4-Difference Cordial Labeling of Cycle and Wheel Related Graphs

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Abstract — Let $G$ be a $(p, q)$ graph. Let $k$ be an integer with $2 \leq k \leq p$ and $f : V(G) \to \{1, 2, \ldots, k\}$ be a map. For each edge $uv$, assign the label $|f(u) - f(v)|$. The function $f$ is called a $k$-difference cordial labeling of $G$ if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ where $v_f(x)$ denotes the number of vertices labelled with $x$ ($x \in \{1, 2, \ldots, k\}$), $e_f(1)$ and $e_f(0)$ respectively denote the number of edges labelled with 1 and not labelled with 1. A graph with a $k$-difference cordial labeling is called a $k$-difference cordial graph. In this paper we discuss 4-difference cordial labeling for cycle, wheel, crown, helm and gear graph.

Key words : Difference cordial labeling, 4-difference cordial labeling.
Subject classification number: 05C78.

I. INTRODUCTION

We consider simple, finite, undirected graph $G = (V, E)$. R. Ponraj, M. Maria Adaickalam and R. Kala [6] introduced $k$-difference cordial labeling of graphs. In [6], they investigated $k$-difference cordial labeling behavior of star, $m$ copies of star and proved that every graph is a subgraph of a connected $k$-difference cordial graph. In [7], R. Ponraj and M. Maria Adaickalam discussed the 3-difference cordial labeling behavior of path, cycle, star, bistar, complete graph, complete bipartite graph, comb, double comb, quadrilateral snake. For the standard terminology and notations we follow Harary[1].

II. MAIN RESULTS

In this paper we have proved that cycle, wheel, helm, crown and gear graph are 4-difference cordial graphs.

Definition II.1. A cycle $C_n (n \in \mathbb{N}, n \geq 3)$ is closed path with $n$ vertices.

Theorem II.1. Cycle $C_n$ is a 4-difference cordial graph.

Proof. Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$. We define labeling function $f : V(C_n) \to \{1, 2, 3, 4\}$ as follows.

Case 1: $n$ is odd.

$$f(v_{4i+1}) = 1; 0 \leq i \leq \left\lfloor \frac{n-1}{4} \right\rfloor.$$  
$$f(v_{4i+2}) = 2; 0 \leq i \leq \left\lfloor \frac{n-3}{4} \right\rfloor.$$  
$$f(v_{4i+3}) = 3; 1 \leq i \leq \left\lfloor \frac{n-1}{4} \right\rfloor.$$  
$$f(v_{4i+4}) = 4; 0 \leq i \leq \left\lfloor \frac{n-3}{4} \right\rfloor.$$  

Case 2: $n$ is even.

Subcase 1: $n \equiv 0(mod4)$.

$$f(v_{4i}) = 1; 1 \leq i \leq \frac{n}{4}.$$  
$$f(v_{4i+1}) = 2; 0 \leq i \leq \frac{n-4}{4}.$$  
$$f(v_{4i+2}) = 3; 0 \leq i \leq \frac{n-4}{4}.$$  
$$f(v_{4i+3}) = 4; 0 \leq i \leq \frac{n-4}{4}.$$
Subcase 2: \( n \equiv 2(\text{mod} 4) \).

\[
\begin{align*}
    f(v_1) &= 2, \\
    f(v_2) &= 1, \\
    f(v_{4i+1}) &= 1; \ 1 \leq i \leq \left\lfloor \frac{n-2}{4} \right\rfloor. \\
    f(v_{4i+2}) &= 2; \ 1 \leq i \leq \left\lfloor \frac{n-2}{4} \right\rfloor. \\
    f(v_{4i}) &= 3; \ 1 \leq i \leq \left\lfloor \frac{n-2}{4} \right\rfloor. \\
    f(v_{4i+3}) &= 4; \ 0 \leq i \leq \left\lfloor \frac{n-6}{4} \right\rfloor.
\end{align*}
\]

In each case cycle \( C_n \) satisfies the conditions for 4-difference cordial labeling. Hence \( C_n \) is a 4-difference cordial graph.

**Example 1.** The 4-difference cordial labeling of \( C_{18} \) is shown in Figure 1.

![Fig. 1](image1.png)

**Definition II.2.** The wheel \( W_n (n \in \mathbb{N}, n \geq 3) \) is a join of the graphs \( C_n \) and \( K_1 \), i.e. \( W_n = C_n + K_1 \). Here vertices corresponding to \( C_n \) are called rim vertices and \( C_n \) is called rim of \( W_n \). The vertex corresponding to \( K_1 \) is called apex vertex.

