On $S_{\frac{1}{2}} \mod I$ spaces and $\theta^I$ closed sets

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Abstract

In this paper we will introduce $S_{\frac{1}{2}} \mod I$ spaces and discuss their properties. We also introduce $\theta^I$ closed sets using the local closure function and obtain the sufficient conditions for a set to be $\theta^I$ closed.

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1 Introduction and Preliminaries

The subject of ideals in topological spaces has been studied by Kuratowski³ and Vaidyanathaswamy⁵. An ideal $I$ on a topological space $(X, \tau)$ is a collection of subsets of $X$ which satisfies that (i) $A \in I$ and $B \in I$ implies $A \cup B \in I$ and (ii) $A \in I$ and $B \subseteq A$ implies $B \in I$. Given a topological space $(X, \tau)$ with an ideal $I$ on $X$ known as ideal topological space and $(\cdot)^*: \wp(X) \rightarrow \wp(X)$, called a local function³ of $A$ with respect to $I$ and $\tau$, is defined as follows: for $A \subseteq X$, $A^*(I, \tau) = \{x \in X : U \cap A \notin I \text{ for every open nhd. } U \text{ of } x \text{ in } X\}$. A Kuratowski closure operator $cl^*(\cdot)$ for a topology $\tau^*(I, \tau)$, called the *-topology, finer than $\tau$, is defined by $cl^*(A) = A \cup A^*(I, \tau)$⁴. A topological $(X, \tau)$ is said to be $S_{\frac{1}{2}}$ if for any two distinct points $x, y$ of $X$, whenever one of them has open set not containing the other then there exist open sets $U$ and $V$ such that $x \in U, y \in V$ and $\overline{U} \cap \overline{V} = \emptyset$. When there is no chance of confusion, we will simply write $A^*$ for $A^*(I, \tau)$ and $\tau^*(I, \tau)$ for $\tau^*(I, \tau)$.

Throughout this paper $(X, \tau)$ will denote topological space on which no separation axioms are assumed. If $I$ is an ideal on $X$, then $(X, \tau, I)$ is called an ideal space. For a subset $A$ of $X$, $cl(A)$ and $int(A)$ will denote the closure of $A$, interior of $A$ in $(X, \tau)$, respectively, $cl^*(A)$ and $int^*(A)$ will denote the closure of $A$, interior of $A$ in $(X, \tau^*)$, respectively, and $A^C$ will denote the complement of $A$ in $X$.

Lemma 1.1. [x] Let $(X, \tau, I)$ be an ideal space. Then for any subset $A$ of $X$ the following holds:

(a) $A^* \subset \Gamma(A)(I, \tau) \subset cl^*(A)$.
(b) $\Gamma(A)(I, \tau) = cl(\Gamma(A)(I, \tau))$.

2 Results

We begin by defining $S_{\frac{1}{2}} \mod I$ spaces.

Definition 2.1. An ideal space $(X, \tau, I)$ is said to be $S_{\frac{1}{2}} \mod I$ if for any two distinct points $x, y$ of $X$, whenever one of them has open set not containing the other then there exist open sets $U$ and $V$ such that $x \in U, y \in V$ and $\overline{U} \cap \overline{V} \in I$. 

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Since $\emptyset \in \mathcal{I}$, therefore, $S_{2 \frac{1}{2}}$ space is $S_{2 \frac{1}{2}} \mod \mathcal{I}$, but the following Example 2.1 shows that the converse need not be true.

**Example 2.1.** Let $X = \{x, y, z\}$, $\tau = \{\emptyset, \{x\}, \{y, z\}, \{x, y, z\}\}$, $\mathcal{I} = \{\emptyset, \{y\}, \{y, z\}\}$. Then $X$ is $S_{2 \frac{1}{2}}$ mod $\mathcal{I}$ but not $S_{2 \frac{1}{2}}$.

**Theorem 2.1.** If an ideal space $(X, \tau, \mathcal{I})$ is $S_{2 \frac{1}{2}}$ mod $\mathcal{I}$ and $\mathcal{I} \subset \mathcal{J}$ then $(X, \tau, \mathcal{J})$ is $S_{2 \frac{1}{2}}$ mod $\mathcal{J}$.

**Proof.** Proof is obvious and hence is omitted. $\Box$

The following Example 2.2 shows that if $(X, \tau^*)$ is $S_{2 \frac{1}{2}}$, then $X$ need not be $S_{2 \frac{1}{2}}$ mod $\mathcal{I}$.

