k-Harmonic Mean Labeling of Some Graphs

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Abstract - Somasundaram and Ponraj [12] have introduced the notion of mean labeling of graphs. R. Ponraj and D. Ramya introduced Super mean labeling of graphs in [5]. S. Somasundaram and S.S. Sandhya introduced the concept of Harmonic mean labeling in [6] and studied their behavior in [7,8,9]. In this paper, we introduce the concept of k-harmonic mean labeling and we investigate k-harmonic mean labeling of some graphs.

Key words - Harmonic mean labeling, harmonic mean graph, k-harmonic mean labeling, k-harmonic mean graph.

1. INTRODUCTION

By a graph $G = (V(G), E(G))$ with $p$ vertices and $q$ edges we mean a simple, connected and undirected graph. In this paper a brief summary of definitions and other information is given in order to maintain compactness. The term not defined here are used in the sense of Harary [3].

A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. A useful survey on graph labeling by J.A. Gallian (2016) can be found in [1]. If the domain of the mapping is the set of vertices (or edges) then the labeling is called a vertex labeling (or an edge labeling).

Somasundaram and Ponraj [12] have introduced the notion of mean labeling of graphs. R. Ponraj and D. Ramya introduced Super mean labeling of graphs in [5]. S. Somasundaram and S.S. Sandhya introduced the concept of Harmonic mean labeling in [6] and studied their behavior in [7,8,9]. In this paper, we introduce the concept of k-harmonic mean labeling and we investigate k-harmonic mean labeling of some graphs.

Definition 1.1

Let $G$ be a $(p, q)$ graph. A function $f$ is called a harmonic mean labeling of a graph $G$ if $f : V(G) \rightarrow \{1, 2, 3, ..., q + 1\}$ is injection and the induced edge function $f^* : E(G) \rightarrow \{1, 2, ..., q\}$ defined as

$$f^*(e = uv) = \frac{2f(u)f(v)}{f(u) + f(v)}$$

or

$$f^*(e = uv) = \frac{2f(u)f(v)}{f(u) + f(v)}$$

is bijective. The graph which admits harmonic mean labeling is called harmonic mean graph.

Definition 1.2

Let $G$ be a $(p, q)$ graph. A function $f$ is called a k-harmonic mean labeling of a graph $G$ if $f : V(G) \rightarrow \{k, k+1, k+2, ..., k + q\}$ is injection and the induced edge function $f^* : E(G) \rightarrow \{k, k+1, k+2, ..., k+q - 1\}$ defined as

$$f^*(e = uv) = \frac{2f(u)f(v)}{f(u) + f(v)}$$

or

$$f^*(e = uv) = \frac{2f(u)f(v)}{f(u) + f(v)}$$

is bijective. The graph which admits k-harmonic mean labeling is called k-harmonic mean graph.

Definition 1.3

The product $P_2 \times P_n$ is called a Ladder and it is denoted by $L_n$.

Definition 1.4

A Twig graph is a tree obtained from a path by attaching exactly two pendent edges to each internal vertex of the path.

Definition 1.5

A Triangular ladder $TL_n$, $n \geq 2$ is a graph obtained from a ladder $L_n$ by adding the edges $u_i v_{i+1}$ for $1 \leq i \leq \frac{n-1}{2}$, where $u_i$ and $v_i$, $1 \leq i \leq n$, are the vertices of $L_n$ such that $u_1, u_2, ..., u_n$ and $v_1, v_2, ..., v_k$ are two paths of length $n$ in $L_n$. 

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Definition 1.6
If G has order n, the Corona of G with H, G \( \cup \) H is the graph obtained by taking one copy of G and n copies of H and joining the \( i \)-th vertex of G with an edge to every vertex in the \( i \)-th copy of H.

Definition 1.7
Let \( P_n \) be a path on \( n \) vertices denoted by \( (1, 1), (1, 2), ... (1, n) \) and with \( n - 1 \) edges denoted by \( e_1, e_2, ..., e_{n-1} \) where \( e_i \) is the edge joining the vertices \( (1, i) \) and \( (1, i+1) \). On each edge \( e_i \), \( 1 \leq i \leq n - 1 \), we erect a ladder with \( n - (i - 1) \) steps including the edge \( e_i \). The graph obtained is called a step ladder graph and denoted by \( S(T_n) \) where \( n \) denotes the number of vertices in the base.

2. MAIN RESULTS

Theorem 2.1
The path \( P_n \) is a k-harmonic mean graph for all \( k \) and \( n \geq 2 \).

