Common tripled fixed point results for hybrid pair of mappings under new condition

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Abstract. We introduce the concept of (EA) property and occasionally $w-$compatibility for hybrid pair $g : X \rightarrow X$ and $F : X \times X \times X \rightarrow 2^X$. We establish some common tripled fixed point theorems for hybrid pair of mappings satisfying, (EA) property and occasionally $w-$compatibility conditions, under weak $\psi-\varphi$ contraction. It is to be noted that to find tripled coincidence point, we do not employ completeness on space and not partially orderdness. Also the condition of continuity is not necessary for any mapping involved therein. An example is also given to validate our results. We extend and generalize several known results.

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1. Introduction and Preliminaries

The study of fixed points for multivalued contraction mappings using the Hausdorff metric was initiated by Nadler [23]. The theory of multivalued mappings has wide range of applications. It is applied in control theory, convex optimization, differential inclusions, and economics.

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Let \((X, d)\) be a metric space and \(CB(X)\) be the set of all nonempty closed bounded subsets of \(X\). Let \(D(x, A)\) denote the distance from \(x\) to \(A \subset X\) and \(H\) denote the Hausdorff metric induced by \(d\), that is, 
\[
D(x, A) = \inf_{a \in A} d(x, a)
\]
and 
\[
H(A, B) = \max \left\{ \sup_{a \in A} D(a, B), \sup_{b \in B} D(b, A) \right\}, \text{ for all } A, B \in CB(X).
\]

The existence of fixed points for various multivalued contractive mappings has been studied by many authors under different conditions and a significant number of papers have been reported. For details, we refer to \([4, 22, 23, 25, 27]\) and the references therein.

Samet and Vetro \([26]\) introduced the concept of coupled fixed point for multivalued mapping and later several authors proved existence of coupled fixed points for multivalued mappings under different conditions. Subsequently, many results in this direction were given (see, e.g., \([14, 15, 16, 18, 21, 26]\)).

Berinde and Borcut \([9]\) introduced the concept of triple fixed points. In \([9]\); Berinde and Borcut established the existence of tripled fixed point of single valued mappings in partially ordered metric spaces. For more details on tripled fixed point theory, we also refer the reader to \([3, 5, 6, 7, 8, 10, 12, 24]\).

Deshpande et al. in \([17]\) introduced Triple fixed, Triple coincidence and Triple common fixed points for multivalued maps.

**Definition 1.1.** \([17]\) Let \(X\) be a non empty set, \(F : X \times X \times X \to 2^X\) (Collection of all non empty subsets of \(X\)), \(g : X \to X\).

(i) The point \((x, y, z) \in X \times X \times X\) is called a tripled fixed point of \(F\) if \(x \in F(x, y, z), y \in F(y, z, x)\) and \(z \in F(z, x, y)\).

(ii) The point \((x, y, z) \in X \times X \times X\) is called a tripled coincident point of \(F\) and \(g\) if \(gx \in F(x, y, z), gy \in F(y, z, x)\) and \(gz \in F(z, x, y)\).

(iii) The point \((x, y, z) \in X \times X \times X\) is called a tripled common fixed point of \(F\) and \(g\) if \(x = gx \in F(x, y, z), y = gy \in F(y, z, x)\) and \(z = gz \in F(z, x, y)\).

We denote the set of tripled coincidence points of mappings \(F\) and \(g\) by \(C(F, g)\).

Note that if \((x, y, z) \in C(F, g)\), then \((y, z, x)\) and \((z, x, y)\) are also in \(C(F, g)\).

**Definition 1.2.** \([17]\) Let \(F : X \times X \times X \to 2^X\) be a multivalued map and \(g\) be a self map on \(X\). The Hybrid pair \(\{F, g\}\) is called \(w\)- compatible if \(g(F(x, y, z)) \subseteq F(gx, gy, gz)\) whenever \((x, y, z)\) is a tripled coincidence point of \(F\) and \(g\).

**Definition 1.3.** \([17]\) Let \(F : X \times X \times X \to 2^X\) be a multivalued mapping and \(g\) be a self-map on \(X\). The mapping \(g\) is called \(F\)-weakly commuting at some point \((x, y, z) \in X \times X \times X\) if \(g^2 x \in F(gx, gy, gz), g^2 y \in F(gy, gz, gx)\) and \(g^2 z \in F(gz, gx, gy)\).
Aamri and ElMoutawakil [1] defined (EA) property for self-mappings which contained the class of non-compatible mappings. Kamran [20] extended the (EA) property for hybrid pair $g : X \to X$ and $F : X \to 2^X$. Abbas and Rhoades [2] extended the concept of occasionally weakly compatible mappings for hybrid pair $g : X \to X$ and $F : X \to 2^X$. Deshpande and Handa [15] introduced the concept of (EA) property and occasionally $w-$compatibility for hybrid pair $g : X \to X$ and $F : X \times X \to 2^X$.

