Regular Elements of the Complete Semigroups of Binary Relations

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Abstract—*In this paper, we investigate regular elements properties in given semilattice* $Q=\{T_1,T_2,\ldots,T_{m-3},T_{m-2},T_{m-1},T_m\}$. *Additionally, we will calculate the number of regular elements of* $B_X(D)$ *for a finite set* X.

Keywords—*Semigroups, Binary relation, Regular elements.*

I. Introduction

Let X be an arbitrary nonempty set and B_X be semigroup of all binary relations on the set X. If D is a nonempty set of subsets of X which is closed under the union then D is called a *complete* $X-$ *semilattice of unions*.

Let $x,y\in X$, $Y\subseteq X$, $\alpha\in B_X$, $T\in D$, $\varnothing\neq D^{'}\subseteq D$ and $t\in \breve{D}$. Then we have the following notations,

$$y\alpha=\{x\in X\mid (y,x)\in\alpha\},Y\alpha=\bigcup_{y\in Y}y\alpha, V(D,\alpha)=\{Y\alpha\mid Y\in D\}$$

$$D_t=\{Z^{'}\in D\mid t\in Z^{'}\}, D^{'}_T=\{Z^{'}\in D^{'}\mid T\subseteq Z^{'}\}, \ddot{D}^{'}_T=\{Z^{'}\in D^{'}\mid Z^{'}\subseteq T\}$$

$$N(D,D^{'})=\{Z\in D\mid Z\subseteq Z^{'} \text{for any} Z^{'}\in D^{'}\}, \Lambda(D,D^{'})=\cup N(D,D^{'})$$

Let f be an arbitrary mapping from X into D. Then one can construct a binary relation α_f on X by $\alpha_f=\bigcup_{x\in X}(\{x\}\times f(x))$. The set of all such binary relations is denoted by $B_X(D)$ and called a *complete semigroup of binary relations* defined by an $X-$semilattice of unions D. This structure was comprehensively investigated in Diasamidze [1].

A complete $X-$semilattice of unions D is an $XI-$ semilattice of unions if $\Lambda(D,D_t)\in D$ for any $t\in\breve{D}$ and $Z=\bigcup_{t\in Z}\Lambda(D,D_t)$ for any nonempty element Z of D.

$\alpha\in B_X(D)$ is *idempotent* if $\alpha\circ\alpha=\alpha$ and $\alpha\in B_X(D)$ is *regular* if $\alpha\circ\beta\circ\alpha=\alpha$ for some $\beta\in B_X(D)$.

Let $D^{'}$ be an arbitrary nonempty subset of the complete $X-$ semilattice of unions D. Set $l(D^{'},T)=\cup(D^{'}\setminus D^{'}_T)$. We say that a nonempty element T is a *nonlimiting element* of $D^{'}$ if $T\setminus l(D^{'},T)\neq\varnothing$. Also, a nonempty element T said to be *limiting element* of $D^{'}$ if $T\setminus l(D^{'},T)=\varnothing$.

Let $D=\{\breve{D},Z_1,Z_2,\ldots,Z_{m-1}\}$ be finite $X-$ semilattice of unions and $C(D)=\{P_0,P_1,P_2\ldots,P_{m-1}\}$ be the family of pairwise nonintersecting subsets of X. If $\varphi=\begin{pmatrix}\breve{D} & Z_1 & \cdots & Z_{m-1}\\ P_0 & P_1 & \cdots & P_{m-1}\end{pmatrix}$ is a mapping from D on $C(D)$ then $\breve{D}=P_0\cup P_1\cup P_2\cup\ldots\cup P_{m-1}$ and $Z_i=P_0\cup\bigcup_{T\in D\setminus D_Z}\varphi(T)$ satisfy.

Definitions and properties of $\Phi(D,D^{'})$, $\Omega(D)$, $R(D^{'})$ and $R_\varphi(D,D^{'})$ can be found in [1].

In this paper, we take in particular $Q=\{T_1,T_2,\ldots,T_{m-3},T_{m-2},T_{m-1},T_m\}$ subsemilattice of $X-$semilattice of unions D which the elements are satisfying

$T_1\subset T_3\subset T_5\subset\ldots\subset T_{m-3}\subset T_{m-1}\subset T_m$, $T_1\subset T_3\subset T_5\subset\ldots\subset T_{m-3}\subset T_{m-2}\subset T_m$,
$T_2\subset T_4\subset T_5\subset\ldots\subset T_{m-3}\subset T_{m-1}\subset T_m$, $T_2\subset T_4\subset T_5\subset\ldots\subset T_{m-3}\subset T_{m-2}\subset T_m$,
$T_2\subset T_3\subset T_5\subset\ldots\subset T_{m-3}\subset T_{m-1}\subset T_m$, $T_2\subset T_3\subset T_5\subset\ldots\subset T_{m-3}\subset T_{m-2}\subset T_m$,
$T_2\cup T_1=T_3,$ $T_4\cup T_3=T_5$, $T_{m-2}\cup T_{m-1}=T_m$, $T_2\setminus T_1\neq\varnothing$, $T_1\setminus T_2\neq\varnothing$, $T_4\setminus T_3\neq\varnothing$
and $T_3\setminus T_4\neq\varnothing$, $T_{m-2}\setminus T_{m-1}\neq\varnothing$, $T_{m-1}\setminus T_{m-2}\neq\varnothing$.

We will investigate the properties of regular element $\alpha\in B_X(D)$ satisfying $V(D,\alpha)=Q$. Moreover, we will calculate the number of regular elements of $B_X(D)$ for a finite set X.

