotations at nodes 2 and 3 respectively of the problem 2.

VI. CONCLUSIONS:
The static responses of some Euler Bernoulli’s beam problems using Fuzzy finite element method has been studied here. When the practical problems involve complicated shapes together with the loads involving uncertainties, the Fuzzy finite element method discussed here in a smooth way. In this paper we considered the loads as triangular fuzzy number only. This study can be extended to the other beam problems with loads as interval, trapezoidal and Type-2 fuzzy numbers. Instead of fuzzy loads one may consider the uncertainties in geometric and material properties. Matlab has been used to depict the results in terms of plots.

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VIII. REFERENCES:


