Deriving Shape Functions and Verified for One Dimensional Hermite Polynomials by Taking Natural Coordinate System 0 To 1

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Abstract — In this paper, I derived shape functions for one dimensional Hermite Polynomials by taking natural coordinate system 0 to 1 and also I verified three verification conditions for shape functions. First verification condition is at node 1 is \( N_1 = 1 \) and \( N_2 = 0, N_3 = 0, N_4 = 0 \) and also \( \frac{\partial N_1}{\partial x} = 1 \) and \( \frac{\partial N_2}{\partial x} = 0, \frac{\partial N_3}{\partial x} = 0, \frac{\partial N_4}{\partial x} = 0 \). Second Verification condition is at node 2 \( N_3 = 1 \) and \( N_1 = 0, N_2 = 0, N_4 = 0 \) and also \( \frac{\partial N_3}{\partial x} = 1 \) and \( \frac{\partial N_1}{\partial x} = 0, \frac{\partial N_2}{\partial x} = 0, \frac{\partial N_4}{\partial x} = 0 \). Third Verification condition is \( N_1 + N_4 = 1 \). For computational purpose I used Mathematica 9 Software [2].

Keywords — Hermite Polynomials, Natural Coordinate System, Shape functions.

I. INTRODUCTION

In Finite Element Analysis any domain of geometry can be split into finite number of domains. Each domain has a particular shape of geometry for example like Rectangular shape, Triangle shape, Circular shape. To study the analysis of these geometries first we need shape functions.

II. GEOMETRICAL DESCRIPTION

![Figure 1: Beam element with natural coordinates varying from 0 to 1](image)

A two noded beam element shown in Figure 1 in which nodal unknowns are the displacement \( W \) and Slope \( \frac{\partial W}{\partial x} \).

III. DERIVING SHAPE FUNCTIONS FOR ONE DIMENSIONAL HERMITE POLYNOMIALS

Since the element in figure 1 has four degrees of freedom, we have to select the polynomial with only 4 constants. In this polynomial after boundary conditions we get shape functions this we can take as first order (cubic) Hermitian Polynomials as shape functions.

\[
W(x) = A_1 + A_2 x + A_3 x^2 + A_4 x^3 \quad (1)
\]

Where \( W \) is the transverse displacement and \( A_1, A_2, A_3, A_4 \) are polynomial Coefficients.
Differentiating eq(1) w.r.t. 'x' 

\[
(1) \Rightarrow \frac{\partial W}{\partial x} = 0 + A_2(1) + A_3(2x) + A_4(3x^2)
\]

\[
\frac{\partial W}{\partial x} = A_2 + 2A_3x + 3A_4x^2
\]  \hspace{1cm} (2)

Applying the nodal conditions such that 
\[W=W_i\] and \[\frac{\partial W}{\partial x} = \theta_1\] at \[x=0\]

and \[W=W_2\] and \[\frac{\partial W}{\partial x} = \theta_2\] at \[x=l\]

in equations (1) and (2), we get

When \(W=W_i\) and \(x=0\)

\[(1) \Rightarrow W_i = A_1 + A_2(0) + A_3(0)^2 + A_4(0)^3\]

\[W_i = A_1 + 0 + 0 + 0\]

\[W_i = A_1\]  \hspace{1cm} (3)

When \(\frac{\partial W}{\partial x} = \theta_1\) and \(x=0\)

\[(2) \Rightarrow \theta_1 = A_2 + 2A_3(0) + 3A_4(0)^2\]

\[\theta_1 = A_2 + 0 + 0\]

\[\theta_1 = A_2\]  \hspace{1cm} (4)

When \(W=W_2\) and \(x=l\)

\[(1) \Rightarrow W_2 = A_1 + A_2(l) + A_3l^2 + A_4l^3\]  \hspace{1cm} (5)

When \(\frac{\partial W}{\partial x} = \theta_2\) and \(x=l\)

\[(2) \Rightarrow \theta_2 = A_2 + 2A_3l + 3A_4l^2\]  \hspace{1cm} (6)

Using Mathematica 9 Software Solving (3), (4), (5) and (6) we get \(A_1, A_2, A_3, A_4\)

Input

\[\text{Solve}\{A_1 - W_1 = 0 & & A_2 - \theta_1 = 0 & & A_3 + (A_2 * (0)) + (A_3 * l^2) + (A_4 * l^3) - W_2 = 0 & & A_4 + (2 * A_3 * l^2) + (3 * A_4 * l^5) - \theta_2 = 0, \{A_1, A_2, A_3, A_4\}\}\]

Output

\[\{\{A_1 -> W_1, A_2 -> \theta_1, A_3 -> \frac{3W_1 - 3W_2 + 2l\theta_1 + l\theta_2}{l^2}, A_4 -> -\frac{-2W_1 + 2W_2 - l\theta_1 - l\theta_2}{l^3}\}\}\]

Substituting \(A_1, A_2, A_3, A_4\) in eq(1)

\[A_1 := W_1\]

\[A_2 := \theta_1\]

\[A_3 := \frac{3W_1 - 3W_2 + 2l\theta_1 + l\theta_2}{l^2}\]

\[A_4 := -\frac{-2W_1 + 2W_2 - l\theta_1 - l\theta_2}{l^3}\]

\[W(\xi) := A_1 + A_2 * x + A_3 * x^2 + A_4 * x^3\]

Expand[W(x)]

Output

\[W_i = \frac{3x^2W_1}{l^2} + \frac{2x^3W_1}{l^3} + \frac{3x^2W_2}{l^2} - \frac{2x^3W_2}{l^3} + x\theta_1 - \frac{2x^2\theta_1}{l} + \frac{x^3\theta_1}{l^2} - \frac{x^2\theta_2}{l} + \frac{x^3\theta_2}{l^2}\]
\[ W(x) = W_i \left(1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3}\right) + \theta_1 \left(x - \frac{2x^2}{l} + \frac{x^3}{l^2}\right) \]
\[ + W_2 \left(\frac{3x^2}{l^2} - \frac{2x^3}{l^3}\right) + \theta_2 \left(-\frac{x^2}{l} + \frac{x^3}{l^2}\right) \]  
(7)

\[ i.e., \, W = N_1W_i + N_2\theta_1 + N_3W_2 + N_4\theta_2 \]
\[ = N_1\delta_1 + N_2\delta_2 + N_3\delta_3 + N_4\delta_4 \]  
(8)

Where \( N_1, N_2, N_3, N_4 \) are shape functions for the beam elements and \( \delta_1, \delta_2, \delta_3, \delta_4 \) are the nodal displacements.

\[ \{ \delta \} = \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} = \begin{bmatrix} W_i \\ \theta_1 \\ W_2 \\ \theta_2 \end{bmatrix} \]

Comparing (7) and (8) we get

\[ N_1 = 1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3} \]  
(9)

\[ N_2 = x - \frac{2x^2}{l} + \frac{x^3}{l^2} \]  
(10)

\[ N_3 = \frac{3x^2}{l^2} - \frac{2x^3}{l^3} \]  
(11)

\[ N_4 = -\frac{x^2}{l} + \frac{x^3}{l^2} \]  
(12)

Substituting length \( l = 1 - 0 = 1 \) and \( x = s \) in eqs (9),(10),(11), and (12) in general we get

\[ N_1 = H_{i0}^1(s) = 1 - 3s^2 + 2s^3 \]  
(13)

\[ N_2 = H_{i1}^1(s) = s - 2s^2 + s^3 = ls(s - 1)^2 \]

(Including length of beam element)

\[ N_3 = H_{i0}^2(s) = s(s - 1)^2 \]  
(14)

\[ N_3 = H_{i1}^2(s) = 3s^2 - 2s^3 \]  
(15)

\[ N_4 = H_{i2}^1(s) = -s^2 + s^3 = ls^2(s - 1) \]

(Including length \( l \) of beam element)

\[ = s^2(s - 1) \]  
(16)

In \( H_{i0}^1(s) \), 0 represents Zeroth order derivative, 1 represents node number one and power 1 represents first order Hermitian function.

In \( H_{i1}^1(s) \), 1 represents first order derivative, 1 represents node number one and power 1 represents first order Hermitian function.

In \( H_{i0}^2(s) \), 0 represents Zeroth order derivative, 2 represents node number two and power 1 represents first order Hermitian function.