Theorem 1.1 *[2, Theorem 10] Let α and σ be binary relations of the semigroup $B_X(D)$ such that $\alpha\circ\sigma\circ\alpha=\alpha$. If $D(\alpha)$ is some generating set of the semilattice $V(D,\alpha)\setminus\{\varnothing\}$ and $\alpha=\bigcup_{T\in V(D,\alpha)}(Y_T^\alpha\times T)$ is a quasinormal representation of the relation α, then $V(D,\alpha)$ is a complete $XI-$ semilattice of unions. Moreover, there exists a complete isomorphism φ between the semilattice $V(D,\alpha)$ and $D'=\{T\sigma\mid T\in V(D,\alpha)\}$, that satisfies the following conditions:*

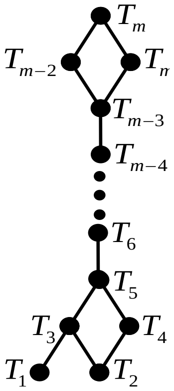
1. $\varphi(T)=T\sigma$ and $\varphi(T)\alpha=T$ for all $T\in V(D,\alpha)$
2. $\bigcup_{T'\in\ddot{D}(\alpha)_T} Y_{T'}^\alpha\supseteq\varphi(T)$ for any $T\in D(\alpha)$,
3. $Y_T^\alpha\cap\varphi(T)\neq\varnothing$ for all nonlimiting element T of the set $\ddot{D}(\alpha)_T$,
4. If T is a limiting element of the set $\ddot{D}(\alpha)_T$, then the equality $\cup B(T)=T$ is always holds for the set $B(T)=\{Z\in\ddot{D}(\alpha)_T\mid Y_Z^\alpha\cap\varphi(T)\neq\varnothing\}$.

On the other hand, if $\alpha\in B_X(D)$ such that $V(D,\alpha)$ is a complete $XI-$ semilattice of unions and if some complete $\alpha-$isomorphism φ from $V(D,\alpha)$ to a subsemilattice D' of D satisfies the conditions $b)-d)$ of the theorem, then α is a regular element of $B_X(D)$.

Theorem 1.2 *[1, Theorem 6.3.5] Let X be a finite set. If φ is a fixed element of the set $\Phi(D,D')$ and $|\Omega(D)|=m_0$ and q is a number of all automorphisms of the semilattice D then $|R(D')|=m_0\cdot q\cdot|R_\varphi(D,D')|$.*

II. RESULTS

Let X be a finite set, D be a complete $X-$ semilattice of unions and $Q=\{T_1,T_2,T_3,\ldots,T_{m-3},T_{m-2},T_{m-1},T_m\}$ be a $X-$subsemilattice of unions of D satisfies the following conditions.The diagram of the Q is shown in the following figure.



$$T_1 \subset T_3 \subset T_5 \subset T_6 \subset \ldots \subset T_{m-3} \subset T_{m-1} \subset T_m,$$
$$T_1 \subset T_3 \subset T_5 \subset T_6 \subset \ldots \subset T_{m-3} \subset T_{m-2} \subset T_m,$$
$$T_2 \subset T_4 \subset T_5 \subset T_6 \subset \ldots \subset T_{m-3} \subset T_{m-1} \subset T_m,$$
$$T_2 \subset T_4 \subset T_5 \subset T_6 \subset \ldots \subset T_{m-3} \subset T_{m-2} \subset T_m,$$
$$T_2 \subset T_3 \subset T_5 \subset T_6 \subset \ldots \subset T_{m-3} \subset T_{m-1} \subset T_m,$$
$$T_2 \subset T_3 \subset T_5 \subset T_6 \subset \ldots \subset T_{m-3} \subset T_{m-2} \subset T_m,$$
$$T_2 \setminus T_1 \neq \varnothing, T_1 \setminus T_2 \neq \varnothing, T_4 \setminus T_3 \neq \varnothing, T_3 \setminus T_4 \neq \varnothing,$$
$$T_{m-2} \setminus T_{m-1} \neq \varnothing, T_{m-1} \setminus T_{m-2} \neq \varnothing, T_2 \cup T_1 = T_{3,}$$
$$T_4 \cup T_3 = T_{5,} \ldots, T_{m-2} \cup T_{m-1} = T_{m.}$$

Let $C(Q) = \{P_i \mid i = 1,2,\ldots,m\}$. Then

$$T_m = P_m \cup P_{m-1} \cup P_{m-2} \cup \cdots \cup P_1,$$
$$T_{m-1} = P_m \cup P_{m-2} \cup \cdots \cup P_1$$
$$T_{m-2} = P_m \cup P_{m-1} \cup \cdots \cup P_1, \ldots,$$
$$T_4 = P_m \cup P_3 \cup P_2 \cup P_1,$$
$$T_3 = P_m \cup P_4 \cup P_2 \cup P_1,$$
$$T_2 = P_m \cup P_1,$$
$$T_1 = P_m \cup P_4 \cup P_2$$

are obtained.

