Abstract: In this paper, we define the total degree of a vertex in the Cartesian product of two interval-valued fuzzy graphs (IVFG) and investigate the total regularity of the Cartesian product. In general, the Cartesian product of two totally regular interval-valued fuzzy graphs need not be a totally regular interval-valued fuzzy graph (TRIVFG). The necessary and sufficient conditions for the Cartesian product of two TRIVFGs to be totally regular under some restrictions are obtained.

Keywords: Interval-valued fuzzy graph, Cartesian Product, Total regularity.

I. INTRODUCTION

Graph theory has so many applications in almost all real world problems. But since the world is full of uncertainty, fuzzy graph has a separate importance in many real life applications. Fuzzy graph theory is finding an increasing number of applications in modeling real time systems where the level of information inherent in the system varies with different levels of precision. In the applied field, the success of the use of fuzzy set theory depends on the choice of the membership function that we make. However, there are applications in which experts do not have precise knowledge of the function that should be taken. In these cases, it is appropriate to represent the membership degree of each element of the fuzzy set by means of an interval. From these considerations arises the extension of fuzzy sets called the theory of interval-valued fuzzy sets, that is fuzzy sets such that the membership degree of each element of the fuzzy set is given by a closed subinterval of the interval [0,1]. Replacing the membership functions of vertices and edges in fuzzy graphs by interval valued fuzzy sets such that they satisfy some particular condition, interval valued fuzzy graphs were defined. Thus IVFG provide a more description of vagueness and uncertainty within the specific interval than the traditional fuzzy graph. The basic concepts of fuzzy sets and interval valued fuzzy sets can be found in [43] and [44].

The first definition of fuzzy graph was by Kaufmann [14] in 1973. But it was Azriel Rosenfeld [34] who considered fuzzy relations on fuzzy sets and developed the theory of fuzzy graphs as a generalization of Euler graph theory in 1975. The works of Bhattacharya [7], Bhutani [8], Bhutani and Battou [9], Bhutani and Rosenfeld [10]–[12], Mordeson [15], Mordeson and Nair [16],[17], Mordeson and Peng [18], Sunitha and Vijayakumar [37]–[40], Nagoor Gani and Basheer Ahmed [19], Nagoor Gani and Malavizhi [20], Nagoor Gani and Radha [21],[22] form the foundation of all researches in fuzzy graph theory.

In 2009, Hongmei and Lianhua [13] introduced IVFG as an extension of fuzzy graphs. Since then, a lot of research work is being done in this area. Muhammad Akram and Wieslaw A. Dudek [3] defined the operations of Cartesian product, composition, union and join of IVFGs and investigated some properties. They also introduced the notion of interval valued fuzzy complete graphs and presented some properties of self complementary and self weak complementary interval valued fuzzy complete graphs. Now IVFG is growing fast and has wide applications in many fields. The various works done in this area can be seen in [1]–[6],[25]–[33],[35]–[36] and [41]–[42]. H. Rashmanlou and Madhumangal Pal [25] defined regular and totally regular IVFGs. Total regularity of the join of two IVFGs was discussed by the author in [35]. The author also studied about regular and edge regular IVFGs [36]. Totally regular property of the Cartesian Product of two intuitionistic fuzzy graphs were studied by A. Nagoor Gani and H. Sheik Mujibur Rahman [23],[24].

In this paper, we introduce and analyse the notion of total degree of a vertex in the Cartesian Product of two IVFGs. Also we obtain the necessary and sufficient conditions for the Cartesian product of two TRIVFG to be totally regular under some restrictions.

II. BASIC CONCEPTS

Graph theoretic terms and results used in this work are either standard or are explained as and when they first appear. We consider only simple graphs. That is, graphs with multiple edges and loops are not considered.

Definition 2.1 [34].

Let $V$ be a non empty set. A fuzzy graph is a pair of functions $G: (\sigma, \mu)$ where $\sigma$ is a fuzzy subset of $V$ and $\mu$ is a symmetric fuzzy relation on $\sigma$. That is, $\sigma: V \rightarrow [0,1]$ and $\mu: V \times V \rightarrow [0,1]$ such that $\mu(u, v) \leq \sigma(u) \land \sigma(v)$ for all $u, v \in V$ where $\sigma(u) \land \sigma(v)$ denotes minimum of $\sigma(u)$ and $\sigma(v)$.
Definition 2.2[3].
An interval number \( D \) is an interval \([a^-, a^+]\) with \( 0 \leq a^- \leq a^+ \leq 1 \).

Remark 2.1.
(i) The interval number \([a, a]\) is identified with the number \( a \in [0,1] \).
(ii) \( D[0,1] \) denotes the set of all interval numbers.