In this paper, we introduce the concept of (EA) property and occasionally $w-$compatibility for hybrid pair $g : X \to X$ and $F : X \times X \to 2^X$. We establish some common tripled fixed point theorems for hybrid pair of mappings satisfying, (EA) property and occasionally $w-$compatibility conditions, under weak $\psi - \varphi$ contraction. It is to be noted that to find tripled coincidence point, we do not employ completeness on space and not partially orderdness. Also the condition of continuity is not necessary for any mapping involved therein. An example is also given to validate our results. We improve, extend and generalize the results of Bhaskar and Lakshmikantham [11], Ciric et al. [13], Ding et al. [18], Gordji et al. [19], Deshpande and Handa [15] and Lakshmikantham and Ciric [21]. The effectiveness of our generalization demonstrated with the help of an example.

2. Main results

First we introduce the following

**Definition 2.1.** Mappings $g : X \to X$ and $F : X \times X \times X \to CB(X)$ are said to satisfy the (EA) property if there exist sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ in $X$, some $r, s, t$ in $X$ and $A, B, C$ in $CB(X)$ such that

$$
\lim_{n \to \infty} gx_n = r \in A = \lim_{n \to \infty} F(x_n, y_n, z_n),
$$

$$
\lim_{n \to \infty} gy_n = s \in B = \lim_{n \to \infty} F(y_n, z_n, x_n),
$$

$$
\lim_{n \to \infty} gz_n = t \in C = \lim_{n \to \infty} F(z_n, x_n, y_n).
$$

**Example 2.1.** Let $X = [1, +\infty)$ with the usual metric. Define $g : X \to X$ and $F : X \times X \times X \to CB(X)$ by

$$
gx = 2 + x \quad \text{and} \\
F(x, y, z) = [2, 3 + 2x + y + z] \quad \text{for all } x, y, z \in X.
$$

Consider the sequences

$$
\{x_n\} = \{2 + \frac{1}{n}\}, \{y_n\} = \{4 + \frac{1}{n}\} \quad \text{and} \quad \{z_n\} = \{6 + \frac{1}{n}\}.
$$
Clearly,
\[ \lim_{n \to \infty} gx_n = 4 \in A = \lim_{n \to \infty} F(x_n, y_n, z_n) = [2, 17], \]
\[ \lim_{n \to \infty} gy_n = 6 \in B = \lim_{n \to \infty} F(y_n, z_n, x_n) = [2, 19], \]
\[ \lim_{n \to \infty} gz_n = 8 \in C = \lim_{n \to \infty} F(z_n, x_n, y_n) = [2, 21]. \]

Hence \( g \) and \( F \) satisfy (EA) property.

**Definition 2.2.** Mappings \( F : X \times X \times X \to 2^X \) and \( g : X \to X \) are said to be occasionally \( w \)-compatible if and only if there exists some point \((x, y, z) \in X \times X \times X\) such that \( gx \in F(x, y, z), gy \in F(y, z, x), gz \in F(z, x, y), \) and \( gF(x, y, z) \subseteq F(gx, gy, gz). \)

Following example shows that, occasionally \( w \)-compatibility is weaker condition than \( w \)-compatibility.

**Example 2.2.** Let \( X = [0, +\infty) \) with usual metric. Define \( g : X \to X, \)
\( F : X \times X \times X \to CB(X), \) for all \( x, y, z \in X, \) by
\[
    gx = \begin{cases} 
        0, & 0 \leq x \leq 1, \\
        4x, & 1 \leq x < \infty,
    \end{cases}
\]
\[
    F(x, y, z) = \begin{cases} 
        [0, 1 + 2x + y + z], & (x, y, z) \neq (0, 0, 0), \\
        \{x\}, & (x, y, z) = (0, 0, 0).
    \end{cases}
\]

It can be easily verified that \((0, 0, 0)\) and \((1, 1, 1)\) are tripled coincidence points of \( g \) and \( F, \) but \( gF(0, 0, 0) \nsubseteq F(g0, g0, g0) \) and \( gF(1, 1, 1) \nsubseteq F(g1, g1, g1) \)

So \( g \) and \( F \) are not \( w \)-compatible. However, the pair \( \{F, g\} \) is occasionally \( w \)- compatible.

Let \( \Psi \) denote the set of all functions \( \psi : [0, +\infty) \to [0, +\infty) \) satisfying

(i) \( \psi \) is continuous and non-decreasing,
(ii) \( \psi(t) = 0 \Leftrightarrow t = 0, \)
(iii) \( \limsup_{s \to 0^+} \psi(s) < \infty, \)

and \( \Phi \) denote the set of all functions \( \varphi : [0, +\infty) \to [0, +\infty) \) satisfying

(i) \( \varphi \) is lower semi-continuous and non-decreasing,
(ii) \( \varphi(t) = 0 \Leftrightarrow t = 0, \)
(iii) \( \text{for any sequence } \{t_n\} \text{ with } \lim_{n \to \infty} t_n = 0, \) there exist \( k \in (0, 1) \)

and \( n_0 \in \mathbb{N}, \) such that \( \varphi(t_n) \geq kt_n \) for each \( n \geq n_0, \)

and \( \Theta \) denote the set of all functions \( \theta : [0, +\infty) \to [0, +\infty) \) satisfying

