Composition Operators on Weighted Orlicz Spaces

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Abstract. In this paper, we study the boundedness of composition operators between any two weighted Orlicz spaces.

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1 Introduction

Let \( X = (X, \Sigma, \mu) \) be a \( \sigma \)-finite complete measure space. Any measurable nonsingular transformation \( \tau \) induces a composition operator \( C_\tau \) from \( L^0(X) \) to itself defined by

\[
C_\tau f(x) = f(\tau(x)), \quad x \in X, \ f \in L^0(X),
\]

where \( L^0(X) \) denotes the linear space of all equivalence classes of all real valued \( \Sigma \)-measurable function on \( X \), where we identify any two functions that are equal \( \mu \)-almost everywhere on \( X \).

A nondecreasing continuous convex function \( \phi : [0, \infty) \to [0, \infty) \) for which \( \phi(0) = 0 \) and \( \lim_{x \to \infty} \phi(x) = \infty \) is called a Young function. A function \( \Phi : X \times [0, \infty) \to [0, \infty) \) is said to be a generalized Young function or Musielak–Orlicz function if

(i) \( \Phi(x, \cdot) \) is a Young function for almost every \( x \in X \) and

(ii) \( \Phi(\cdot, u) \) is \( \Sigma \)-measurable for every \( u \geq 0 \).

For any generalized Young function \( \Phi \), the Musielak – Orlicz space associated with \( \Phi \), denoted by \( L^\Phi(X) \), is defined as the set of all \( f \in L^0(X) \) such that

\[
I_\Phi(\lambda f) = \int_X \Phi(x, \lambda |f(x)|)d\mu(x) < \infty \quad \text{for some } \lambda = \lambda(f) > 0.
\]

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The function $I_{\Phi}$ is called the modular. The space $L^{\Phi}(X)$ is a Banach space with the Luxemburg – Nakano norm
\[
\|f\|_{\Phi} = \inf \{\lambda > 0 \mid I_{\Phi}(f/\lambda) \leq 1\}.
\]
Let $\phi$ be an Orlicz function and $w$ be a weight in $X$ i.e. an a.e. positive and integrable real valued function in $X$. Then $\Phi : X \times [0, \infty) \rightarrow [0, \infty)$ defined by
\[
\Phi(x, u) = \phi(u)w(x), \quad x \in X, u \geq 0,
\]
is a generalized Young function. The resulting Musielak-Orlicz space is called weighted Orlicz space and is denoted by $L_{\phi}^{\Phi}(X)$. In this case, the modular $I_{\Phi}$ is given by
\[
I_{\Phi}(f) = \int_X \phi(|f(x)|)w(x)d\mu(x).
\]
If $\Phi$ is independent of $x$, then the resulting Musielak-Orlicz space is simply called Orlicz space and is denoted by $L^{\Phi}(X)$.
Composition operators on Orlicz spaces have also been studied in [3], [4], [5], [8] and [14]. The techniques used in this paper essentially depend on the conditions of embedding of one Orlicz space into another (see, [11] Page 45 for details).

## 2 Boundedness of Composition Operators

In this section, we study the boundedness of composition operators on weighted Orlicz spaces.

**Lemma 2.1.** ( [11] Lemma 8.3 ) Let $(X, \Sigma, \mu)$ be a $\sigma$-finite nonatomic measure space, \{\alpha_n\} a sequence of positive numbers and \{s_n\} a sequence of measurable, finite, non-negative functions on $X$ such that for $n = 1, 2, \ldots$
\[
\int_X s_n(x)d\mu(x) \geq 2^n\alpha_n.
\]
Then there exist an increasing sequence \{n_k\} of integers and a sequence \{A_k\} of pairwise disjoint measurable sets such that for $k = 1, 2, \ldots$
\[
\int_{A_k} s_{n_k}(x)d\mu(x) = \alpha_{n_k}.
\]

**Theorem 2.2.** Let $(X, \Sigma, \mu)$ be a $\sigma$-finite nonatomic measure space, $w_1$ and $w_2$ be weights in $X$ and $\tau : X \rightarrow X$ be a measurable non-singular transformation such that $\tau(X) = X$. Denote by $g_\tau$ the Raydon-Nikodym derivative $d\mu \circ \tau^{-1}/d\mu$. Then the composition operator $C_\tau : L_{w_1}^{\phi_1}(X) \rightarrow L_{w_2}^{\phi_2}(X)$ is bounded if and only if there exist $a, b > 0$ and $0 \leq h \in L^1(X)$ such that
\[
\phi_2(au)(w_2 \circ \tau^{-1})(x)g_\tau(x) \leq b\phi_1(u)w_1(x) + h(x)
\]
for almost all $x \in X$ and for all $u \geq 0$. 