**Theorem II.2.** \( W_n \) is a 4-difference cordial graph.

**Proof.** Let \( v_0 \) be the apex vertex and \( v_1, v_2, \ldots, v_n \) be the rim vertices of \( W_n \). We define labeling function \( f : V(W_n) \to \{1, 2, 3, 4\} \) as follows.

**Case 1:** \( n \) is odd.

\[
\begin{align*}
    f(v_{4i+1}) &= 1; \ 1 \leq i \leq \left\lfloor \frac{n-1}{4} \right\rfloor. \\
    f(v_{4i+2}) &= 2; \ 0 \leq i \leq \left\lfloor \frac{n-1}{4} \right\rfloor. \\
    f(v_{4i}) &= 3; \ 0 \leq i \leq \left\lfloor \frac{n-3}{4} \right\rfloor. \\
    f(v_{4i+3}) &= 4; \ 0 \leq i \leq \left\lfloor \frac{n-3}{4} \right\rfloor.
\end{align*}
\]

**Case 2:** \( n \) is even.

\[
\begin{align*}
    f(v_1) &= 2, \\
    f(v_2) &= 3, \\
    f(v_3) &= 4. \\
    f(v_{4i+3}) &= 1; \ 1 \leq i \leq \left\lfloor \frac{n-3}{4} \right\rfloor. \\
    f(v_{4i}) &= 2; \ 1 \leq i \leq \left\lfloor \frac{n}{4} \right\rfloor. \\
    f(v_{4i+1}) &= 3; \ 1 \leq i \leq \left\lfloor \frac{n-2}{4} \right\rfloor. \\
    f(v_{4i+2}) &= 4; \ 1 \leq i \leq \left\lfloor \frac{n-2}{4} \right\rfloor.
\end{align*}
\]

In each case wheel graph \( W_n \) satisfies the conditions of 4-difference cordial labeling. Hence \( W_n \) is 4-difference cordial graph.

**Example 2.** 4-difference cordial labeling of \( W_{11} \) is shown in Figure 2.

![Fig. 2](image2.png)
Definition II.3. The crown \( C_n \odot K_1(n \in \mathbb{N}, n \geq 3) \) is obtained by joining a pendant edge to each vertex of \( C_n \).

Theorem II.3. Crown \( C_n \odot K_1 \) is a 4-difference cordial graph.

Proof. Let \( V(C_n \odot K_1) = \{v_1, v_2, \ldots, v_n, v'_1, v'_2, \ldots, v'_n\} \), where \( v_1, v_2, \ldots, v_n \) are rim vertices and \( v'_1, v'_2, \ldots, v'_n \) are pendant vertices.

We define labeling function \( f : V(C_n \odot K_1) \rightarrow \{1, 2, 3, 4\} \) as follows.

Case 1: \( n \) is odd.

\[ f(v_{2i+1}) = 1; \quad 0 \leq i \leq \frac{n-1}{2}. \]
\[ f(v_{2i}) = 3; \quad 1 \leq i \leq \frac{n-1}{2}. \]
\[ f(v'_{2i+1}) = 2; \quad 0 \leq i \leq \frac{n-2}{2}. \]
\[ f(v'_{2i}) = 4; \quad 1 \leq i \leq \frac{n-2}{2}. \]

Case 2: \( n \) is even.

\[ f(v_{2i+1}) = 1; \quad 0 \leq i \leq \frac{n-2}{2}. \]
\[ f(v_{2i}) = 3; \quad 1 \leq i \leq \frac{n}{2}. \]
\[ f(v'_{2i+1}) = 2; \quad 0 \leq i \leq \frac{n-2}{2}. \]
\[ f(v'_{2i}) = 4; \quad 1 \leq i \leq \frac{n}{2}. \]

In each case the crown graph \( C_n \odot K_1 \) satisfies the conditions of 4-difference cordial labeling. Hence it is 4-difference cordial graph. \( \square \)

Example 3. 4-difference cordial labeling of crown \( C_9 \odot K_1 \) is shown in Figure 3.

Definition II.4. A helm \( H_n(n \geq 3) \) is the graph obtained from the wheel \( W_n \) by adding a pendant edge at each vertex on the rim of \( W_n \).

Theorem II.4. \( H_n \) is a 4-difference cordial graph.