First, we investigate that in which conditions Q is $XI-$ semilattice of unions. We determine the greatest lower bounds of the each semilattice Q_t in Q for $t \in T_m$. We get,

$$Q_t = \begin{cases} Q & ,t \in P_m \\ \{T_m, T_{m-2}\} & ,t \in P_{m-1} \\ \{T_m, T_{m-1}\} & ,t \in P_{m-2} \\ \{T_m, T_{m-1}, T_{m-2}\} & ,t \in P_{m-3} \\ \{T_m, T_{m-1}, T_{m-2}, T_{m-3}\} & ,t \in P_{m-4} \\ \vdots & \\ \{T_m, \ldots, T_5, T_3, T_1\} & ,t \in P_4 \\ \{T_m, \ldots, T_4\} & ,t \in P_3 \\ \{T_m, \ldots, T_3, T_1\} & ,t \in P_2 \\ \{T_m, \ldots, T_3, T_2\} & ,t \in P_1 \end{cases} \qquad (2.1)$$

From the Equation (2.1) the greatest lower bounds for each semilattice Q_t

$$\begin{array}{lll}
t \in P_m & \Rightarrow N(Q,Q_t) = \varnothing & \Rightarrow \Lambda(Q,Q_t) = \varnothing \\
t \in P_{m-1} & \Rightarrow N(Q,Q_t) = \{T_{m-2}, T_{m-3}, \ldots, T_1\} & \Rightarrow \Lambda(Q,Q_t) = T_{m-2} \\
t \in P_{m-2} & \Rightarrow N(Q,Q_t) = \{T_{m-1}, T_{m-3}, \ldots, T_1\} & \Rightarrow \Lambda(Q,Q_t) = T_{m-1} \\
t \in P_{m-3} & \Rightarrow N(Q,Q_t) = \{T_{m-3}, \ldots, T_1\} & \Rightarrow \Lambda(Q,Q_t) = T_{m-3} \\
t \in P_{m-4} & \Rightarrow N(Q,Q_t) = \{T_{m-3}, \ldots, T_1\} & \Rightarrow \Lambda(Q,Q_t) = T_{m-3} \\
\vdots & \vdots & \vdots \\
t \in P_4 & \Rightarrow N(Q,Q_t) = \{T_1\} & \Rightarrow \Lambda(Q,Q_t) = T_1 \\
t \in P_3 & \Rightarrow N(Q,Q_t) = \{T_4, \ldots, T_1\} & \Rightarrow \Lambda(Q,Q_t) = T_4 \\
t \in P_2 & \Rightarrow N(Q,Q_t) = \varnothing & \Rightarrow \Lambda(Q,Q_t) = \varnothing \\
t \in P_1 & \Rightarrow N(Q,Q_t) = \{T_2\} & \Rightarrow \Lambda(Q,Q_t) = T_2
\end{array} \tag{2.2}$$

are obtained. If $t \in P_m$ or $t \in P_2$, then $\Lambda(D,D_t) = \varnothing \notin D$. So, $P_m \cup P_2 = \varnothing$. Also using the Equation (2.2), we have seen easily $\bigcup_{t \in T_i} \Lambda(Q,Q_t) \in D$.

Lemma 2.1 *Q is $XI-$ semilattice of unions if and only if $T_1 \cap T_4 = \varnothing$*

Proof. $\Rightarrow$: Let Q be a $XI-$ semilattice of unions. Then $P_m \cup P_2 = \varnothing$ and $T_1 = P_2$, $T_4 = P_1 \cup P_3$ by Equation (2.1). Therefore $T_1 \cap T_4 = \varnothing$ since P_1, P_2 and P_3 are pairwise disjoint sets.

$\Leftarrow$: If $T_1 \cap T_4 = \varnothing$, then $P_m \cup P_2 = \varnothing$. Using the Equation (2.2), we see that $\bigcup_{t \in T_i} \Lambda(Q,Q_t) = T_i$. So, we have Q is $XI-$ semilattice of unions.

Lemma 2.2 *Let $G = \{T_1, T_2, \ldots, T_{m-1}\}$ be a generating set of Q. Then the elements $T_1, T_2, T_4, T_6, \ldots, T_{m-1}$ are nonlimiting elements of the sets $\ddot{G}_{T_1}$, $\ddot{G}_{T_2}$, $\ddot{G}_{T_4}$, $\ddot{G}_{T_6}, \ldots, \ddot{G}_{T_{m-1}}$ respectively and T_3, T_5 is limiting element sof the sets $\ddot{G}_{T_3}, \ddot{G}_{T_5}$ respectively.*

Proof. Definition of $\ddot{D}_T$ and $l(\ddot{G}_{T_i}, T_i) = \cup(\ddot{G}_{T_i} \setminus \{T_i\})$, $i \in \{1, 2, \ldots, m-1\}$, we find nonlimiting and limiting elements of $\ddot{G}_{T_i}$.

$$\begin{array}{ll}
T_1 \setminus l(\ddot{G}_{T_1}, T_1) = T_1 \setminus \varnothing \neq \varnothing, & T_1 \text{ nonlimiting element of } \ddot{G}_{T_1} \\
T_2 \setminus l(\ddot{G}_{T_2}, T_2) = T_2 \setminus \varnothing \neq \varnothing, & T_2 \text{ nonlimiting element of } \ddot{G}_{T_2} \\
T_3 \setminus l(\ddot{G}_{T_3}, T_3) = T_3 \setminus T_3 = \varnothing, & T_3 \text{ limiting element of } \ddot{G}_{T3} \\
T_4 \setminus l(\ddot{G}_{T_4}, T_4) = T_4 \setminus T_2 \neq \varnothing, & T_4 \text{ nonlimiting element of } \ddot{G}_{T_4} \\
T_5 \setminus l(\ddot{G}_{T_5}, T_5) = T_5 \setminus T_5 = \varnothing, & T_5 \text{ limiting element of } \ddot{G}_{T_5} \\
T_6 \setminus l(\ddot{G}_{T_6}, T_6) = T_6 \setminus T_5 \neq \varnothing, & T_6 \text{ nonlimiting element of } \ddot{G}_{T_6} \\
\vdots & \vdots \\
T_{m-4} \setminus l(\ddot{G}_{T_{m-4}}, T_{m-4}) = T_{m-4} \setminus T_{m-5} \neq \varnothing, & T_{m-4} \text{ nonlimiting element of } \ddot{G}_{T_{m-4}} \\
T_{m-3} \setminus l(\ddot{G}_{T_{m-3}}, T_{m-3}) = T_{m-3} \setminus T_{m-4} \neq \varnothing, & T_{m-3} \text{ nonlimiting element of } \ddot{G}_{T_{m-3}} \\
T_{m-2} \setminus l(\ddot{G}_{T_{m-2}}, T_{m-2}) = T_{m-2} \setminus T_{m-3} \neq \varnothing, & T_{m-2} \text{ nonlimiting element of } \ddot{G}_{T_{m-2}} \\
T_{m-1} \setminus l(\ddot{G}_{T_{m-1}}, T_{m-1}) = T_{m-1} \setminus T_{m-3} \neq \varnothing, & T_{m-1} \text{ nonlimiting element of } \ddot{G}_{T_{m-1}}
\end{array}$$