(i) \( \theta \) is continuous,
(ii) \( \theta(t) = 0 \Leftrightarrow t = 0. \)
For simplicity, we define

\[(I)\] \[M(x, y, z, u, v, w) = \max \begin{cases} d(gx, gu), D(gx, F(x, y, z)), D(gu, F(u, v, w)), \\ d(gy, gv), D(gy, F(y, z, x)), D(gv, F(v, w, u)), \\ d(gz, gw), D(gz, F(z, x, y)), D(gw, F(w, u, v)), \\ \frac{D(gx, F(u, v, w)) + D(gu, F(x, y, z))}{2}, \\ \frac{D(gy, F(v, w, u)) + D(gv, F(y, z, x))}{2}, \\ \frac{D(gz, F(w, u, v)) + D(gw, F(z, x, y))}{2} \end{cases},\]

\[(II)\] \[N(x, y, z, u, v, w) = \min \begin{cases} D(gx, F(x, y, z)), D(gu, F(u, v, w)), \\ D(gy, F(u, v, w)), D(gu, F(x, y, z)), \\ D(gy, F(y, z, x)), D(gv, F(v, w, u)), \\ D(gz, F(z, x, y)), D(gw, F(w, u, v)), \\ D(gz, F(w, u, v)), D(gw, F(z, x, y)). \end{cases} \]

**Theorem 2.1.** Let \((X, d)\) be a metric space, \(F : X \times X \times X \to CB(X)\) and \(g : X \to X\) be two mappings. Suppose that there exist some \(\psi \in \Psi, \varphi \in \Phi\) and \(\theta \in \Theta\) such that

\[
\psi(H(F(x, y, z), F(u, v, w))) \\
\leq \psi(M(x, y, z, u, v, w)) - \varphi(M(x, y, z, u, v, w)) + \theta(N(x, y, z, u, v, w)),
\]

for all \(x, y, z, u, v, w \in X\). Furthermore, assume that \(\{F, g\}\) satisfies the (EA) property. Then \(F\) and \(g\) have a tripled coincidence point. Moreover, \(F\) and \(g\) have a common tripled fixed point, if one of the following conditions holds:

\(a\) \(F\) and \(g\) are \(w\)-compatible. \(\lim_{n \to \infty} g^n x = u, \lim_{n \to \infty} g^n y = v\) and \(\lim_{n \to \infty} g^n z = w\) for some \((x, y, z) \in C\{F, g\}\) and for some \(u, v, w \in X\) and \(g\) is continuous at \(u, v\) and \(w\).

\(b\) \(g\) is \(F\)-weakly commuting for some \((x, y, z) \in C\{F, g\}\) and \(g\) is continuous at \(x, y\) and \(z\).

\(c\) \(g\) is continuous at \(x, y\) and \(z\), \(\lim_{n \to \infty} g^n u = x, \lim_{n \to \infty} g^n v = y\) and \(\lim_{n \to \infty} g^n w = z\) for some \((x, y, z) \in C\{F, g\}\) and for some \(u, v, w \in X\).

\(d\) \(g(C\{F, g\})\) is a singleton subset of \(C\{F, g\}\).

**Proof.** Since \(\{F, g\}\) satisfies the (EA) property, therefore there exist sequences \(\{x_n\}, \{y_n\}\) and \(\{z_n\}\) in \(X\), some \(r, s, t, \in X\) and \(A, B, C\) in \(CB(X)\) such that

\[
\lim_{n \to \infty} gx_n = r \in A = \lim_{n \to \infty} F(x_n, y_n, z_n), \\
\lim_{n \to \infty} gy_n = s \in B = \lim_{n \to \infty} F(y_n, z_n, x_n), \\
\lim_{n \to \infty} gz_n = t \in B = \lim_{n \to \infty} F(z_n, x_n, y_n). \tag{2.2}
\]
Since \( g(X) \) is a subset of \( X \), then there exist \( x, y, z \in X \), we have
\[
r = gx, s = gy \text{ and } t = gz
\]
Now, by using condition (2.1) and \((i_\varphi)\), we get
\[
\psi (H(F(x_n, y_n, z_n), F(x, y, z)))
\leq \psi (M(x_n, y_n, z_n, x, y, z)) - \varphi (\psi (M(x_n, y_n, z_n, x, y, z)))
+ \theta (N(x_n, y_n, z_n, x, y, z)),
\]
where
\[
M(x_n, y_n, z_n, x, y, z) = \max \left\{ \frac{d(gx_n, gx)}{D(gx_n, F(x_n, y_n, z_n))}, \frac{d(gy_n, gy)}{D(gy_n, F(y_n, z_n, x_n))}, \frac{d(gz_n, gz)}{D(gz_n, F(z_n, x_n, y_n))}, \right. \\
\left. \frac{d(gx_n, F(x_n, y_n, z_n)) + D(gx, F(x, y, z))}{d(gy_n, F(y_n, z_n, x_n)) + D(gy, F(y, z, x)),} \right. \\
\left. \frac{D(gz_n, F(z_n, x_n, y_n)) + D(gz, F(z, x, y))}{d(gn, F(n, x, y)) + D(gn, F(z_n, x_n, y_n))} \right\}
\]
and
\[
N(x_n, y_n, z_n, x, y, z) = \min \left\{ D(gx_n, F(x_n, y_n, z_n)), D(gx, F(x, y, z)), \\
D(gy_n, F(y_n, z_n, x_n)), D(gy, F(y, z, x)), \\
D(gz_n, F(z_n, x_n, y_n)), D(gz, F(z, x, y)), \\
D(gn, F(n, x, y)), D(gn, F(z_n, x_n, y_n)) \right\}.
\]
Letting \( n \to \infty \) in the above inequality, by using \((i_\varphi)\), \((i_\varphi)\), \((ii_\varphi)\), \((2.2)\), \((2.3)\), \( gx \in A, gy \in B \) and \( gz \in C \), we get
\[
\psi (D(gx, F(x, y, z)))
\leq \psi (\max \{D(gx, F(x, y, z)), D(gy, F(y, z, x)), D(gz, F(z, x, y))\})
- \varphi (\psi (\max \{D(gx, F(x, y, z)), D(gy, F(y, z, x)), D(gz, F(z, x, y))\})).
\]
Similarly, we can obtain that
\[
\psi (D(gy, F(y, z, x)))
\leq \psi (\max \{D(gx, F(x, y, z)), D(gy, F(y, z, x)), D(gz, F(z, x, y))\})
- \varphi (\psi (\max \{D(gx, F(x, y, z)), D(gy, F(y, z, x)), D(gz, F(z, x, y))\})),
\]
and
\[
\psi (D(gz, F(z, x, y)))
\leq \psi (\max \{D(gx, F(x, y, z)), D(gy, F(y, z, x)), D(gz, F(z, x, y))\})
- \varphi (\psi (\max \{D(gx, F(x, y, z)), D(gy, F(y, z, x)), D(gz, F(z, x, y))\})).
\]
Since which, by it follows that (that is, non empty.