Proof
Let \( V(P_n) = \{ v_i \mid 1 \leq i \leq n \} \) and 
\[ E(P_n) = \{ e_i = (v_i, v_{i+1}) \mid 1 \leq i \leq n - 1 \} \]

Define a function \( f: V(P_n) \rightarrow \{ k, k+1, k+2, ..., k+q \} \) by 
\[ f(v_i) = k + i - 1 \quad 1 \leq i \leq n \]

Then the induced edge labels are 
\[ f^*(e_i) = k + i - 1 \quad 1 \leq i \leq n - 1 \]

The above defined function \( f \) provides k-harmonic mean labeling of the graph.

Hence \( P_n \) is a k-harmonic mean graph.

Example 2.1

The Twig graph \( T_n \) is k-harmonic mean graph for all \( k \) and \( n \geq 3 \).

Proof
Let \( V(T_n) = \{ v_i \mid 0 \leq i \leq n - 1, u_i, w_i \mid 1 \leq i \leq n - 2 \} \) and 
\[ E(T_n) = \{ v_i u_i, v_i w_i \mid 1 \leq i \leq n - 2, v_i v_{i+1}; 0 \leq i \leq n-2 \} \]

The ordinary labeling is

First we label the vertices as follows

Define a function \( f: V(T_n) \rightarrow \{ k, k+2, k+2, ..., k+q \} \) by
\[ f(v_0) = k \]
\[ f(v_i) = k + 3i - 2, \quad \text{for} \ 1 \leq i \leq n - 1 \]
\[ f(w_i) = k + 3i - 1, \quad \text{for} \ 1 \leq i \leq n - 2 \]
\[ f(u_i) = k + 3i, \quad \text{for} \ 1 \leq i \leq n - 2 \]

Then the induced edge labels are
\[ f*(v_i w_{i+1}) = k + 3i, \quad \text{for} \ 0 \leq i \leq n - 2 \]
\[ f*(v_i u_i) = k + 3i - 1, \quad \text{for} \ 1 \leq i \leq n - 2 \]
\[ f*(v_i w_i) = k + 3i - 2, \quad \text{for} \ 1 \leq i \leq n - 2 \]
The above defined function $f$ provides $k$-harmonic mean labeling of the graph. Hence $T_n$ is a $k$-harmonic mean graph.

**Example 2.2**

The above defined function $f$ provides $k$-harmonic mean labeling of the graph. Hence $T_n$ is a $k$-harmonic mean graph.

**Example 2.3**

The Triangular ladder $T_L n$ is a $k$-harmonic mean graph for all $k$ and $n \geq 2$.

**Theorem 2.4**

Let $V(L_n \circ k) = \{u_i, v_i, w_i, x_i; 1 \leq i \leq n \}$ and $E(L_n \circ k) = \{u_i w_i, v_i x_i, u_i v_i + 1; 1 \leq i \leq n - 1, u_i v_i + 1; 1 \leq i \leq n \}$

The ordinary labeling is

First we label the vertices as follows

Define a function $f : V(TL_n) \rightarrow \{k, k + 1, k + 2, \ldots, k + q\}$ by

- $f(u_i) = k + 4i – 3$, for $1 \leq i \leq n$
- $f(v_i) = k$
- $f(w_i) = k + 4i – 5$, for $2 \leq i \leq n$.

Then the induced edge labels are

- $f^*(u_i w_i) = k + 4i – 1$, for $1 \leq i \leq n - 1$
- $f^*(v_i v_i + 1) = k + 4i – 3$, for $1 \leq i \leq n - 1$
- $f^*(u_i v_i) = k + 4i – 4$, for $1 \leq i \leq n$
- $f^*(u_i v_i + 1) = k + 4i – 2$, for $1 \leq i \leq n - 1$

The above defined function $f$ provides $k$-harmonic mean labeling of the graph. Hence $T_L n$ is a $k$-harmonic mean graph.

**Example 2.3**

The above defined function $f$ provides $k$-harmonic mean labeling of the graph. Hence $T_L n$ is a $k$-harmonic mean graph.
$E(L_n \circ k_1) = \{u_i v_i, u_i w_i, v_i x_i; 1 \leq i \leq n, \ u_{i+1}, v_{i+1}, 1 \leq i \leq n-1\}$

The ordinary labeling is

First we label the vertices as follows:

Define a function $f : V(L_n \circ k_1) \rightarrow \{k, k+1, k+2, \ldots, k+q\}$ by

- $f(u_i) = k + 5i - 3$, for $1 \leq i \leq n$
- $f(v_i) = k + 5i - 4$, for $1 \leq i \leq n$
- $f(w_i) = k + 5i - 2$, for $1 \leq i \leq n$
- $f(x_i) = k + 5i - 5$, for $1 \leq i \leq n$

Then the induced edge labels are

- $f^*(u_i u_{i+1}) = k + 5i - 1$, for $1 \leq i \leq n - 1$
- $f^*(v_i v_{i+1}) = k + 5i - 2$, for $1 \leq i \leq n - 1$
- $f^*(u_i v_i) = k + 5i - 4$, for $1 \leq i \leq n$
- $f^*(u_i w_i) = k + 5i - 3$, for $1 \leq i \leq n$
- $f^*(v_i x_i) = k + 5i$, for $1 \leq i \leq n$

The above defined function $f$ provides $k$-harmonic mean labeling of the graph. Hence $L_n \circ k_1$ is a $k$-harmonic mean graph.