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Proof. Suppose that the given condition holds. Let $0 \neq f \in L^p_{w_1}(X)$. Let $M \geq 1$ be a real number satisfying $M(b + \|h\|_1) \geq 1$. Then

$$I_{\Phi_2} \left( \frac{C_\tau f}{(M(b + \|h\|_1)^{1/2} + \|f\|_{\Phi_1})/a} \right)$$

$$= \int_X \phi_2 \left( \frac{a|C_\tau f(x)|}{M(b + \|h\|_1)^{1/2} + \|f\|_{\Phi_1}} \right) w_2(x) d\mu(x)$$

$$\leq \frac{1}{M(b + \|h\|_1)^{1/2}} \int_X \phi_2 \left( \frac{a|f(\tau(x))|}{\|f\|_{\Phi_1}} \right) w_2(x) d\mu(x)$$

$$= \frac{1}{M(b + \|h\|_1)^{1/2}} \int_{\tau(X)} \phi_2 \left( \frac{a|f(y)|}{\|f\|_{\Phi_1}} \right) (w_2 \circ \tau^{-1})(y) d\mu(y)$$

$$\leq \frac{1}{M(b + \|h\|_1)^{1/2}} \int_X \phi_2 \left( \frac{a|f(y)|}{\|f\|_{\Phi_1}} \right) (w_2 \circ \tau^{-1})(y) (\mu \circ \tau^{-1})(y)$$

Thus $\|C_\tau f\|_{\Phi_2} \leq \frac{M}{a}(b + \|h\|_1)^{1/2} \|f\|_{\Phi_1}$. This shows that $C_\tau$ is bounded.

Consider the function

$$h_n(x) = \sup_{u \geq 0} \left( \phi_2(2^{-n}u)(w_2 \circ \tau^{-1})(x)g_\tau(x) - 2^n \phi_1(u)w_1(x) \right).$$

Write $X = \bigcup_{i=1}^{\infty} X_i$, where $\{X_i\}_{i=1}^{\infty}$ is a pairwise disjoint sequence of measurable subsets of $X$ with $\mu(X_i) < \infty$ for every $i = 1, 2, \ldots$

For every $q \in \mathbb{Q}^+$, we put $f_{q,i}(x) = q \chi_{X_i}(x)$, where $\chi_{X_i}$ is the characteristic function of $X_i$. Then it can be shown that

$$h_n(x) = \sup_{q \in \mathbb{Q}^+ \atop i \in \mathbb{N}} \left( \phi_2(2^{-n}q,i(x))(w_2 \circ \tau^{-1})(x)g_\tau(x) - 2^n \phi_1(f_{q,i}(x))w_1(x) \right).$$

Taking $(f_k)$ to be a rearrangement of $(f_{q,i})$ with $f_1 = f_{0,i}$, the above equation can be rewritten as

$$h_n(x) = \sup_{k \in \mathbb{N}} \left( \phi_2(2^{-n}f_k(x))(w_2 \circ \tau^{-1})(x)g_\tau(x) - 2^n \phi_1(f_k(x))w_1(x) \right).$$

Then $h_n$ are measurable and $h_n(x) \geq 0$ for each $x \in X$. To complete the proof, we have to show that $\int_X h_n(x) d\mu(x) < \infty$ for some $n$. Suppose this is not true.

Denote

$$r_{m,n}(x) = \max_{1 \leq k \leq m} \left( \phi_2(2^{-n}f_k(x))(w_2 \circ \tau^{-1})(x)g_\tau(x) - 2^n \phi_1(f_k(x))w_1(x) \right).$$
Then \( r_{m,n} \) are measurable, \( r_{m,n}(x) \geq 0 \) and \( r_{m,n}(x) \) is a nondecreasing sequence tending to \( h_n(x) \) as \( m \to \infty \) for every \( x \in X \). Thus for any \( n \), there exists \( m_n \) such that \( \int_X r_{m,n}(x) d\mu(x) \geq 2^n \). Taking \( r_n = r_{m_n,n} \), we have \( \int_X r_n(x) d\mu(x) \geq 2^n \) for \( n = 1, 2, \ldots \).