\[ u;v;w \in F \]

As

\[ \psi (D(gx, F(x, y, z)), D(gy, F(y, z, x)), D(gz, F(z, x, y))) \]

\[ \leq \psi (\max \{D(gx, F(x, y, z)), D(gy, F(y, z, x)), D(gz, F(z, x, y))\}) \]

\[ -\varphi (\psi (\max \{D(gx, F(x, y, z)), D(gy, F(y, z, x)), D(gz, F(z, x, y))\})) . \]

Since \( \psi \) is non-decreasing, therefore,

\[ \psi (\max \{D(gx, F(x, y, z)), D(gy, F(y, z, x)), D(gz, F(z, x, y))\}) \leq \psi (\max \{D(gx, F(x, y, z)), D(gy, F(y, z, x)), D(gz, F(z, x, y))\}) \]

\[ -\varphi (\psi (\max \{D(gx, F(x, y, z)), D(gy, F(y, z, x)), D(gz, F(z, x, y))\})) . \]

which, by \((ii_\psi)\) and \((ii_\psi)\), implies that

\[ \max \{D(gx, F(x, y, z)), D(gy, F(y, z, x)), D(gz, F(z, x, y))\} = 0, \]

it follows that

\[ gx \in F(x, y, z), gy \in F(y, z, x) \text{ and } gz \in F(z, x, y) \]

that is, \((x, y, z)\) is a tripled coincidence point of \( F \) and \( g \). That is \( C\{F, g\} \) is non empty.

Suppose now that \((a)\) holds. Assume that for some \((x, y, z) \in C\{F, g\}, \]

\[ \lim_{n \to \infty} g^n x = u, \lim_{n \to \infty} g^n y = v, \text{ and } \lim_{n \to \infty} g^n z = w, \quad (2.4) \]

where \( u, v, w \in X \). Since \( g \) is continuous at \( u, v \) and \( w \). We have, by \((2.4)\), that \( u, v \) and \( w \) are fixed points of \( g \), that is,

\[ gu = u, gv = v \text{ and } gw = w. \quad (2.5) \]

As \( F \) and \( g \) are \( w \)-compatible, so

\[ (g^n x, g^n y, g^n z) \in C\{F, g\}, \text{ for all } n \geq 1, \]

that is,

\[ g^n x \in F(g^{n-1} x, g^{n-1} y, g^{n-1} z), \]

\[ g^n y \in F(g^{n-1} y, g^{n-1} z, g^{n-1} x) \text{ and } \]

\[ g^n z \in F(g^{n-1} z, g^{n-1} x, g^{n-1} y), \text{ for all } n \geq 1. \quad (2.6) \]

Now, by using \((2.1)\), \((2.6)\) and \((i_\psi)\), we obtain

\[ \psi (D(g^n x, F(u, v, w))) \]

\[ \leq \psi (H(F(g^{n-1} x, g^{n-1} y, g^{n-1} z), F(u, v, w))) \]

\[ \leq \psi (M(g^{n-1} x, g^{n-1} y, g^{n-1} z, u, v, w)) - \varphi (\psi (M(g^{n-1} x, g^{n-1} y, g^{n-1} z, u, v, w))) \]

\[ + \theta (N(g^{n-1} x, g^{n-1} y, g^{n-1} z, u, v, w)), \]