Definition 2.3[3].
For interval numbers \( D_1 = [a_1^-, b_1^+] \) and \( D_2 = [a_2^-, b_2^+] \)
- \( \min(D_1, D_2) = [\min(a_1, a_2), \min(b_1, b_2)] \)
- \( \max(D_1, D_2) = [\max(a_1, a_2), \max(b_1, b_2)] \)
- \( D_1 + D_2 = [a_1 + a_2 - a_1 + a_2, b_1 + b_2 - b_1 + b_2] \)
- \( D_1 \leq D_2 \iff a_1 \leq a_2 \) and \( b_1 \leq b_2 \)
- \( D_1 = D_2 \iff a_1 = a_2 \) and \( b_1 = b_2 \)
- \( \lambda D_1 = \lambda [a_1, b_1] = [\lambda a_1, \lambda b_1] \) where \( 0 \leq \lambda \leq 1 \).

Then \( (D[0,1]) \) is a complete lattice with \([0,0]\) as the least element and \([1,1]\) as the greatest. Here \( V \) denotes maximum and \( A \) denotes minimum.

Definition 2.4[3].
The interval – valued fuzzy set (IVFS) \( A \) in \( V \) is defined by \( A = \{x, [\mu_A(x), \mu_B(x)] : x \in V\} \) where \( \mu_A(x) \) and \( \mu_B(x) \) are fuzzy subsets of \( V \) such that \( \mu_A(x) \leq \mu_B(x) \) for all \( x \in V \). We shall sometimes denote the IVFS \( A \) by \([\mu_A(x), \mu_B(x)]\).

For any two IVFSs \( A = [\mu_A(x), \mu_B(x)] \) and \( B = [\mu_A(x), \mu_B(x)] \) in \( V \), we define
- \( A \cup B = \left\{ x, \max\left(\mu_A(x), \mu_B(x), \mu_B(x), \mu_B(x)\right) \right\} : x \in V \)
- \( A \cap B = \left\{ x, \min\left(\mu_A(x), \mu_B(x), \mu_B(x), \mu_B(x)\right) \right\} : x \in V \)

Definition 2.5[3].
If \( G' = (V, E) \) is a graph, then by an interval – valued fuzzy relation (IVFR) \( B \) on the set \( E \) we mean an IVFS such that \( \mu_B(xy) \leq \min(\mu_A(x), \mu_B(y)) \) and \( \mu_B(xy) \leq \min(\mu_A(x), \mu_B(y)) \) for all \( xy \in E \).

Definition 2.6[3].
By an interval – valued fuzzy graph (IVFG) of a graph \( G = (V, E) \), we mean a pair \( G = (A, B) \), where \( A = [\mu_A, \mu_B] \) is an IVFS on \( V \) and \( B = [\mu_B, \mu_B] \) is an IVFR on \( E \).

Definition 2.7 [25].
The negative degree of a vertex \( u \in V \) is defined by \( d^{-}(u) = \sum_{uv \in E} \mu_A(u) \). Similarly, positive degree of a vertex \( u \in V \) is defined by \( d^{+}(u) = \sum_{uv \in E} \mu_B(u) \). Then the degree of the vertex \( u \in V \) is defined as \( d(u) = [d^{-}(u), d^{+}(u)] \).

Definition 2.8 [25].
If \( d^{-}(u) = k_1 \), \( d^{+}(u) = k_2 \) for all \( u \in V \) where \( k_1, k_2 \) are real numbers, then the graph \( G \) is called \([k_1, k_2]\) – regular interval – valued fuzzy graph (RIVFG) or regular interval – valued fuzzy graph of degree \([k_1, k_2]\).

Definition 2.9 [25].
The total degree of the vertex \( u \in V \) is defined as \( td(u) = [td^{-}(u), td^{+}(u)] \) where,
\[
\begin{align*}
\text{if } \mu_B(xy) &\leq \mu_A(x), \\
\mu_B(xy) &\leq \mu_A(y), \\
\mu_B(xy) &\leq \min(\mu_A(x), \mu_B(y)),
\end{align*}
\]

Definition 2.10 [25].
Let \( G = (A, B) \) be an IVFG. If each vertex of \( G \) has the same total degree \([k_1, k_2]\) then the graph \( G \) is called \([k_1, k_2]\) – totally regular interval valued fuzzy graph (TRIVFG) or TRIVFG of degree \([k_1, k_2]\).

Definition 2.11 [36].
If the underlying graph \( G' \) is regular, then \( G \) is said to be a partially regular interval – valued fuzzy graph (PRIVFG).