In \( H_{i1}^2(s) \), 0 represents Zeroth order derivative, 2 represents node number two and power 1 represents first order Hermitian function.

\[ \text{IV. VERIFICATION} \]

(i). 1st VERIFICATION CONDITION

First verification condition at node 1 is
\[ N_1 = 1 \text{ and } N_2 = 0, \ N_3 = 0, N_4 = 0 \text{ and also } \]
\[ N_4 := s^2(s - 1) \]

\[ \frac{\partial N_2}{\partial s} = 1 \text{ and } \frac{\partial N_1}{\partial s} = 0, \frac{\partial N_3}{\partial s} = 0, \frac{\partial N_4}{\partial s} = 0 \]
\[ \partial_s(N_1) \]
\[ \partial_s(N_2) \]

At Node 1 \( s = 0 \)
\[ N_1 := 1 - 3s^2 + 2s^3 \]  \( (17) \)
\[ \partial_s(N_1) = -6s + s^2 \]  \( (21) \)

\[ N_2 := s(s - 1)^2 \]  \( (18) \)
\[ \partial_s(N_2) = (-1 + s)^2 + 2(-1 + s)s \]  \( (22) \)

\[ N_3 := s^2(3 - 2s) \]  \( (19) \)
\[ \partial_s(N_3) = 2(3 - 2s)s - 2s^2 \]  \( (23) \)

\[ N_4 := s^2(s - 1) \]  \( (20) \)
\[ \partial_s(N_4) = 2(-1 + s)s + s^2 \]  \( (24) \)

Output
\[ 2(3 - 2s)s - 2s^2 \]
\[ 2(-1 + s)s + s^2 \]
\[ (-1 + s)^2 + 2(-1 + s)s \]
\[ -6s + s^2 \]

Partial derivative condition at node 1, \( s = 0 \)
\[ \frac{\partial N_1}{\partial s} = -6s + s^2 \]

Finding first derivatives for (17), (18), (19) and (20)
\[ \partial_s(N_1) = (-1 + s)^2 + 2(-1 + s)s \]
\[ \partial_s(N_3) = 2(3 - 2s)s - 2s^2 \]
\[ \partial_s(N_4) = 2(-1 + s)s + s^2 \]
\[ s = 0 \]
\[ \frac{\partial N_1}{\partial s} \]

\[ N_2 \]

\[ \frac{\partial N_2}{\partial s} \]

\[ N_3 \]

\[ \frac{\partial N_3}{\partial s} \]

\[ N_4 \]

\[ \frac{\partial N_4}{\partial s} \]  

**Output**

\[ 0 \]

\[ 1 \]

\[ 0 \]

\[ 0 \]

(ii) 2nd VERIFICATION CONDITION

Second Verification condition is at node 2

\[ N_3 = 1 \] and \[ N_4 = 0 \], \[ N_1 = 0 \] and also

\[ \frac{\partial N_3}{\partial s} = 2(3 - 2s)s - 2s^2 \]

\[ \frac{\partial N_4}{\partial s} = 2(-1 + s)s + s^2 \]

\[ s = 1 \]

At Node 2, \( s = 1 \)

\[ N_1 := 1 - 3s^2 + 2s^3 \]

\[ \frac{\partial N_1}{\partial s} \]

\[ N_2 := s(s - 1)^2 \]

\[ \frac{\partial N_2}{\partial s} \]

\[ N_3 := s^2(3 - 2s) \]

\[ \frac{\partial N_3}{\partial s} \]

\[ N_4 := s^2(s - 1) \]

\[ \frac{\partial N_4}{\partial s} \]

\[ s := 1 \]

**Output**
Third Verification Condition

3rd verification condition is $N_1 + N_3 = 1$

\[
N_1 := 1 - 3s^2 + 2s^3
\]

\[
N_3 := s^2(3 - 2s)
\]

\[\text{FullSimplify}[N_1 + N_3]\]

Output

\[1\]

V. CONCLUSIONS

1. Derived Shape Functions for Hermite Polynomials.

2. Verified First verification condition at node 1, $N_1 = 1$ and $N_2 = 0, N_3 = 0, N_4 = 0$

and also

\[
\frac{\partial N_1}{\partial x} = 1 \quad \text{and} \quad \frac{\partial N_1}{\partial y} = 0, \quad \frac{\partial N_1}{\partial z} = 0, \quad \frac{\partial N_4}{\partial x} = 0.
\]

3. Verified Second Verification condition at node 2 $N_3 = 1$ and $N_1 = 0, N_2 = 0, N_4 = 0$

and also

\[
\frac{\partial N_4}{\partial x} = 1 \quad \text{and} \quad \frac{\partial N_4}{\partial y} = 0, \quad \frac{\partial N_4}{\partial z} = 0, \quad \frac{\partial N_3}{\partial x} = 0.
\]

4. Verified Third verification condition

$N_1 + N_3 = 1$.

REFERENCES