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where

\[
M(g^{n-1}x, g^{n-1}y, g^{n-1}z, u, v, w)
= \max \left\{ \begin{array}{l}
d(g^n x, gu), D(g^n x, F(g^{n-1} x, g^{n-1} y, g^{n-1} z)), D(g u, F(u, v, w)), \\
d(g^n y, gv), D(g^n y, F(g^{n-1} y, g^{n-1} z, g^{n-1} x)), D(g v, F(v, w, u)), \\
d(g^n z, gw), D(g^n z, F(g^{n-1} z, g^{n-1} x, g^{n-1} y)), D(g w, F(w, u, v)), \\
\frac{D(g^n x, F(u, v, w)) + D(g w, F(g^{n-1} x, g^{n-1} y, g^{n-1} z))}{2} \\
\frac{D(g^n y, F(v, w, u)) + D(g v, F(g^{n-1} y, g^{n-1} z, g^{n-1} x))}{2} \\
\frac{D(g^n z, F(w, u, v)) + D(g w, F(g^{n-1} z, g^{n-1} x, g^{n-1} y))}{2}
\end{array} \right.
\]

\[
\leq \max \left\{ \begin{array}{l}
d(g^n x, gu), d(g^n x, g^n x), D(g u, F(u, v, w)), \\
d(g^n y, gv), d(g^n y, g^n y), D(g v, F(v, w, u)), \\
d(g^n z, gw), d(g^n z, g^n z), D(g w, F(w, u, v)), \\
\frac{D(g^n x, F(u, v, w)) + D(g w, g^n x)}{2}, \\
\frac{D(g^n y, F(v, w, u)) + D(g v, g^n y)}{2}, \\
\frac{D(g^n z, F(w, u, v)) + D(g w, g^n z)}{2}
\end{array} \right.
\]

and

\[
N(g^{n-1}x, g^{n-1}y, g^{n-1}z, u, v, w)
= \min \left\{ \begin{array}{l}
D(g^n x, F(g^{n-1} x, g^{n-1} y, g^{n-1} z)), D(g u, F(u, v, w)), \\
D(g^n x, F(u, v, w)), D(g u, F(g^{n-1} x, g^{n-1} y, g^{n-1} z)), \\
D(g^n x, F(u, v, w)), D(g u, F(g^{n-1} x, g^{n-1} y, g^{n-1} z)), \\
D(g^n y, F(v, w, u)), D(g v, F(v^{n-1} y, g^{n-1} z, g^{n-1} x)), \\
D(g^n z, F(w, u, v)), D(g w, F(g^{n-1} z, g^{n-1} x, g^{n-1} y)), \\
D(g^n z, F(w, u, v)), D(g w, F(g^{n-1} z, g^{n-1} x, g^{n-1} y))
\end{array} \right. = 0.
\]

On taking limit as \( n \to \infty \) in the above inequality, by using (i\( _\psi \)), (i\( _\varphi \)), (ii\( _\theta \)), (2.4), (2.5) and (2.6), we get

\[
\psi (D(g u, F(u, v, w))) \\
\leq \psi (\max \{D(g u, F(u, v, w)), D(g v, F(v, w, u)), D(g w, F(w, u, v))\}) \\
- \varphi (\psi (\max \{D(g u, F(u, v, w)), D(g v, F(v, w, u)), D(g w, F(w, u, v))\})).
\]

Similarly

\[
\psi (D(g v, F(v, w, u))) \\
\leq \psi (\max \{D(g u, F(u, v, w)), D(g v, F(v, w, u)), D(g w, F(w, u, v))\}) \\
- \varphi (\psi (\max \{D(g u, F(u, v, w)), D(g v, F(v, w, u)), D(g w, F(w, u, v))\})).
\]

and

\[
\psi (D(g w, F(w, u, v))) \\
\leq \psi (\max \{D(g u, F(u, v, w)), D(g v, F(v, w, u)), D(g w, F(w, u, v))\}) \\
- \varphi (\psi (\max \{D(g u, F(u, v, w)), D(g v, F(v, w, u)), D(g w, F(w, u, v))\})).
\]
Combining them, we get
\[
\max \{\psi(D(gu, F(u, v, w))), \psi(D(gv, F(v, w, u))), \psi(D(gw, F(w, u, v)))\} \\
\leq \psi \left( \max \{D(gu, F(u, v, w)), D(gv, F(v, w, u)), D(gw, F(w, u, v))\} \right) \\
- \varphi \left( \psi \left( \max \{D(gu, F(u, v, w)), D(gv, F(v, w, u)), D(gw, F(w, u, v))\} \right) \right).
\]

Since \(\psi\) is non-decreasing, therefore
\[
\psi \left( \max \{D(gu, F(u, v, w)), D(gv, F(v, w, u)), D(gw, F(w, u, v))\} \right) \\
\leq \psi \left( \max \{D(gu, F(u, v, w)), D(gv, F(v, w, u)), D(gw, F(w, u, v))\} \right) \\
- \varphi \left( \psi \left( \max \{D(gu, F(u, v, w)), D(gv, F(v, w, u)), D(gw, F(w, u, v))\} \right) \right),
\]
which, by (ii\(\varphi\)) and (ii\(\psi\)), implies that
\[
\max \{D(gu, F(u, v, w)), D(gv, F(v, w, u)), D(gw, F(w, u, v))\} = 0,
\]
it follows that
\[
gu \in F(u, v, w), \quad gv \in F(v, w, u) \quad \text{and} \quad gw \in F(w, u, v). \quad (2.7)
\]
Now, from (2.5) and (2.7), we have
\[
u = gu \in F(u, v, w), \quad v = gv \in F(v, w, u) \quad \text{and} \quad w = gw \in F(w, u, v),
\]
that is, \((u, v, w)\) is a common tripled fixed point of \(F\) and \(g\).