Now, we determine properties of a regular element α of $B_X(Q)$ where $V(D,\alpha)=Q$ and $\alpha=\bigcup_{i=1}^{m}(Y_i^{\alpha}\times T_i)$.

Theorem 2.3 *Let* $\alpha\in B_X(Q)$ *with a quasinormal representation of the form* $\alpha=\bigcup_{i=1}^{m}(Y_i^{\alpha}\times T_i)$ *such that* $V(D,\alpha)=Q$. *Then* $\alpha\in B_X(D)$ *is a regular iff* $T_1\cap T_4=\varnothing$ *and for some complete* α *-isomorphism* $\varphi:Q\to D^{'}\subseteq D$, *the following conditions are satisfied:*

$$\begin{aligned}
&Y_1^{\alpha}\supseteq\varphi(T_1), Y_2^{\alpha}\supseteq\varphi(T_2), Y_2^{\alpha}\cup Y_4^{\alpha}\supseteq\varphi(T_4),\\
&Y_1^{\alpha}\cup Y_2^{\alpha}\cup Y_3^{\alpha}\cup Y_4^{\alpha}\cup Y_5^{\alpha}\cup Y_6^{\alpha}\supseteq\varphi(T_6),\\
&\vdots\\
&Y_1^{\alpha}\cup Y_2^{\alpha}\cup\cdots\cup Y_{m-4}^{\alpha}\supseteq\varphi(T_{m-4}),\\
&Y_1^{\alpha}\cup Y_2^{\alpha}\cup\cdots\cup Y_{m-3}^{\alpha}\supseteq\varphi(T_{m-3}),\\
&Y_1^{\alpha}\cup Y_2^{\alpha}\cup\cdots\cup Y_{m-3}^{\alpha}\cup Y_{m-2}^{\alpha}\supseteq\varphi(T_{m-2}),\\
&Y_1^{\alpha}\cup Y_2^{\alpha}\cup\cdots\cup Y_{m-3}^{\alpha}\cup Y_{m-1}^{\alpha}\supseteq\varphi(T_{m-1}),\\
&Y_4^{\alpha}\cap\varphi(T_4)\neq\varnothing, Y_6^{\alpha}\cap\varphi(T_6)\neq\varnothing, Y_{m-4}^{\alpha}\cap\varphi(T_{m-4})\neq\varnothing,\\
&Y_{m-3}^{\alpha}\cap\varphi(T_{m-3})\neq\varnothing, Y_{m-2}^{\alpha}\cap\varphi(T_{m-2})\neq\varnothing, Y_{m-1}^{\alpha}\cap\varphi(T_{m-1})\neq\varnothing,
\end{aligned}\tag{2.3}$$

Proof. Let $G=\{T_1,T_2,\ldots,T_{m-1}\}$ be a generating set of Q.

$\Rightarrow$: Since $\alpha\in B_X(D)$ is regular and $V(D,\alpha)=Q$ is $XI-$semilattice of unions, by Theorem 1.1, there exits a complete $\alpha-$isomorphism $\varphi:Q\to D^{'}$. By Theorem 1.1 (a), $\varphi(T)\alpha=T$ for all $T\in V(D,\alpha)$. Applying the Theorem 1.1 (b) and

$$\begin{aligned}
&Y_1^{\alpha}\supseteq\varphi(T_1), Y_2^{\alpha}\supseteq\varphi(T_2), Y_2^{\alpha}\cup Y_4^{\alpha}\supseteq\varphi(T_4),\\
&Y_1^{\alpha}\cup Y_2^{\alpha}\cup Y_3^{\alpha}\cup Y_4^{\alpha}\cup Y_5^{\alpha}\cup Y_6^{\alpha}\supseteq\varphi(T_6),\\
&\vdots\\
&Y_1^{\alpha}\cup Y_2^{\alpha}\cup\cdots\cup Y_{m-4}^{\alpha}\supseteq\varphi(T_{m-4}),\\
&Y_1^{\alpha}\cup Y_2^{\alpha}\cup\cdots\cup Y_{m-3}^{\alpha}\supseteq\varphi(T_{m-3}),\\
&Y_1^{\alpha}\cup Y_2^{\alpha}\cup\cdots\cup Y_{m-3}^{\alpha}\cup Y_{m-2}^{\alpha}\supseteq\varphi(T_{m-2}),\\
&Y_1^{\alpha}\cup Y_2^{\alpha}\cup\cdots\cup Y_{m-3}^{\alpha}\cup Y_{m-1}^{\alpha}\supseteq\varphi(T_{m-1})
\end{aligned}$$