Definition 2.12 [3].
Let \( G_1 = (A_1, B_1) \) and \( G_2 = (A_2, B_2) \) be two IVFGs with \( G_1' = (V_1, E_1) \) and \( G_2' = (V_2, E_2) \). Then the Cartesian product \( G_1 \times G_2 \) of \( G_1 \) and \( G_2 \) is a pair of functions \((A_1 \times A_2, B_1 \times B_2)\) with underlying vertex set \( V_1 \times V_2 = \{(u_1, v_1) : u_1 \in V_1 \text{ and } v_1 \in V_2\} \) and underlying edge set \( E_1 \times E_2 = \{(u_1, v_1)(u_2, v_2) : u_1 = u_2, v_1 = v_2 \in E_2 \text{ or } u_1, u_2 \in E_1, v_1, v_2 \in V_2\} \) such that
\[
\begin{align*}
\mu_{A_1}(u_1, v_1) &= \min\left(\mu_{A_1}(u_1), \mu_{A_1}(v_1)\right), \\
\mu_{A_2}(u_1, v_1) &= \min\left(\mu_{A_2}(u_1), \mu_{A_2}(v_1)\right), \\
\mu_{B_1}(u_1, v_1) &= \min\left(\mu_{B_1}(u_1), \mu_{B_1}(v_1)\right), \\
\mu_{B_2}(u_1, v_1) &= \min\left(\mu_{B_2}(u_1), \mu_{B_2}(v_1)\right).
\end{align*}
\]
Remark 2.1.
Clearly $G_1 \times G_2$ is an IVFG.

III. TOTAL DEGREE OF A VERTEX IN CARTESIAN PRODUCT

By definition, for any $(u_1, v_1) \in V_1 \times V_2$,  
$$
\begin{align*}
\text{td}_{G_1 \times G_2}(u_1, v_1) &= \sum_{(u_1,v_1)(u_2,v_2) \in E_1 \times E_2} (\mu_{\bar{B}} \times \mu_{\bar{B}})((u_1,v_1)(u_2,v_2)) \\
&= \sum_{u_1u_2 \in E_1 \times E_2} \min \left( \mu_{\bar{A}}(u_1), \mu_{\bar{B}}(v_1) \right) + \\
&\quad \sum_{u_1u_2 \in E_1 \times E_2} \min \left( \mu_{\bar{A}}(u_1), \mu_{\bar{B}}(v_1) \right) \\
&= \sum_{u_1u_2 \in E_1 \times E_2} \min \left( \mu_{\bar{A}}(u_1), \mu_{\bar{B}}(v_1) \right) + \\
&\quad \sum_{u_1u_2 \in E_1 \times E_2} \min \left( \mu_{\bar{A}}(u_1), \mu_{\bar{B}}(v_1) \right)
\end{align*}
$$

Also,  
$$
\begin{align*}
\text{td}_{G_1 \times G_2}(u_1, v_1) &= \sum_{(u_1,v_1)(u_2,v_2) \in E_1 \times E_2} (\mu_{\bar{B}} \times \mu_{\bar{B}})((u_1,v_1)(u_2,v_2)) \\
&= \sum_{u_1u_2 \in E_1 \times E_2} \min \left( \mu_{\bar{A}}(u_1), \mu_{\bar{B}}(v_1) \right) + \\
&\quad \sum_{u_1u_2 \in E_1 \times E_2} \min \left( \mu_{\bar{A}}(u_1), \mu_{\bar{B}}(v_1) \right) \\
&= \sum_{u_1u_2 \in E_1 \times E_2} \min \left( \mu_{\bar{A}}(u_1), \mu_{\bar{B}}(v_1) \right) + \\
&\quad \sum_{u_1u_2 \in E_1 \times E_2} \min \left( \mu_{\bar{A}}(u_1), \mu_{\bar{B}}(v_1) \right)
\end{align*}
$$

Result 3.1
For any two numbers $a$ and $b$,  
$$
\min(a, b) = a + b - \max(a, b)
$$

Using the above result, equations (3.1) and (3.2) can be rewritten as  
$$
\begin{align*}
\text{td}_{G_1 \times G_2}(u_1, v_1) &= \sum_{u_1u_2 \in E_1 \times E_2} \min \left( \mu_{\bar{A}}(u_1), \mu_{\bar{B}}(v_1) \right) + \\
&\quad \sum_{u_1u_2 \in E_1 \times E_2} \min \left( \mu_{\bar{A}}(u_1), \mu_{\bar{B}}(v_1) \right) \\
&= \sum_{u_1u_2 \in E_1 \times E_2} \min \left( \mu_{\bar{A}}(u_1), \mu_{\bar{B}}(v_1) \right) + \\
&\quad \sum_{u_1u_2 \in E_1 \times E_2} \min \left( \mu_{\bar{A}}(u_1), \mu_{\bar{B}}(v_1) \right)
\end{align*}
$$

Also,  
$$
\begin{align*}
\text{td}_{G_1 \times G_2}(u_1, v_1) &= \sum_{u_1u_2 \in E_1 \times E_2} \min \left( \mu_{\bar{A}}(u_1), \mu_{\bar{B}}(v_1) \right) + \\
&\quad \sum_{u_1u_2 \in E_1 \times E_2} \min \left( \mu_{\bar{A}}(u_1), \mu_{\bar{B}}(v_1) \right)
\end{align*}
$$

Now we find the total degree of a vertex $(u_1, v_1)$ in $G_1 \times G_2$ in some particular cases.