**Example 2.2.** Let $X = \{x, y, z\}$, $\tau = \{\emptyset, \{y\}, \{y, z\}, \{x, y, z\}\}$, $\mathcal{I} = \{\emptyset, \{y\}, \{y, z\}\}$. So $\tau^* = \varphi(X)$ and hence $(X, \tau^*)$ is obviously $S_{2 \frac{1}{2}}$, but $X$ is not $S_{2 \frac{1}{2}}$ mod $\mathcal{I}$. Since $\emptyset$ has an open set not containing $\{y\}$, but $\{y\} \cap \emptyset = \emptyset$.

Even though we have seen that if $(X, \tau^*)$ is $S_{2 \frac{1}{2}}$, then $X$ need not be $S_{2 \frac{1}{2}}$ mod $\mathcal{I}$, but the following Theorem 2.2 shows that for codense ideals $(X, \tau^*)$ is $S_{2 \frac{1}{2}}$ implies $X$ is $S_{2 \frac{1}{2}}$ mod $\mathcal{I}$.

**Theorem 2.2.** Let $(X, \tau, \mathcal{I})$ be an ideal space where $\mathcal{I}$ is codense and $(X, \tau^*)$ is $S_{2 \frac{1}{2}}$ then $X$ is $S_{2 \frac{1}{2}}$ mod $\mathcal{I}$.

**Proof.** Let $x, y \in X$ be any two distinct points such that one of them has $\tau$-open and hence $\tau^*$-open subset not containing the other. Then $(X, \tau^*)$ is $S_{2 \frac{1}{2}}$ implies there exist basic open set $G - I, H - J$ where $G, H$ are open in $x$ and $I, J \in \mathcal{I}$ such that $x \notin G - I, y \notin H - J$ and $cl^I(G - I) \cap cl^J(H - J) = \emptyset$ and so by $cl^I(G - I) \cap cl^J(H - J) = \emptyset$. This implies that $cl^I(G) \cap cl^J(H) = \emptyset$. Therefore, $(cl^I(G) \cap cl^J(H)) \cap (I \cup J) = \emptyset$. Now $\mathcal{I}$ is codense implies that $cl^I(G) = cl^J(G)$ for every open subset $G$ of $X$. Hence $cl^I(G) \cap cl^J(H)$ implies that $X$ is $S_{2 \frac{1}{2}}$ mod $\mathcal{I}$. $\Box$

**Definition 2.2.** An ideal space $(X, \tau, \mathcal{I})$ is said to be $S_{2 \frac{1}{2}}$ mod $\mathcal{I}$ if for any two distinct points $x, y$ of $X$, whenever one of them has $\tau$-open set not containing the other then exist open sets $U$ and $V$ such that $x \in U$, $y \in V$ and $\overline{U} \cap \overline{V} \in \mathcal{I}$.

It can be seen easily that $(X, \tau, \mathcal{I})$ is $S_{2 \frac{1}{2}}$ mod $\mathcal{I}$ implies $S_{2 \frac{1}{2}}$ mod $\mathcal{I}$ but the following Example 2.3 shows that the converse is not true.

**Example 2.3.** Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X\}$, $\mathcal{I} = \{\emptyset, \{a\}\}$. So $\tau^* = \varphi(X)$ and hence $(X, \tau, \mathcal{I})$ is obviously $S_{2 \frac{1}{2}}$ mod $\mathcal{I}$, but $\{a\}$ has a $\tau$-open set not containing $\{a\}$ and $X$ is the only open subset containing $\{a\}$ and $\emptyset$ implies that $(X, \tau, \mathcal{I})$ is not $S_{2 \frac{1}{2}}$ mod $\mathcal{I}$.

In [2], Gupta and Noiri introduced QHC spaces with respect to an ideal written $I$-QHC(An Ideal space $(X, \tau, \mathcal{I})$ is said to be $I$-QHC if for every open cover $\{G_\alpha : \alpha \in \Delta\}$ of $X$, there exists a finite subset $\Delta_0$ of $\Delta$ such that $X - \bigcup\{cl(G_\alpha) : \alpha \in \Delta_0\} \in \mathcal{I}$). We now discussed some properties of $I$-QHC spaces.