Let

\[
E_{n,k} = \left\{ x \in X \mid \phi_2(2^{-n} f_k(x))(w_2 \circ \tau^{-1})(x) g_\tau(x) - 2^n \phi_1(f_k(x)) w_1(x) = r_n(x) \right\}
\]

and

\[
E_n = X \setminus (E_{n,1} \cup E_{n,2} \cup \ldots \cup E_{n,m_n}).
\]

Then \( \mu(E_n) = 0 \).

Let

\[
\tilde{f}_n(x) = \begin{cases} 
0 & \text{if } x \in E_{n,1} \cup E_n \\
 f_k(x) & \text{if } x \in E_{n,k} \setminus \bigcup_{j=1}^{k-1} E_{n,j}, \ k = 2, 3, \ldots, m_n.
\end{cases}
\]

Then

\[
r_n(x) = \phi_2(2^{-n} \tilde{f}_n(x))(w_2 \circ \tau^{-1})(x) g_\tau(x) - 2^n \phi_1(\tilde{f}_n(x)) w_1(x)
\]

\[
\geq 0.
\]

Therefore,

\[
\int_X \phi_2(2^{-n} \tilde{f}_n(x))(w_2 \circ \tau^{-1})(x) g_\tau(x) d\mu(x) = 2^n \int_X \phi_1(\tilde{f}_n(x)) w_1(x) d\mu(x)
\]

\[
+ \int_X r_n(x) d\mu(x)
\]

\[
\geq \int_X r_n(x) d\mu(x)
\]

\[
\geq 2^n.
\]

By Lemma 2.1, we obtain an increasing sequence \( \{n_k\} \) and a sequence \( \{A_k\} \) of pairwise disjoint measurable sets such that

\[
\int_{A_k} \phi_2(2^{-n_k} \tilde{f}_{n_k}(x))(w_2 \circ \tau^{-1})(x) g_\tau(x) d\mu(x) = 1, \quad k = 1, 2, \ldots
\]

Put

\[
f(x) = \begin{cases} 
\tilde{f}_{n_k}(x) & \text{if } x \in A_k \\
 0 & \text{otherwise.}
\end{cases}
\]
Let $\lambda > 0$. Choose $p$ large enough that $2^{-np} \leq \lambda$. Then

\[
\int_X \phi_2(\lambda C_f(x)) w_2(x) d\mu(x) = \int_X \phi_2(\lambda f(\tau(x))) w_2(x) d\mu(x)
\]
\[
= \int_{\tau(X)} \phi_2(\lambda f(y))(w_2 \circ \tau^{-1})(y) d(\mu \circ \tau^{-1})(y)
\]
\[
= \int_X \phi_2(\lambda f(y))(w_2 \circ \tau^{-1})(y) g_\tau(y) d\mu(y)
\]
\[
= \sum_{k=1}^{\infty} A_k \int_{A_k} \phi_2(\lambda \tilde{f}_{nk}(y))(w_2 \circ \tau^{-1})(y) g_\tau(y) d\mu(y)
\]
\[
\geq \sum_{k=p}^{\infty} A_k \int_{A_k} \phi_2(2^{-nk} \tilde{f}_{nk}(y))(w_2 \circ \tau^{-1})(y) g_\tau(y) d\mu(y)
\]
\[
= \infty.
\]

And

\[
\int_X \phi_1(f(x)) w_1(x) d\mu(x) = \sum_{k=1}^{\infty} 2^{-nk} \int_{A_k} \phi_2(2^{-nk} \tilde{f}_{nk}(x))(w_2 \circ \tau^{-1})(x) g_\tau(x) d\mu(x)
\]
\[
- \sum_{k=1}^{\infty} 2^{-nk} \int_{A_k} r_{nk}(x) d\mu(x)
\]
\[
\leq \sum_{k=1}^{\infty} 2^{-nk} \int_{A_k} \phi_2(2^{-nk} \tilde{f}_{nk}(x))(w_2 \circ \tau^{-1})(x) g_\tau(x) d\mu(x)
\]
\[
= \sum_{k=1}^{\infty} 2^{-nk}
\]
\[
\leq 1.
\]

Thus, $f \in L_{w_1}^{\phi_1}(X)$ but $C_f \notin L_{w_2}^{\phi_2}(X)$, which is a contradiction. Hence,

\[
\int_X h_n(x) d\mu(x) < \infty \text{ for some } n.
\]

This completes the proof. \qed

References