Theorem 3.1.
Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two IVFGs. If $A_1 \geq B_2$ and $A_2 \geq B_1$, then  
$$
\text{td}_{G_1 \times G_2}(u_1, v_1) = d_{G_1}(u_1) + d_{G_2}(v_1) + \\
\min \left( \mu_{\bar{A}}(u_1), \mu_{\bar{A}}(v_1) \right)
$$

Proof:
$$
\begin{align*}
A_1 \geq B_2 &\Rightarrow \left[ \mu_{\bar{A}}, \mu_{\bar{A}} \right] \geq \left[ \mu_{\bar{B}}, \mu_{\bar{B}} \right] \\
&\Rightarrow \mu_{\bar{A}} \geq \mu_{\bar{B}} \text{ and } \mu_{\bar{A}} \geq \mu_{\bar{B}}
\end{align*}
$$

Again, $A_2 \geq B_1$  
$$
\begin{align*}
A_2 \geq B_1 &\Rightarrow \left[ \mu_{\bar{A}}, \mu_{\bar{A}} \right] \geq \left[ \mu_{\bar{B}}, \mu_{\bar{B}} \right] \\
&\Rightarrow \mu_{\bar{A}} \geq \mu_{\bar{B}} \text{ and } \mu_{\bar{A}} \geq \mu_{\bar{B}}
\end{align*}
$$

Similarly,  
$$
\begin{align*}
\min \left( \mu_{\bar{A}}(u_1), \mu_{\bar{A}}(v_1) \right) &= \mu_{\bar{B}}(v_1) \\
\min \left( \mu_{\bar{A}}(u_1), \mu_{\bar{A}}(v_1) \right) &= \mu_{\bar{B}}(v_1) \\
\min \left( \mu_{\bar{A}}(u_1), \mu_{\bar{A}}(v_1) \right) &= \mu_{\bar{B}}(v_1) \\
\min \left( \mu_{\bar{A}}(u_1), \mu_{\bar{A}}(v_1) \right) &= \mu_{\bar{B}}(v_1)
\end{align*}
$$

Theorem 3.2.
Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two IVFGs. If $A_1 \geq B_2$ and $A_2 \geq B_1$, then  
$$
\text{td}_{G_1 \times G_2}(u_1, v_1) = d_{G_1}(u_1) + d_{G_2}(v_1) + \\
\min \left( \mu_{\bar{A}}(u_1), \mu_{\bar{A}}(v_1) \right)
$$

Proof:
$$
\begin{align*}
A_1 \geq B_2 &\Rightarrow \left[ \mu_{\bar{A}}, \mu_{\bar{A}} \right] \geq \left[ \mu_{\bar{B}}, \mu_{\bar{B}} \right] \\
&\Rightarrow \mu_{\bar{A}} \geq \mu_{\bar{B}} \text{ and } \mu_{\bar{A}} \geq \mu_{\bar{B}}
\end{align*}
$$

Again, $A_2 \geq B_1$  
$$
\begin{align*}
A_2 \geq B_1 &\Rightarrow \left[ \mu_{\bar{A}}, \mu_{\bar{A}} \right] \geq \left[ \mu_{\bar{B}}, \mu_{\bar{B}} \right] \\
&\Rightarrow \mu_{\bar{A}} \geq \mu_{\bar{B}} \text{ and } \mu_{\bar{A}} \geq \mu_{\bar{B}}
\end{align*}
$$

Similarly,  
$$
\begin{align*}
\min \left( \mu_{\bar{A}}(u_1), \mu_{\bar{A}}(v_1) \right) &= \mu_{\bar{B}}(v_1) \\
\min \left( \mu_{\bar{A}}(u_1), \mu_{\bar{A}}(v_1) \right) &= \mu_{\bar{B}}(v_1) \\
\min \left( \mu_{\bar{A}}(u_1), \mu_{\bar{A}}(v_1) \right) &= \mu_{\bar{B}}(v_1)
\end{align*}
$$

Then from equation (3.3), we have  
$$
\begin{align*}
\text{td}_{G_1 \times G_2}(u_1, v_1) &= \sum_{u_1u_2 \in E_1 \times E_2} \min \left( \mu_{\bar{A}}(u_1), \mu_{\bar{A}}(v_1) \right) + \\
&\quad \sum_{u_1u_2 \in E_1 \times E_2} \min \left( \mu_{\bar{A}}(u_1), \mu_{\bar{A}}(v_1) \right) \\
&= \sum_{u_1u_2 \in E_1 \times E_2} \min \left( \mu_{\bar{A}}(u_1), \mu_{\bar{A}}(v_1) \right) + \\
&\quad \sum_{u_1u_2 \in E_1 \times E_2} \min \left( \mu_{\bar{A}}(u_1), \mu_{\bar{A}}(v_1) \right)
\end{align*}
$$
\[ td_{G_1}(u_1) + td_{G_2}(v_1) - \max(\mu_{\tilde{A}_1}(u_1), \mu_{\tilde{A}_2}(v_1)) \]