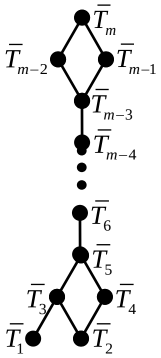
Moreover, considering that the elements $T_1,T_2,T_4,T_6,\ldots,T_{m-1}$ are nonlimiting elements of the sets $\ddot{G}_{T_1}$, $\ddot{G}_{T_2}$, $\ddot{G}_{T_4}$, $\ddot{G}_{T_6},\ldots,\ddot{G}_{T_{m-1}}$ respectively and using the Theorem 1.1 (c), following properties

$$Y_4^{\alpha}\cap\varphi(T_4)\neq\varnothing, Y_6^{\alpha}\cap\varphi(T_6)\neq\varnothing, Y_{m-4}^{\alpha}\cap\varphi(T_{m-4})\neq\varnothing,\ldots, Y_{m-1}^{\alpha}\cap\varphi(T_{m-1})\neq\varnothing$$

are obtained. Therefore there exists a complete $\alpha-$isomorphism φ which holds given conditions.

$\Leftarrow$: Since $V(D,\alpha)=Q$, $V(D,\alpha)$ is $XI-$semilattice of unions. Let $\varphi:Q\to D^{'}\subseteq D$ be complete $\alpha-$isomorphism which holds given conditions. So, considering Equation (2.3), satisfying Theorem 1.1 $(a)-(c)$. Remembering that T_3 and T_5 are limiting elements of the sets $\ddot{G}_{T_3}$ and $\ddot{G}_{T_5}$, we constitute the set $B(T_3)=\{Z\in\ddot{G}_{T_3}\mid Y_Z^{\alpha}\cap\varphi(T_3)\neq\varnothing\}$ and $B(T_5)=\{Z\in\ddot{G}_{T_5}\mid Y_Z^{\alpha}\cap\varphi(T_5)\neq\varnothing\}$ It has been proved that $\cup B(T_3)=T_3$ and $\cup B(T_5)=T_5$ in [?, Theorem 3.4]. By Theorem 1.1, we conclude that α is the regular element of the $B_X(D)$.

Now we calculate the number of regular elements α, satisfying the hyphothesis of Theorem 2.3. Let $\alpha \in B_X(D)$ be a regular element which is quasinormal representation of the form $\alpha = \bigcup_{i=1}^{m}(Y_i^\alpha \times T_i)$ and $V(D,\alpha) = Q$. Then there exist a complete $\alpha-$ isomorphism $\varphi : Q \to D' = \{\varphi(T_1), \varphi(T_2), \ldots, \varphi(T_m)\}$ satisfying the hyphothesis of Theorem 2.3. So, $\alpha \in R_\varphi(Q, D')$. We will denote $\varphi(T_i) = \overline{T}_i, i = 1,2,\ldots m.$ Diagram of the $D' = \{\overline{T}_1, \overline{T}_2, \ldots, \overline{T}_m\}$ is shown in the following figure. Then the Equation (2.3) reduced to below equation.



$$\begin{aligned}
&Y_1^\alpha \supseteq \varphi(\overline{T}_1), Y_2^\alpha \supseteq \varphi(\overline{T}_2), Y_2^\alpha \cup Y_4^\alpha \supseteq \varphi(\overline{T}_4),\\
&Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \cup Y_4^\alpha \cup Y_5^\alpha \cup Y_6^\alpha \supseteq \varphi(\overline{T}_6),\\
&\vdots\\
&Y_1^\alpha \cup Y_2^\alpha \cup \cdots \cup Y_{m-3}^\alpha \cup Y_{m-2}^\alpha \supseteq \varphi(\overline{T}_{m-2}),\\
&Y_1^\alpha \cup Y_2^\alpha \cup \cdots \cup Y_{m-3}^\alpha \cup Y_{m-1}^\alpha \supseteq \varphi(\overline{T}_{m-1}),\\
&Y_4^\alpha \cap \varphi(\overline{T}_4) \neq \varnothing, Y_6^\alpha \cap \varphi(\overline{T}_6) \neq \varnothing,\\
&Y_{m-4}^\alpha \cap \varphi(\overline{T}_{m-4}) \neq \varnothing, Y_{m-3}^\alpha \cap \varphi(\overline{T}_{m-3}) \neq \varnothing,\\
&Y_{m-2}^\alpha \cap \varphi(\overline{T}_{m-2}) \neq \varnothing, Y_{m-1}^\alpha \cap \varphi(\overline{T}_{m-1}) \neq \varnothing
\end{aligned} \tag{2.4}$$

On the other hand, $\overline{T}_1, \overline{T}_2, \overline{T}_3 \setminus \overline{T}_4, \ldots, \overline{T}_{k+1} \setminus \overline{T}_k$ $(k = 3,5,6,7,\ldots,m-5,m-2)$,…, $(\overline{T}_{m-1} \cap \overline{T}_{m-2}) \setminus \overline{T}_{m-5}$, $\overline{T}_{m-1} \setminus \overline{T}_{m-2}$, $\overline{T}_{m-2} \setminus \overline{T}_{m-1}$, $X \setminus \overline{T}_m$ are also pairwise disjoint sets and union of these sets equals X.

Lemma 2.4 *For every* $\alpha \in R_\varphi(Q, D')$, *there exists an ordered system of disjoint mappings* $\{\overline{T}_1, \overline{T}_2, \overline{T}_3 \setminus \overline{T}_4, \ldots, \overline{T}_{k+1} \setminus \overline{T}_k (k = 3,5,6,7,\ldots,m-5,m-2)$,…, $(\overline{T}_{m-1} \cap \overline{T}_{m-2}) \setminus \overline{T}_{m-5}, \overline{T}_{m-2} \setminus \overline{T}_{m-1}, X \setminus \overline{T}_m\}$.