**Example 2.4**

![Diagram of 800-Harmonic Mean Labeling of $L_7 \circ k_1$]

**Theorem 2.5**

A graph obtained by attaching a triangle at each pendent vertex of a comb is $k$-harmonic mean graph for all $k$.

**Proof**

Let $G$ be a graph obtained by attaching a triangle $K_3$ at each pendent vertex of $P_n \circ k_1$. 

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Let \( u_i, v_i \) be the vertices of the comb \( P_n O k_1 \) in which \( v_i \) is joined with the vertex \( u_i \) of \( P_n \) and let \( x_i, y_i, z_i \) be the vertices of \( k^i \) copy of \( K_3 \). Identify \( z_i \) with \( v_i \), \( 1 \leq i \leq n \). The resultant graph is \( G \) whose edge set is

\[
E = \{u_iu_{i+1} ; 1 \leq i \leq n-1\} \cup \{u_iv_i, v_ix_i, v_iy_i ; 1 \leq i \leq n\}
\]

The ordinary labeling is

\[
define \text{a function } f: V(G) \to \{k, k+1, k+2, \ldots, k+q\} \text{ by} \\
f(u_i) = k + 5i - 3, \text{ for } 1 \leq i \leq n \\
f(v_i) = k + 5i - 2, \text{ for } 1 \leq i \leq n \\
f(x_i) = k + 5i - 5, \text{ for } 1 \leq i \leq n \\
f(y_i) = k + 5i - 4, \text{ for } 1 \leq i \leq n
\]

Then the induced edge labels are

\[
define \text{induced edge labels are} \\
f^*(u_iu_{i+1}) = k + 5i - 1, \text{ for } 1 \leq i \leq n-1 \\
f^*(u_iv_i) = k + 5i - 2, \text{ for } 1 \leq i \leq n \\
f^*(v_ix_i) = k + 5i - 4, \text{ for } 1 \leq i \leq n. \\
f^*(v_iy_i) = k + 5i - 3, \text{ for } 1 \leq i \leq n. \\
f^*(x_iy_i) = k + 5i - 5, \text{ for } 1 \leq i \leq n.
\]

The above defined function \( f \) provides \( k \)-harmonic mean labeling of the graph.

Hence the graph \( G \) is \( k \)-harmonic mean graph.

**Example 2.5:**

**Theorem 2.6:**

The Step ladder \( S(T_n) \) is \( k \)-harmonic mean graph for all \( k \).

**Proof:**

\[V(S(T_n)) = \{(1, 1), (1, 2), \ldots, (1, n), (2, 1), (2, 2), \ldots, (2, n), (3, 1), (3, 2), \ldots, (3, n-1), \ldots, (n, 1), (n, 2)\}

The ordinary labeling is
First we label vertices as follows
Define a function \( f: V(S(T_n)) \to \{k, k + 1, k + 2, \ldots, k + q\} \) by
\[
\begin{align*}
    f(i, 1) &= k + n^2 + i - 2, & \text{for } 1 \leq i \leq n \\
    f(1, j) &= k + (n - j + 1)^2 - 1, & \text{for } 2 \leq i \leq n \\
    f(i, j) &= k + (n - j + 1)^2 + i - 2, & \text{for } 2 \leq i \leq n, 2 \leq j \leq n - j + 2
\end{align*}
\]

Then the induced edge labels are
\[
\begin{align*}
    f^*((i, 1), (i + 1, 1)) &= k + n^2 + i - 2, & \text{for } 1 \leq i \leq n - 1 \\
    f^*((1, j), (1, j + 1)) &= k + (n - j) (n - j + 1) + i - 1, & \text{for } 1 \leq j \leq n - 1 \\
    f^*((i, j), (i, j + 1)) &= k + (n - j) (n - j + 1) + i - 2, & \text{for } 2 \leq i \leq n, 1 \leq j \leq n - j + 1 \\
    f^*((i, j), (i + 1, j)) &= k + (n - j + 1)^2 + i - 2, & \text{for } 2 \leq j \leq n, 1 \leq i \leq n - j + 1
\end{align*}
\]

The above defined function \( f \) provides \( k \)-harmonic mean labeling of the graph.
Hence \( S(T_n) \) is \( k \)-harmonic mean graph.

**Example 2.6**

\[100 \text{- harmonic mean labeling of } S(T_8)\]
REFERENCES