Again from equation (3.4), we have
\[ \sum_{v_1 \in E_2} \mu_{\tilde{A}_2}(v_2) + \sum_{u_1 \in E_1} \mu_{\tilde{A}_1}(u_1) + \mu_{\tilde{A}_2}(v_1) - \max(\mu_{\tilde{A}_1}(u_1), \mu_{\tilde{A}_2}(v_1)) \]

\[ = d_{G_1}(u_1) + d_{G_2}(v_1) - \max(\mu_{\tilde{A}_1}(u_1), \mu_{\tilde{A}_2}(v_1)) \]

**Lemma 3.1.**

Let \( G_1 = (A_1, B_1) \) and \( G_2 = (A_2, B_2) \) be two IVFGs. If \( A_1 \leq B_2 \), then \( A_2 \geq B_1 \) and vice versa.

**Proof:**

By the definition of an IVFG,
\[ \mu_{\tilde{A}_1}(u, v) \leq \mu_{\tilde{A}_1}(u, \mu_{\tilde{A}_1}(v)) \]

and
\[ \mu_{\tilde{A}_2}(u, v) \leq \mu_{\tilde{A}_2}(u, \mu_{\tilde{A}_2}(v)) \]

for all \( (u, v) \in E_1 \) where \( i = 1, 2 \).

\[ \sum_{v_1 \in E_2} \mu_{\tilde{A}_2}(v_2) + \sum_{u_1 \in E_1} \mu_{\tilde{A}_1}(u_1) + \mu_{\tilde{A}_2}(v_1) - \max(\mu_{\tilde{A}_1}(u_1), \mu_{\tilde{A}_2}(v_1)) \]

\[ = d_{G_1}(u_1) + d_{G_2}(v_1) - \max(\mu_{\tilde{A}_1}(u_1), \mu_{\tilde{A}_2}(v_1)) \]

**Theorem 3.3.**

Let \( G_1 = (A_1, B_1) \) and \( G_2 = (A_2, B_2) \) be two IVFGs such that \( A_1 \leq B_2 \). Then,
\[ td_{G_1} \times G_2(u_1, v_1) = d_{G_1}(u_1) + \mu_{\tilde{A}_1}(u_1) d_{G_2}(v_1) + \mu_{\tilde{A}_2}(u_1) \]

\[ + \mu_{\tilde{A}_1}(u_1) d_{G_2}(v_1) + \mu_{\tilde{A}_2}(u_1) \]

**Proof:**

Since \( A_1 \leq B_2 \), by lemmas 3.1 and 3.2, we have \( A_2 \geq B_1 \) and \( A_1 \leq A_2 \) respectively. Using these conditions in equation 3.1, \( td_{G_1} \times G_2(u_1, v_1) \)
\[ = \sum_{v_1 \in E_2} \mu_{\tilde{A}_2}(v_2) + \sum_{u_1 \in E_1} \mu_{\tilde{A}_1}(u_1) + \mu_{\tilde{A}_2}(v_1) - \max(\mu_{\tilde{A}_1}(u_1), \mu_{\tilde{A}_2}(v_1)) \]

\[ = d_{G_1}(u_1) + \mu_{\tilde{A}_1}(u_1) d_{G_2}(v_1) + \mu_{\tilde{A}_2}(u_1) \]

Similarly, \( td_{G_1} \times G_2(u_1, v_1) = d_{G_1}(u_1) + \mu_{\tilde{A}_1}(u_1) d_{G_2}(v_1) + \mu_{\tilde{A}_2}(u_1) \)

**Theorem 3.4.**

Let \( G_1 = (A_1, B_1) \) and \( G_2 = (A_2, B_2) \) be two IVFGs such that \( A_1 \leq B_2 \). Then,
\[ td_{G_1} \times G_2(u_1, v_1) = d_{G_1}(u_1) + \mu_{\tilde{A}_1}(u_1) d_{G_2}(v_1) \]

and
\[ td_{G_1} \times G_2(u_1, v_1) = d_{G_1}(u_1) + \mu_{\tilde{A}_1}(u_1) d_{G_2}(v_1) \]