Proof. Let $f_\alpha : X \to D$ be a mapping satisfying the condition $f_\alpha(t) = t\alpha$ for all $t \in X$. We consider the restrictions of the mapping f_α as $f_{1\alpha}$, $f_{2\alpha}$, $f_{4\alpha}, \ldots,$ $f_{k\alpha}, \ldots,$ $f_{(m-3)\alpha}$, $f_{(m-2)\alpha}$, $f_{(m-1)\alpha}$, $f_{m\alpha}$ on the sets $\overline{T}_1, \overline{T}_2, \overline{T}_3 \setminus \overline{T}_4, \ldots, \overline{T}_{k+1} \setminus \overline{T}_k$ $(k = 3,5,6,7,\ldots,m-5,m-2)$,…, $(\overline{T}_{m-1} \cap \overline{T}_{m-2}) \setminus \overline{T}_{m-5}, \overline{T}_{m-2} \setminus \overline{T}_{m-1}, X \setminus \overline{T}_m$ respectively.

Now, considering the definition of the sets Y_i^α, $(i = 1,2,\ldots,m-1)$ together with the Equation (2.4) we have,

$$\begin{aligned}
&t \in \overline{T}_1 \Rightarrow t \in Y_1^\alpha \Rightarrow f_{1\alpha}(t) = T_1, \forall t \in \overline{T}_1\\
&t \in \overline{T}_2 \Rightarrow t \in Y_2^\alpha \Rightarrow f_{2\alpha}(t) = T_2, \forall t \in \overline{T}_2\\
&t \in \overline{T}_3 \setminus \overline{T}_4 \Rightarrow t \in Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \Rightarrow f_{4\alpha}(t) = \{T_1, T_2, T_3\}, \forall t \in \overline{T}_3 \setminus \overline{T}_4\\
&t \in \overline{T}_{k+1} \setminus \overline{T}_k \quad \begin{aligned}&\Rightarrow t \in \overline{T}_{k+1} \setminus \overline{T}_k \subseteq \overline{T}_{k+1} \subseteq Y_1^\alpha \cup Y_2^\alpha \cup \cdots \cup Y_{k+1}^\alpha\\ &\Rightarrow f_{k\alpha}(t) \in \{T_1, T_2, \ldots, T_{k+1}\}, \forall t \in \overline{T}_{k+1} \setminus \overline{T}_k\end{aligned}
\end{aligned}$$

$$\Rightarrow t \in \overline{T}_{m-1} \cap \overline{T}_{m-2} \subseteq Y_1^\alpha \cup Y_2^\alpha \cup \cdots \cup Y_{m-3}^\alpha$$

$$t \in (\overline{T}_{m-1} \cap \overline{T}_{m-2}) \setminus \overline{T}_{m-5} \Rightarrow f_{(m-3)\alpha}(t) \in \{T_1, \ldots, T_{m-3}\},$$

$$\forall t \in (\overline{T}_{m-1} \cap \overline{T}_{m-2}) \setminus \overline{T}_{m-5}$$

$$t \in \overline{T}_{m-2} \setminus \overline{T}_{m-1} \quad \begin{array}{l} \Rightarrow t \in \overline{T}_{m-2} \subseteq Y_1^\alpha \cup \cdots \cup Y_{m-3}^\alpha \cup Y_{m-2}^\alpha \\ \Rightarrow f_{(m-1)\alpha}(t) \in \{T_1, \ldots, T_{m-3}, T_{m-2}\}, \forall t \in \overline{T}_{m-2} \setminus \overline{T}_{m-1} \end{array}$$

$$t \in X \setminus \overline{T}_m \Rightarrow t \in X \setminus \overline{T}_m \subseteq X = \bigcup_{i=1}^{m} Y_i^\alpha \Rightarrow f_{m\alpha}(t) \in Q, \forall t \in X \setminus \overline{T}_m$$

Besides, $Y_{k+1}^\alpha \cap \overline{T}_{k+1} \neq \varnothing$ so there is an element $t_{k+1} \in Y_{k+1}^\alpha \cap \overline{T}_{k+1}$. Then $t_{k+1}\alpha = T_{k+1}$ and $t_{k+1} \in \overline{T}_{k+1}$. If $t_{k+1} \in \overline{T}_k$ then $t_{k+1} \in \overline{T}_k \subseteq Y_1^\alpha \cup \cdots \cup Y_k^\alpha$. Thus $t_{k+1}\alpha \in \{T_1, \ldots, T_k\}$ which is in contradiction with the equality $t_{k+1}\alpha = T_{k+1}$. So, there is an element $t_{k+1} \in \overline{T}_{k+1} \setminus \overline{T}_k$ such that $f_{k\alpha}(t_{k+1}) = T_{k+1}$.

Similarly, $f_{(m-3)\alpha}(t_{m-3}) = T_{m-3}$ for some $t_{m-3} \in (\overline{T}_{m-1} \cap \overline{T}_{m-2}) \setminus \overline{T}_{m-5}$, $f_{(m-1)\alpha}(t_{m-2}) = T_{m-2}$ for some $t_{m-2} \in \overline{T}_{m-2} \setminus \overline{T}_{m-1}$. Therefore, for every $\alpha \in R_\varphi(Q, D')$ there exists an ordered system $(f_{1\alpha}, f_{2\alpha}, \ldots, f_{m\alpha})$.