Suppose now that (b) holds. Assume that for some \((x, y, z) \in C\{F, g\}\), \(g\) is \(F\)-weakly commuting, that is \(g^2x \in F(gx, gy, gz), g^2y \in F(gy, gz, gx), g^2z \in F(gz, gx, gy)\) and \(g^2x = gx, g^2y = gy, g^2z = gz\). Thus \(gx = g^2x \in F(gx, gy, gz), gy = g^2y \in F(gy, gz, gx)\) and \(gz = g^2z \in F(gz, gy, gx)\), that is, \((gx, gy, gz)\) is a common tripled fixed point of \(F\) and \(g\).

Suppose now that (c) holds. Assume that for some \((x, y, z) \in C\{F, g\}\) and for some \(u, v, w \in X\),
\[
\lim_{n \to -\infty} g^n u = x, \quad \lim_{n \to -\infty} g^n v = y \quad \text{and} \quad \lim_{n \to -\infty} g^n w = z. \quad (2.8)
\]
Since \(g\) is continuous at \(x, y\) and \(z\). Therefore, by (2.8), we obtain that \(x, y\) and \(z\) are fixed points of \(g\), that is,
\[
\begin{align*}
gx &= x, \quad gy = y \quad \text{and} \quad gz = z. \quad (2.9)
\end{align*}
\]
Since \((x, y, z) \in C\{F, g\}\). Therefore, by (2.9), we obtain \(x = gx \in F(x, y, z), y = gy \in F(y, z, x)\) and \(z = gz \in F(z, x, y)\), that is, \((x, y, z)\) is a common tripled fixed point of \(F\) and \(g\).

Finally, suppose that (d) holds. Let \(g(C\{F, g\}) \subseteq \{(x, x, x)\}\). Then \(\{x\} = \{gx\} = F(x, x, x)\). Hence \((x, x, x)\) is a common tripled fixed point of \(F\) and \(g\).
If we put $\theta(t) = 0$ in the Theorem 2.1, we get the following result:

**Corollary 2.2.** Let $(X, d)$ be a metric space, $F : X \times X \times X \to CB(X)$ and $g : X \to X$ be two mappings. Suppose that there exist some $\psi \in \Psi$ and $\varphi \in \Phi$ such that

$$\psi(H(F(x, y, z), F(u, v, w))) \leq \psi(M(x, y, z, u, v, w)) - \varphi(\psi(M(x, y, z, u, v, w))),$$

for all $x, y, z, u, v, w \in X$. Furthermore, assume that $\{F, g\}$ satisfies the (EA) property. Then $F$ and $g$ have a tripled coincidence point. Moreover, $F$ and $g$ have a common tripled fixed point, if one of the conditions $(a)$ to $(d)$ of Theorem 2.1 holds.

If we put $\varphi(t) = t - t\varphi(t)$ for all $t \geq 0$ in Corollary 2.2, then we get the following result:

**Corollary 2.3.** Let $(X, d)$ be a metric space, $F : X \times X \times X \to CB(X)$ and $g : X \to X$ be two mappings. Suppose that there exist some $\psi \in \Psi$ and $\varphi \in \Phi$ such that

$$\psi(H(F(x, y, z), F(u, v, w))) \leq \varphi(\psi(M(x, y, z, u, v, w))) \psi(M(x, y, z, u, v, w)),$$

for all $x, y, z, u, v, w \in X$. Furthermore, assume that $\{F, g\}$ satisfies the (EA) property. Then $F$ and $g$ have a tripled coincidence point. Moreover, $F$ and $g$ have a common tripled fixed point, if one of the conditions $(a)$ to $(d)$ of Theorem 2.1 holds.

If we put $\psi(t) = 2t$ for all $t \geq 0$ in Corollary 2.3, then we get the following result:

**Corollary 2.4.** Let $(X, d)$ be a metric space, $F : X \times X \times X \to CB(X)$ and $g : X \to X$ be two mappings. Suppose that there exists some $\varphi \in \Phi$ such that

$$H(F(x, y, z), F(u, v, w)) \leq \varphi(2M(x, y, z, u, v, w)) 2M(x, y, z, u, v, w),$$

for all $x, y, z, u, v, w \in X$. Furthermore, assume that $\{F, g\}$ satisfies the (EA) property. Then $F$ and $g$ have a tripled coincidence point. Moreover, $F$ and $g$ have a common tripled fixed point, if one of the conditions $(a)$ to $(d)$ of Theorem 2.1 holds.