Proof. Let \( V(H_n) = \{v_0, v_1, \ldots, v_n, v'_1, v'_2, \ldots, v'_n\} \), where \( v_0 \) is apex vertex, \( \{v_1, v_2, \ldots, v_n\} \) are rim vertices and \( \{v'_1, v'_2, \ldots, v'_n\} \) are pendant vertices.

We define labeling function \( f : V(H_n) \rightarrow \{1, 2, 3, 4\} \) as follows.

Case 1: \( n \) is odd.

\[ f(v_{4i}) = 1; \quad 1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor. \]
\[ f(v_{4i+1}) = 2; \quad 0 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor. \]
\[ f(v_{4i+2}) = 3; \quad 0 \leq i \leq \left\lfloor \frac{n-3}{4} \right\rfloor. \]
\[ f(v_{4i+3}) = 4; \quad 0 \leq i \leq \left\lfloor \frac{n-3}{4} \right\rfloor. \]
\[ f(v'_{4i+1}) = 1; \quad 0 \leq i \leq \left\lfloor \frac{n-3}{4} \right\rfloor. \]
\[ f(v'_{4i+2}) = 2; \quad 0 \leq i \leq \left\lfloor \frac{n-3}{4} \right\rfloor. \]
\[ f(v'_{4i+3}) = 3; \quad 0 \leq i \leq \left\lfloor \frac{n-1}{4} \right\rfloor. \]
\[ f(v'_{4i}) = 4; \quad 1 \leq i \leq \left\lfloor \frac{n-1}{4} \right\rfloor. \]
Case 2: \( n \) is even.

\[
\begin{align*}
f(v_{2i+1}) &= 2; \quad 0 \leq i \leq \frac{n-2}{2}. \\
f(v_{2i}) &= 4; \quad 1 \leq i \leq \frac{n}{2}. \\
f(v'_{2i+1}) &= 1; \quad 0 \leq i \leq \frac{n-2}{2}. \\
f(v'_{2i}) &= 3; \quad 1 \leq i \leq \frac{n}{2}.
\end{align*}
\]

In each case the helm graph \( H_n \) satisfies the conditions of 4-difference cordial labeling. Hence \( H_n \) is 4-difference cordial graph.

Example 4. 4-difference cordial labeling of helm \( H_9 \) is shown in Figure 4.

![Fig. 4](image)

Definition II.5. A gear graph \( G_n \) \((n \geq 3)\) is obtained from the wheel \( W_n \) by adding a vertex between every pair of adjacent vertices of rim of \( W_n \).

Theorem II.5. Gear \( G_n \) is a 4-difference cordial graph.

Proof. Let \( G_n = \{v_0, v_1, \ldots, v_{2n}\}\), where \( v_0 \) is apex vertex, \( \{v_1, v_3, \ldots, v_{2n-1}\} \) are the vertices of degree 3 and \( \{v_2, v_4, \ldots, v_{2n}\} \) are the vertices of degree 2.

We define labeling function \( f : V(G_n) \rightarrow \{1, 2, 3, 4\} \) as follows.

Case 1: \( n \) is odd.

\[
\begin{align*}
v_0 &= 3. \\
f(v_{4i+1}) &= 1; \quad 0 \leq i \leq \frac{n-1}{2} - 1. \\
f(v_{4i+2}) &= 2; \quad 0 \leq i \leq \frac{n}{2} - 1. \\
f(v_{4i+3}) &= 3; \quad 0 \leq i \leq \frac{n-3}{2} - 1. \\
f(v_{4i+4}) &= 4; \quad 0 \leq i \leq \frac{n-3}{2} - 1.
\end{align*}
\]

Case 2: \( n \) is even.

\[
\begin{align*}
v_0 &= 1. \\
f(v_{4i+1}) &= 1; \quad 0 \leq i \leq \frac{n}{2} - 1. \\
f(v_{4i+2}) &= 2; \quad 0 \leq i \leq \frac{n}{2} - 1. \\
f(v_{4i+3}) &= 3; \quad 0 \leq i \leq \frac{n}{2} - 1. \\
f(v_{4i+4}) &= 4; \quad 0 \leq i \leq \frac{n}{2} - 1.
\end{align*}
\]

In each case the gear graph \( G_n \) satisfies the conditions of 4-difference cordial labeling. Hence \( G_n \) is 4-difference cordial graph.

Example 5. 4-difference cordial labeling of \( G_5 \) is shown in Figure 5.

![Fig. 5](image)
REFERENCES