On the other hand, suppose that for $\alpha, \beta \in R_\varphi(Q, D')$ which $\alpha \neq \beta$, be obtained $f_\alpha = (f_{1\alpha}, f_{2\alpha}, \ldots, f_{m\alpha})$ and $f_\beta = (f_{1\beta}, f_{2\beta}, \ldots, f_{m\beta})$. İf $f_\alpha = f_\beta$, we get

$$f_\alpha = f_\beta \Rightarrow f_\alpha(t) = f_\beta(t), \forall t \in X \Rightarrow t\alpha = t\beta, \forall t \in X \Rightarrow \alpha = \beta$$

which contradicts to $\alpha \neq \beta$. Therefore different binary relations's ordered systems are different.

Lemma 2.5 *Let Q be an $XI-$semilattice of unions and $f = (f_1, f_2, \ldots, f_m)$ be ordered system from X in the semilattice D such that*

$$\begin{array}{l}
f_1 : \overline{T}_1 \to \{T_1\}, f_1(t) = T_1, \\
f_2 : \overline{T}_2 \to \{T_2\}, f_2(t) = T_2, \\
f_4 : \overline{T}_3 \setminus \overline{T}_4 \to \{T_1, T_2, T_3\}, f_4(t) \in \{T_1, T_2, T_3\}, \\
f_k : \overline{T}_{k+1} \setminus \overline{T}_k \to \{T_1, \ldots, T_{k+1}\}, f_k(t) \in \{T_1, \ldots, T_{k+1}\} \\
\text{and } f_k(t_{k+1}) = T_{k+1} \exists t_{k+1} \in \overline{T}_{k+1} \setminus \overline{T}_k, \\
f_{m-3} : (\overline{T}_{m-1} \cap \overline{T}_{m-2}) \setminus \overline{T}_{m-5} \to \{T_1, \ldots, T_{m-3}\}, f_{m-3}(t) \in \{T_1, \ldots, T_{m-3}\} \\
\text{and } f_{m-3}(t_{m-3}) = T_{m-3} \exists t_{m-3} \in (\overline{T}_{m-1} \cap \overline{T}_{m-2}) \setminus \overline{T}_{m-5}, \\
f_{m-1} : \overline{T}_{m-2} \setminus \overline{T}_{m-1} \to \{T_1, \ldots, T_{m-3}, T_{m-2}\}, f_{m-1}(t) \in \{T_1, \ldots, T_{m-3}, T_{m-2}\} \\
\text{and } f_{m-1}(t_{m-2}) = T_{m-2} \exists t_{m-2} \in \overline{T}_{m-2} \setminus \overline{T}_{m-1}, \\
f_m : X \setminus \overline{T}_m \to Q, f_{m-1}(t) \in Q.
\end{array}$$

Then $\beta = \bigcup_{x \in X} (\{x\} \times f(x)) \in B_X(D)$ is regular and φ is complete $\beta-$ isomorphism. So $\beta \in R_\varphi(Q, D')$.

Proof. First we see that $V(D,\beta)=Q$. Considering $V(D,\beta)=\{Y\beta \mid Y\in D\}$, the properties of f mapping, $\overline{T}_i\beta=\bigcup_{x\in\overline{T}_i} x\beta$ and $D^{'}\subseteq D$, we get $V(D,\beta)=Q$.

Also, $\beta=\bigcup_{T\in V(X^*,\beta)}\left(Y_T^{\beta}\times T\right)$ is quasinormal representation of β since $\varnothing\notin Q$. From the definition of β, $f(x)=x\beta$ for all $x\in X$. It is easily seen that $V(X^*,\beta)=V(D,\beta)=Q$. We get $\beta=\bigcup_{i=1}^{m}\left(Y_i^{\beta}\times T_i\right)$.

On the other hand

$$t\in\overline{T}_1\Rightarrow t\beta=f(t)=T_1\Rightarrow t\in Y_1^{\beta}\Rightarrow\overline{T}_1\subseteq Y_1^{\beta},$$
$$t\in\overline{T}_2\Rightarrow t\beta=f(t)=T_2\Rightarrow t\in Y_2^{\beta}\Rightarrow\overline{T}_2\subseteq Y_2^{\beta},$$
$$t\in\overline{T}_4\Rightarrow t\beta=f(t)=\{T_2,T_4\}\Rightarrow t\in Y_2^{\beta}\cup Y_4^{\beta}\Rightarrow\overline{T}_4\subseteq Y_2^{\alpha}\cup Y_4^{\alpha}$$
$$t\in\overline{T}_k,(k=3,5,6,\ldots,m-5,m-2)\Rightarrow t\beta\in\{T_1,T_2,\ldots,T_k\}\begin{array}{l}\Rightarrow t\in Y_1^{\beta}\cup Y_2^{\beta}\cup\cdots\cup Y_k^{\beta}\\ \Rightarrow Y_1^{\beta}\cup Y_2^{\beta}\cup\cdots\cup Y_k^{\beta}\supseteq\overline{T_k}\end{array}$$
$$t\in\overline{T}_{m-3}\Rightarrow t\beta\in\{T_1,\ldots,T_{m-3}\}\begin{array}{l}\Rightarrow t\in Y_1^{\alpha}\cup Y_2^{\alpha}\cup\cdots\cup Y_{m-3}^{\alpha}\\ \Rightarrow Y_1^{\alpha}\cup Y_2^{\alpha}\cup\cdots\cup Y_{m-3}^{\alpha}\supseteq\overline{T}_{m-3}\end{array}$$
$$t\in\overline{T}_{m-1}\Rightarrow t\beta\in\{T_1,\ldots,T_{m-3},T_{m-1}\}\begin{array}{l}\Rightarrow t\in Y_1^{\alpha}\cup Y_2^{\alpha}\cup\cdots\cup Y_{m-3}^{\alpha}\cup Y_{m-1}^{\alpha}\\ \Rightarrow Y_1^{\alpha}\cup Y_2^{\alpha}\cup\cdots\cup Y_{m-3}^{\alpha}\cup Y_{m-1}^{\alpha}\supseteq\overline{T}_{m-1}\end{array}$$