**Proof:**

Since \( A_1 \leq B_2 \), by lemmas 3.1 and 3.2, we have \( A_2 \geq B_1 \) and \( A_1 \leq A_2 \) respectively. Using these conditions in equation 3.1, \( td_{G_1} \times G_2(u_1, v_1) \)
\[ = \sum_{v_1 \in E_2} \mu_{\tilde{A}_2}(v_2) + \sum_{u_1 \in E_1} \mu_{\tilde{A}_1}(u_1) + \mu_{\tilde{A}_2}(v_1) - \max(\mu_{\tilde{A}_1}(u_1), \mu_{\tilde{A}_2}(v_1)) \]

\[ = d_{G_1}(u_1) + \mu_{\tilde{A}_1}(u_1) d_{G_2}(v_1) + \mu_{\tilde{A}_2}(u_1) \]

Similarly, \( td_{G_1} \times G_2(u_1, v_1) = d_{G_1}(u_1) + \mu_{\tilde{A}_1}(u_1) d_{G_2}(v_1) + \mu_{\tilde{A}_2}(u_1) \)

IV. TOTAL REGULARITY OF THE CARTESIAN PRODUCT OF TWO INTERVAL-VALUED FUZZY GRAPHS

In general the Cartesian product of two TRIVFGs need not be a TRIVFG which is clear from the following example.

**Example 4.1.**

Consider the IVFG \( G_1 = (A_1, B_1) \) defined on graph \( G_1 = (V_1, E_1) \) such that \( V_1 = \{u_1, u_2, u_3\} \) and \( E_1 = \{u_1 u_2, u_2 u_3, u_3 u_1\} \). The membership degrees of the vertices and edges are as follows:

\[ \begin{align*}
\mu_{\tilde{A}_1}(u_1) &= 0.7, \quad \mu_{\tilde{A}_1}(v_1) = 0.5 \\
\mu_{\tilde{A}_1}(u_2) &= 0.8, \quad \mu_{\tilde{A}_1}(v_2) = 0.6 \\
\mu_{\tilde{A}_1}(u_3) &= 0.9, \quad \mu_{\tilde{A}_1}(v_3) = 0.7 \\
\mu_{\tilde{A}_2}(u_1) &= 0.6, \quad \mu_{\tilde{A}_2}(v_1) = 0.4 \\
\mu_{\tilde{A}_2}(u_2) &= 0.7, \quad \mu_{\tilde{A}_2}(v_2) = 0.5 \\
\mu_{\tilde{A}_2}(u_3) &= 0.8, \quad \mu_{\tilde{A}_2}(v_3) = 0.6 \\
\end{align*} \]
Now consider the IVFG $G_2 = (A_2, B_2)$ defined by

<table>
<thead>
<tr>
<th>$\mu_{\tilde{A}_2}$</th>
<th>$\mu_{\tilde{B}_2}$</th>
<th>$\mu_{\tilde{A}<em>2} \times \mu</em>{\tilde{B}_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.4$</td>
<td>$0.4$</td>
<td>$0.4$</td>
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<tr>
<td>$0.5$</td>
<td>$0.5$</td>
<td>$0.5$</td>
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</table>

Then $G_1 \times G_2$ will be an IVFG defined by

<table>
<thead>
<tr>
<th>$\mu_{\tilde{A}_1}$</th>
<th>$\mu_{\tilde{B}_1}$</th>
<th>$\mu_{\tilde{A}<em>1} \times \mu</em>{\tilde{B}_1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.3$</td>
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</tr>
</tbody>
</table>

Using routine computations, we can see that both $G_1$ and $G_2$ are TRIVFGs, but $G_1 \times G_2$ is not a TRIVFG.

Next we give an example to show that $G_1 \times G_2$ is a TRIVFG need not imply that both $G_1$ and $G_2$ should be TRIVFGs

**Example 4.2.**

Consider the IVFG $G_2 = (A_1, B_1)$ defined by

<table>
<thead>
<tr>
<th>$\mu_{\tilde{A}_1}$</th>
<th>$\mu_{\tilde{B}_1}$</th>
<th>$\mu_{\tilde{A}<em>1} \times \mu</em>{\tilde{B}_1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.4$</td>
<td>$0.4$</td>
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</tr>
<tr>
<td>$0.5$</td>
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<td>$0.5$</td>
</tr>
</tbody>
</table>

Also consider the IVFG $G_2 = (A_2, B_2)$ defined by

<table>
<thead>
<tr>
<th>$\mu_{\tilde{A}_2}$</th>
<th>$\mu_{\tilde{B}_2}$</th>
<th>$\mu_{\tilde{A}<em>2} \times \mu</em>{\tilde{B}_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.4$</td>
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</tbody>
</table>

Then $G_1 \times G_2$ will be an IVFG defined by

<table>
<thead>
<tr>
<th>$\mu_{\tilde{A}_1}$</th>
<th>$\mu_{\tilde{B}_1}$</th>
<th>$\mu_{\tilde{A}<em>1} \times \mu</em>{\tilde{B}_1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.4$</td>
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<tr>
<td>$0.5$</td>
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<td>$0.5$</td>
</tr>
</tbody>
</table>

Using routine computations, we can see that both $G_1 \times G_2$ and $G_1$ are TRIVFGs, but $G_2$ is not a TRIVFG.