If we put $\varphi(t) = k$ where $0 < k < 1$, for all $t \geq 0$ in Corollary 2.4, then we get the following result:

**Corollary 2.5.** Let $(X, d)$ be a metric space. Assume $F : X \times X \times X \to CB(X)$ and $g : X \to X$ be two mappings satisfying

$$H(F(x, y, z), F(u, v, w)) \leq kM(x, y, z, u, v, w),$$

for all $x, y, z, u, v, w \in X$, where $0 < k < 1$. Furthermore, assume that $\{F, g\}$ satisfies the (EA) property. Then $F$ and $g$ have a tripled coincidence point.
Moreover, $F$ and $g$ have a common tripled fixed point, if one of the conditions (a) to (d) of Theorem 2.1 holds.

**Theorem 2.6.** Let $(X, d)$ be a metric space, $F : X \times X \times X \to CB(X)$ and $g : X \to X$ be two mappings. Suppose that there exist some $\psi \in \Psi$, $\varphi \in \Phi$ and $\theta \in \Theta$ satisfying (2.1) and $\{F, g\}$ is occasionally $w$–compatible. Then $F$ and $g$ have a common tripled fixed point.

**Proof.** Since the pair $\{F, g\}$ is occasionally $w$–compatible, therefore there exists some point $(x, y, z) \in X \times X \times X$ such that

$$gx \in F(x, y, z), \quad gy \in F(y, z, x), \quad gz \in F(z, x, y) \quad \text{and} \quad gF(x, y, z) \subseteq F(gx, gy, gz).$$

It follows that

$$g^2x \in F(gx, gy, gz), \quad g^2y \in F(gy, gz, gx) \quad \text{and} \quad g^2z \in F(gz, gx, gy).$$

(2.14)

Now, suppose $u = gx, v = gy$ and $w = gz$, then by (2.15), we get

$$gu \in F(u, v, w), \quad gv \in F(v, w, u) \quad \text{and} \quad gw \in F(w, u, v).$$

(2.16)

Thus, by condition (2.1), we have

$$\psi(H(F(x, y, z), F(u, v, w))) \leq \psi(M(x, y, z, u, v, w)) - \varphi(\psi(M(x, y, z, u, v, w))) + \theta(N(x, y, z, u, v, w)).$$

which, by (2.14), (2.16), (i$\phi$), (i$\varphi$), (i$\psi$), and triangle inequality, implies

$$\psi(d(gx, gu)) \leq \psi(\max\{d(gx, gu), d(gy, gv), d(gz, gw)\}) - \varphi(\psi(\max\{d(gx, gu), d(gy, gv), d(gz, gw)\})).$$

Similarly, we can obtain that

$$\psi(d(gy, gv)) \leq \psi(\max\{d(gx, gu), d(gy, gv), d(gz, gw)\}) - \varphi(\psi(\max\{d(gx, gu), d(gy, gv), d(gz, gw)\})).$$

and

$$\psi(d(gz, gw)) \leq \psi(\max\{d(gx, gu), d(gy, gv), d(gz, gw)\}) - \varphi(\psi(\max\{d(gx, gu), d(gy, gv), d(gz, gw)\})).$$

Combining them, we get

$$\max\{\psi(d(gx, gu)), \psi(d(gy, gv)), \psi(d(gz, gw))\} \leq \psi(\max\{d(gx, gu), d(gy, gv), d(gz, gw)\}) - \varphi(\psi(\max\{d(gx, gu), d(gy, gv), d(gz, gw)\})).$$

Since $\psi$ is non-decreasing, therefore

$$\psi(\max\{d(gx, gu), (gy, gv), (gz, gw)\}) \leq \psi(\max\{d(gx, gu), d(gy, gv), d(gz, gw)\}) - \varphi(\psi(\max\{d(gx, gu), d(gy, gv), d(gz, gw)\})).$$

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which, by \((ii_\varphi)\) and \((ii_\psi)\), implies that
\[
\max \{d(gx, gu), d(gy, gv), d(gz, gw)\} = 0,
\]
it follows that \(d(gx, gu) = d(gy, gv) = d(gz, gw) = 0\). Hence
\[
u = gx = gu, v = gy = gv \text{ and } w = gz = gw.
\]
Thus, by (2.16) and (2.17), we get
\[
u = gu \in F(u, v, w), v = gv \in F(v, w, u) \text{ and } w = gw \in F(w, u, v),
\]
that is, \((u, v, w)\) is a common tripled fixed point of \(F\) and \(g\).

**Example 2.3.** Suppose that \(X = [0, 1]\), equipped with the metric \(d : X \times X \to [0, +\infty)\) defined as \(d(x, y) = \max\{x, y\}\) and \(d(x, x) = 0\) for all \(x, y, \in X\). Let \(F : X \times X \times X \to CB(X)\) be defined as
\[
F(x, y, z) = \begin{cases} 
\{0\}, & \text{for } x, y, z = 1, \\
[0, \frac{x+y+z}{6}], & \text{for } x, y, z \in [0, 1).
\end{cases}
\]
and \(g : X \to X\) be defined as
\[
gx = \frac{x}{2} \text{ for all } x \in X.
\]
Define \(\psi : [0, +\infty) \to [0, +\infty)\) by
\[
\psi(t) = \frac{t}{2}, \text{ for all } t \geq 0,
\]
and \(\varphi : [0, +\infty) \to [0, +\infty)\) by
\[
\varphi(t) = \frac{t}{3}, \text{ for all } t \geq 0,
\]
and \(\theta : [0, +\infty) \to [0, +\infty)\) by
\[
\theta(t) = \frac{t}{4}, \text{ for all } t \geq 0.
\]
Now, for all \(x, y, z, u, v, w \in X\) with \(x, y, z, u, v, w \in [0, 1]\), we have