Also, for $k=4,6$ by using $f_{k-1}(t_k)=T_k,\exists\ t\in\overline{T}_{k+1}\setminus\overline{T}_k$, we obtain $Y_k^{\beta}\cap\overline{T}_k\neq\varnothing$. Similarly, $Y_{m-4}^{\alpha}\cap\varphi(\overline{T}_{m-4})\neq\varnothing$, $Y_{m-3}^{\beta}\cap\overline{T}_{m-3}\neq\varnothing$, $Y_{m-2}^{\beta}\cap\overline{T}_{m-2}\neq\varnothing$ and $Y_{m-1}^{\beta}\cap\overline{T}_{m-1}\neq\varnothing$. Therefore the mapping $\varphi:Q\to D^{'}=\{\overline{T}_1,\overline{T}_2,\ldots,\overline{T}_m\}$ to be defined $\varphi(T_i)=\overline{T}_i$ satisfies the conditions in the Equation (2.4) for β. Hence φ is complete $\beta-$isomorphism because of $\varphi(T)\beta=\overline{T}\beta=T$, for all $T\in V(D,\beta)$. By Theorem 2.3, $\beta\in R_{\varphi}(Q,D^{'})$.

Therefore, there is one to one correspondence between the elements of $R_{\varphi}(Q,D^{'})$ and the set of ordered systems of disjoint mappings.

Theorem 2.6 *Let* X *be a finite set and* Q *be* $XI-$ *semilattice and* $m\geq 7$. *If* $D^{'}=\{\overline{T}_1,\overline{T}_2,\ldots,\overline{T}_m\}$ *is* $\alpha-$ *isomorphic to* Q *and* $\Omega(Q)=m_0$, *then*

$$\left|R(D^{'})\right|=2m_0 3^{\left|\overline{T}_3\setminus\overline{T}_4\right|}\left((k+1)^{\left|\overline{T}_{k+1}\setminus\overline{T}_k\right|}-k^{\left|\overline{T}_{k+1}\setminus\overline{T}_k\right|}\right)$$
$$\left((m-3)^{\left|(\overline{T}_{m-1}\cap\overline{T}_{m-2})\setminus\overline{T}_{m-5}\right|}-(m-4)^{\left|(\overline{T}_{m-1}\cap\overline{T}_{m-2})\setminus\overline{T}_{m-5}\right|}\right)$$
$$\left((m-2)^{\left|\overline{T}_{m-2}\setminus\overline{T}_{m-1}\right|}-(m-3)^{\left|\overline{T}_{m-2}\setminus\overline{T}_{m-1}\right|}\right)m^{\left|X\setminus\overline{T}_m\right|}$$

Proof. Lemma 2.4 and Lemma 2.5 show us that the number of the ordered system of disjoint mappings $(f_{1\alpha},f_{2\alpha},\ldots,f_{(m-1)\alpha})$ is equal to $\left|R_{\varphi}(Q,D^{'})\right|$, which $\alpha\in B_X(D)$ regular element $V(D,\alpha)=Q$ and $\varphi:Q\to D^{'}$ is a complete $\alpha-$isomorphism.

The number of the mappings $f_{1\alpha}, f_{2\alpha}, f_{3\alpha}, f_{4\alpha},\ldots, f_{(m-5)\alpha}, f_{(m-4)\alpha}, f_{(m-3)\alpha}, f_{(m-2)\alpha}$ and $f_{(m-1)\alpha}$ are respectively as

$$1,1,3^{\left|\overline{T}_3\setminus\overline{T}_4\right|},(k+1)^{\left|\overline{T}_{k+1}\setminus\overline{T}_k\right|}-k^{\left|\overline{T}_{k+1}\setminus\overline{T}_k\right|}(k=3,5,6,\ldots,m-5,m-2)$$

$$(m-3)^{\left|(\overline{T}_{m-1}\cap\overline{T}_{m-2})\setminus\overline{T}_{m-5}\right|}-(m-4)^{\left|(\overline{T}_{m-1}\cap\overline{T}_{m-2})\setminus\overline{T}_{m-5}\right|},$$
$$(m-2)^{\left|\overline{T}_{m-2}\setminus\overline{T}_{m-1}\right|}-(m-3)^{\left|\overline{T}_{m-2}\setminus\overline{T}_{m-1}\right|},m^{\left|X\setminus\overline{T}_{m}\right|}$$

The number of all automorphisms of the semilattice Q is $q=2$. Therefore by using, there is one to one correspondence between the elements of $R_{\varphi}(Q,D^{'})$ and the set of ordered systems of disjoint mappings and Theorem 1.2, then

$$\left|R(D^{'})\right|=2m_0 3^{\left|\overline{T}_3\setminus\overline{T}_4\right|}((k+1)^{\left|\overline{T}_{k+1}\setminus\overline{T}_k\right|}-k^{\left|\overline{T}_{k+1}\setminus\overline{T}_k\right|})$$
$$((m-3)^{\left|(\overline{T}_{m-1}\cap\overline{T}_{m-2})\setminus\overline{T}_{m-5}\right|}-(m-4)^{\left|(\overline{T}_{m-1}\cap\overline{T}_{m-2})\setminus\overline{T}_{m-5}\right|})$$
$$((m-1)^{\left|\overline{T}_{m-2}\setminus\overline{T}_{m-1}\right|}-(m-3)^{\left|\overline{T}_{m-2}\setminus\overline{T}_{m-1}\right|})m^{\left|X\setminus\overline{T}_{m}\right|}.$$

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