**Theorem 2.3.** Let $(X, \tau, \mathcal{I})$ be $S_{2 \frac{1}{2}}$ mod $\mathcal{I}$ space and $F$ be $I$-QHC subset of $X$ such that $x \notin \overline{F}$ then there exist open subsets $U$ and $V$ such that $x \in U$, $\overline{U} \notin \mathcal{I}$ and $x \notin \overline{V} \notin \mathcal{I}$.

**Proof.** Let $F$ be any $I$-QHC subset of $X$ and $x \in X$ be any element such that $x \notin \overline{F}$ then $x \in X - \overline{F}$. Therefore, for all $y \in F$, $x$ has an open set $x \notin \overline{F}$ not containing the elements of $F$ and so $X$ is $S_{2 \frac{1}{2}}$ mod $\mathcal{I}$ implies that there exist open subsets $U, V$ containing $x, y$ respectively such that $\overline{U} \cap \overline{V} \in \mathcal{I}$ and $\overline{U} \notin \mathcal{I}$ and $\overline{V} \notin \mathcal{I}$. Further, $F$ is $I$-QHC subset of $X$ implies that there exist finite subset $F_0$ of $F$ such that $F \setminus \bigcup_{y \in F_0} \overline{V_0} \in \mathcal{I}$ and so $F \setminus \bigcup_{y \in F_0} \overline{V_0} \in \mathcal{I}$. Consider $U = \bigcup_{y \in F_0} U_y$ and $V = \bigcup_{y \in F_0} V_y$ then $U$ is the open subset containing $x$ and $\overline{U} \notin \mathcal{I}$ and $\overline{V} \notin \mathcal{I}$. $\Box$

**Theorem 2.4.** Let $(X, \tau, \mathcal{I})$ be an ideal space and $K$ be $I$-QHC subset of $X$ then $cl^I(K)$ is also $I$-QHC.

**Proof.** Let $G_\alpha$ be open cover of $cl^I(K)$ so that $cl^I(K) \subseteq \bigcup_\alpha G_\alpha$ and so $K \subseteq cl^I(K) \subseteq \bigcup_\alpha G_\alpha$. But $K$ is $I$-QHC subset of $X$ implies that $K - \bigcup_{\alpha=1}^\beta G_\alpha \in \mathcal{I}$. Let $G = \bigcup_{\alpha=1}^\beta G_\alpha$, so that $K \setminus \overline{G} \in \mathcal{I}$. Now we will prove that $cl^I(K) \setminus \overline{G} \in \mathcal{I}$. For this we will prove that $cl^I(K) \setminus \overline{G} \subseteq K \setminus \overline{G}$.

Let $x \notin K \setminus \overline{G}$. Then there can be two possibilities: case(i) $x \notin K$ case(ii) $x \notin \overline{G}$. Now if $x \in \overline{G}$ then obviously $x \notin K$ and $x \notin \overline{G}$. Then $x \in (\overline{G})^C$. This implies that $\overline{G}$ is open set containing $x$ and $(\overline{G})^C \cap K \in \mathcal{I}$ implies that $x \notin K^*$. Hence, $x \notin K \cup K^*$ and so $x \notin \overline{K}$. Therefore, $x \notin cl^I(K) \setminus \overline{G}$. Hence $cl^I(K) \setminus \overline{G} \subseteq K \setminus \overline{G} \in \mathcal{I}$ and so $cl^I(K) \setminus \overline{G} \in \mathcal{I}$ implies that $cl^I(K)$ is $I$-QHC. $\Box$
In [1], Al-Omari and Noiri defined the local closure function in ideal topological spaces (where in an ideal topological space $(X, τ, I)$ for a subset $A$ of $X$, the local closure function of $A$ denoted by $Γ(A)(I, τ)$ is defined as $Γ(A)(I, τ) = \{x \in X : U \cap A \notin I \text{ for every } τ\text{-nhd. } U \text{ of } x \in X \}$). Before our further results firstly, we will define $θ^I$ closed sets using the local closure function.

**Definition 2.3.** Let $(X, τ, I)$ be an ideal space and $A$ be any subset of $X$. Then $A$ is said to be $θ^I$ closed if $Γ(A)(I, τ) ⊆ A$.

**Theorem 2.5.** Let $(X, τ, I)$ be *$S_{2\frac{1}{2}}$* mod $I$ space and $K$ be any $I$-QHC subset of $X$. Then $K$ is $θ^I$ closed if and only if $K$ or $K^C$ is union of $τ^*$-closed subsets of $X$.