Now we proceed to obtain some necessary and sufficient conditions for the Cartesian product of two TRIVFGs to be totally regular under some restrictions

**Theorem 4.1.**

Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two IVFGs. If $A_1 \geq B_2$ and $A_2 \geq B_1$ and $rmin(A_1, A_2)$ is a constant, then $G_1 \times G_2$ is totally regular if and only if $G_1$ and $G_2$ are regular.

**Proof:**

Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two IVFGs. Suppose $A_1 \geq B_2$ and $A_2 \geq B_1$. Then by theorem 3.1, $td_{G_1 \times G_2}(u_1, v_1) = d_{G_1}(u_1) + d_{G_2}(v_1) + \min\{-\mu_{\tilde{A}_1}(u_1), -\mu_{\tilde{B}_2}(v_1), 0\}$ and $td_{G_1 \times G_2}(u_1, v_1) = d_{G_1}(u_1) + d_{G_2}(v_1) + \min\{\mu_{\tilde{A}_1}(u_1), \mu_{\tilde{B}_2}(v_1), 0\}$.

Now suppose that $\min(A_1, A_2) = [c_1, c_2]$, a constant. Then, $td_{G_1 \times G_2}(u_1, v_1) = d_{G_1}(u_1) + d_{G_2}(v_1) + c_1$ and $td_{G_1 \times G_2}(u_1, v_1) = d_{G_1}(u_1) + d_{G_2}(v_1) + c_2$

Suppose $G_1$ and $G_2$ are regular IVFGs with degrees $[k_1, k_2]$ and $[l_1, l_2]$ respectively. Then the above equations become $td_{G_1 \times G_2}(u_1, v_1) = k_1 + l_1 + c_1$ and $td_{G_1 \times G_2}(u_1, v_1) = k_2 + l_2 + c_2$ where $(u_1, v_1) \in V_1 \times V_2$ is arbitrary which shows that $G_1 \times G_2$ is totally regular.

Conversely, suppose that $G_1 \times G_2$ is totally regular. We have to prove that $G_1$ and $G_2$ are regular. Then for any two vertices $(u_1, v_1)$ and $(u_2, v_2)$ in $V_1 \times V_2$, $td_{G_1 \times G_2}(u_1, v_1) = td_{G_1 \times G_2}(u_2, v_2)$

$\Rightarrow d_{G_1}(u_1) + d_{G_2}(v_1) + c_1 = d_{G_1}(u_2) + d_{G_2}(v_2) + c_1$ [By theorem 3.1]

$\Rightarrow d_{G_1}(u_1) + d_{G_2}(v_1) = d_{G_1}(u_2) + d_{G_2}(v_2)$

Fix $u \in V_1$ and consider $(u, v_1)$ and $(u, v_2)$ in $V_1 \times V_2$, where $v_1, v_2 \in V_2$ are arbitrary. Then, $d_{G_1}(u) + d_{G_2}(v_1) = d_{G_1}(u) + d_{G_2}(v_2)$ which implies $d_{G_2}(v_1) = d_{G_2}(v_2)$. Similarly, $d_{G_2}(v_1) = d_{G_2}(v_2)$, this is true for all $v_1, v_2 \in V_2$.
Now fix \( v \in V_2 \) and consider \((u_1,v)\) and \((u_2,v)\) in \(V_1 \times V_2\), where \(u_1, u_2 \in V_1\) are arbitrary. Then, 
\[ d^-_{G_1}(u_1) \times d^-_{G_2}(v) = d^-_{G_1}(u_2) + d^-_{G_2}(v) \]  
which implies \( d^-_{G_1}(u_1) = d^-_{G_1}(u_2) \).

Similarly, \( d^+_{G_1}(u_1) = d^+_{G_1}(u_2) \). This is true for all \( u_1, u_2 \in V_1\). Thus \( G_1 \) is a RIVFG

**Theorem 4.2.**

Let \( G_1 = (A_1, B_1) \) and \( G_2 = (A_2, B_2) \) be two IVFGs. If \( A_1 \subseteq B_2 \) and \( A_2 \subseteq B_1 \) and \( r \text{max}(A_1, A_2) \) is a constant, then \( G_1 \times G_2 \) is totally regular if and only if \( G_1 \) and \( G_2 \) are totally regular.