**Case (a).** If \(x + y + z = u + v + w\), then
\[
\psi(H(F(x, y, z), F(u, v, w)))
\]
\[
= \frac{1}{2}H(F(x, y, z), F(u, v, w))
\]
\[
= \frac{1}{12}(u + v + w)
\]
\[
\leq \frac{1}{6}\max\left\{\frac{x}{2}, \frac{u}{2}\right\} + \frac{1}{6}\max\left\{\frac{y}{2}, \frac{v}{2}\right\} + \frac{1}{6}\max\left\{\frac{z}{2}, \frac{w}{2}\right\}
\]
\[
\leq \frac{1}{6}d(gx, gu) + \frac{1}{6}d(gy, gv) + \frac{1}{6}d(gz, gw)
\]
\[
\leq \frac{1}{3}M(x, y, z, u, v, w)
\]
\[
\leq \psi(M(x, y, z, u, v, w)) - \varphi(\psi(M(x, y, z, u, v, w)))
\]
\[
\leq \psi(M(x, y, z, u, v, w)) - \varphi(\psi(M(x, y, z, u, v, w))) + \theta(N(x, y, z, u, v, w)).
\]
Case (b). If \( x + y + z \neq u + v + w \) with \( x + y + z < u + v + w \), then

\[
\psi \left( H(F(x, y, z), F(u, v, w)) \right) = \frac{1}{2} H(F(x, y, z), F(u, v, w)) = \frac{1}{12} (u + v + w)
\]

\[
\leq \frac{1}{6} \max \left\{ \frac{x}{2}, \frac{u}{2} \right\} + \frac{1}{6} \max \left\{ \frac{y}{2}, \frac{v}{2} \right\} + \frac{1}{6} \max \left\{ \frac{z}{2}, \frac{w}{2} \right\}
\]

\[
\leq \frac{1}{6} d(gx, gu) + \frac{1}{6} d(gy, gv) + \frac{1}{6} d(gz, gw)
\]

\[
\leq \frac{1}{3} M(x, y, z, u, v, w)
\]

\[
\leq \psi \left( M(x, y, z, u, v, w) \right) - \varphi \left( \psi \left( M(x, y, z, u, v, w) \right) \right)
\]

\[
\leq \psi \left( M(x, y, z, u, v, w) \right) - \varphi \left( \psi \left( M(x, y, z, u, v, w) \right) \right) + N(x, y, z, u, v, w)
\]

Similarly, we obtain the same result for \( u + v + w < x + y + z \). Thus the contractive condition (2.1) is satisfied for all \( x, y, z, u, v, w \in X \) with \( u, v, w = 1 \) and \( x, v, w = 1 \), we have

\[
\psi \left( H(F(x, y, z), F(u, v, w)) \right) = \frac{1}{2} H(F(x, y, z), F(u, v, w)) = \frac{1}{12} (x + y + z)
\]

\[
\leq \frac{1}{6} \max \left\{ \frac{x}{2}, \frac{u}{2} \right\} + \frac{1}{6} \max \left\{ \frac{y}{2}, \frac{v}{2} \right\} + \frac{1}{6} \max \left\{ \frac{z}{2}, \frac{w}{2} \right\}
\]

\[
\leq \frac{1}{6} d(gx, gu) + \frac{1}{6} d(gy, gv) + \frac{1}{6} d(gz, gw)
\]

\[
\leq \frac{1}{3} M(x, y, z, u, v, w)
\]

\[
\leq \psi \left( M(x, y, z, u, v, w) \right) - \varphi \left( \psi \left( M(x, y, z, u, v, w) \right) \right)
\]

\[
\leq \psi \left( M(x, y, z, u, v, w) \right) - \varphi \left( \psi \left( M(x, y, z, u, v, w) \right) \right) + N(x, y, z, u, v, w)
\]

Thus the contractive condition (2.1) is satisfied for all \( x, y, z, u, v, w \in X \) with \( x, y, z \in [0, 1] \) and \( u, v, w = 1 \). Similarly, we can see that the contractive condition (2.1) is satisfied for all \( x, y, z, u, v, w \in X \) with \( x, y, z, u, v, w = 1 \). Hence, the hybrid pair \( \{ F, g \} \) satisfies the contractive condition (2.1), for all \( x, y, z, u, v, w \in X \). In addition, all the other conditions of Theorem 2.1 and Theorem 2.6 are satisfied and \( z = (0, 0, 0) \) is a common tripled fixed point of hybrid pair \( \{ F, g \} \). The function \( F : X \times X \to CB(X) \) involved in this example is not continuous at the point \((1, 1, 1) \in X \times X \times X\).


[9] V. Berinde and M. Borcut, Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces, Nonlinear Analysis, 74(15), (2011), 4889 - 4897.