**Proof.** Firstly, let $K$ is $θ^I$ closed and so $τ^*$-closed. This implies that $K$ is union of $τ^*$-closed sets. Conversely, let $K = \bigcup_{a \in F_a}$, where $F_a$ are $τ^*$-closed subsets of $X$. Then we will prove that $Γ(K)(I, τ) ⊆ K$. Let $x \in Γ(K)(I, τ)$ be any element then for every open subset $G$ containing $x$, $G \cap K \notin I$. Consider the filter $F$ generated by the filterbase $F(I) = \{G \cap A : G \text{ open subset of } X \text{ containing } x \}$. Then it can be easily seen that $F$ is the filter containing the closure of every open subset containing $x$ and $F \cap I = \emptyset$. Further, $K$ is $I$-QHC subset of $X$ implies that there exists $y \in K$ such that $y \in \bigcap_{F \in F(I)} Γ(F)(I, τ)$. Therefore, there exists $α$ such that $y \in F_a$. Now, let $x \notin K$, then $x \notin F_a$. So $x \in F_a^C$. This implies that $F_a^C$ is $τ^*$-open nhd of $x$ containing $x$ but not $y$. Therefore, $x$ is *$S_{2\frac{1}{2}}$* mod $I$ implies that there exist open sets $U$ and $V$ of $X$ containing $x$ and $y$ respectively such that $U \cap V \in I$ and so $y \notin Γ(U)(I, τ)$. Also $U$ is open subset of $X$ containing $x$ implies that $U \in F$. Therefore, $y \in Γ(U)(I, τ)$ which means that $U \cap V \notin I$, which is a contradiction. Therefore, $x \in K$ and so $Γ(K)(I, τ) \subseteq K$. Hence $K$ is $θ^I$ closed. □

The following Examples show that we can not replace *$S_{2\frac{1}{2}}$* mod $I$ space by $S_{2\frac{1}{2}}$ mod $I$ space or by $(X, τ^*)$ is $S_{2\frac{1}{2}}$.

**Example 2.4.** Let $X$ be any infinite set with indiscrete topology and $I = I_f$ = ideal of finite subsets of $X$. Then $τ^* = \{G \subseteq X | X - G \text{ is finite} \}$ i.e. $τ^*$ is cofinite topology. Now, it can be easily seen that $X$ is $S_{2\frac{1}{2}}$ mod $I$ space since no point of $X$ has a neighbourhood not containing the other. Further, $X$ is the only open subset of $X$ so every subset of $X$ is $I$-QHC. Let $K$ = any infinite subset of $X$ so $K = \bigcup_{x \in K} \{x\}$ where each $x \in K$ is $τ^*$-closed i.e. $K$ is union of $τ^*$-closed subsets of $X$. But $K$ is not $θ^I$ closed. Since $X$ is the only open subset of $X$ and $X \cap K = K \cap K = K \notin I$. Therefore, $Γ(K)(I, τ) = X$ and so $Γ(K)(I, τ) \notin K$. Hence $K$ is not $θ^I$ closed.

**Example 2.5.** Let $X = \{a, b, c\}$ with $τ = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $I = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. So $τ^* = τ(I)(X)$. So it can be easily seen that $(X, τ^*)$ is $S_{2\frac{1}{2}}$ but $X$ is not *$S_{2\frac{1}{2}}$* mod $I$. Since $a$ has $τ^*$-open subset $\{a\}$ not containing $b$ but $\overline{a} \cap \overline{b} = \{a, c\} \cap \{b, c\} = \{c\} \notin I$. Now, $\{c\}$ is $τ^*$-closed but $Γ(\{c\})(I, τ) = \{a, b, c\}$ and so $Γ(\{c\})(I, τ) \notin \{c\}$. Hence $\{c\}$ is not $θ^I$ closed.

Even though we cannot replace *$S_{2\frac{1}{2}}$* mod $I$ space by $S_{2\frac{1}{2}}$ mod $I$ space. But the following Theorem 2.6 holds.

**Theorem 2.6.** Let $(X, τ, I)$ be $S_{2\frac{1}{2}}$ mod $I$ space and $K$ be any $I$-QHC subset of $X$. Then $K$ or $K^C$ is union of closed subsets of $X$ implies that $K$ is $θ^I$ closed.

**Proof.** Proof is similar to Theorem 2.5 and hence is omitted. □

**References**