**Proof:**

Let \( G_1 = (A_1, B_1) \) and \( G_2 = (A_2, B_2) \) be two IVFGs. Suppose \( A_1 \geq B_2 \) and \( A_2 \geq B_1 \). Then by theorem 3.2,
\[ td^-_{G_1 \times G_2}(u_1, v_1) = td^-_{G_1}(u_1) + td^-_{G_2}(v_1) - \max(\mu^-_{A_1}(u_1), \mu^-_{B_1}(v_1)) \]
and
\[ td^+_{G_1 \times G_2}(u_1, v_1) = td^+_{G_1}(u_1) + td^+_{G_2}(v_1) - \max(\mu^+_{A_1}(u_1), \mu^+_{B_1}(v_1)) \]
Now suppose that \( r \text{max}(A_1, A_2) = [c_1, c_2] \), a constant. Then
\[ td^-_{G_1 \times G_2}(u_1, v_1) = td^-_{G_1}(u_1) + td^-_{G_2}(v_1) - c_1 \]
and
\[ td^+_{G_1 \times G_2}(u_1, v_1) = td^+_{G_1}(u_1) + td^+_{G_2}(v_1) - c_2 \]
Suppose \( G_1 \) and \( G_2 \) are TRIVFGs with degrees \([k_1, k_2]\) and \([l_1, l_2]\) respectively. Then the above equations become
\[ td^-_{G_1 \times G_2}(u_1, v_1) = k_1 + l_1 - c_1 \]
and
\[ td^+_{G_1 \times G_2}(u_1, v_1) = k_2 + l_2 - c_2 \] which shows that \( G_1 \times G_2 \) is totally regular.

Conversely, suppose that \( G_1 \times G_2 \) is totally regular. We have to prove that \( G_1 \) and \( G_2 \) are totally regular. Then for any two vertices \((u_1, v_1)\) and \((u_2, v_2)\) in \(V_1 \times V_2\),
\[ td^-_{G_1 \times G_2}(u_1, v_1) = td^-_{G_1}(u_1) + td^-_{G_2}(v_1) \]
\[ td^-_{G_1 \times G_2}(u_2, v_1) = td^-_{G_1}(u_2) + td^-_{G_2}(v_1) \]
implies \( td^-_{G_1}(u_1) = td^-_{G_1}(u_2) \), similarly \( td^-_{G_2}(v_2) = td^-_{G_2}(v_1) \).

Similarly, \( td^+_{G_1 \times G_2}(u_1, v_1) = td^+_{G_1}(u_1) + td^+_{G_2}(v_1) \) which is true for all \( v_1, v_2 \in V_2 \). Thus \( G_1 \) is a TRIVFG.

**Theorem 4.3.**

Let \( G_1 = (A_1, B_1) \) and \( G_2 = (A_2, B_2) \) be two IVFGs such that \( A_1 \leq B_2 \) and \( A_1 \) is a constant function. Then \( G_1 \times G_2 \) is totally regular if and only if \( G_1 \) is regular and \( G_2 \) is partially regular.

**Proof:**

Let \( G_1 = (A_1, B_1) \) and \( G_2 = (A_2, B_2) \) be two IVFGs such that \( A_1 \leq B_2 \). Then, by theorem 3.3,
\[ td^-_{G_1 \times G_2}(u_1, v_1) = d^-_{G_1}(u_1) + \mu^-_{A_1}(u_1) dG^2_2(v_1) + \mu^-_{B_1}(v_1) \]
and
\[ td^+_{G_1 \times G_2}(u_1, v_1) = d^+_{G_1}(u_1) + \mu^+_{A_1}(u_1) dG^2_2(v_1) + \mu^+_{B_1}(v_1) \]
Let \( A_1 = [c_1, c_2] \), a constant. Then the above equations become
\[ td^-_{G_1 \times G_2}(u_1, v_1) = d^-_{G_1}(u_1) + c_1 dG^2_2(v_1) + c_1 \]
and
\[ td^+_{G_1 \times G_2}(u_1, v_1) = d^+_{G_1}(u_1) + c_1 dG^2_2(v_1) + c_2 \]
Suppose \( G_1 \) is regular with degree \( k_1, k_2 \) and \( G_2 \) is partially regular with degree \( m \). Then the above equations become
\[ td^-_{G_1 \times G_2}(u_1, v_1) = k_1 + c_1 m + c_1 = k_1 + (m + 1) c_1 \]
and
\[ td^+_{G_1 \times G_2}(u_1, v_1) = k_2 + c_2 m + c_2 = k_2 + (m + 1) c_2 \] which shows that \( G_1 \times G_2 \) is totally regular.

Conversely, suppose that \( G_1 \times G_2 \) is totally regular. We have to prove that \( G_1 \) is regular and \( G_2 \) is partially regular. Then for any two points \((u_1, v_1)\) and \((u_2, v_2)\) in \(V_1 \times V_2\),
\[ td^-_{G_1 \times G_2}(u_1, v_1) = td^-_{G_1 \times G_2}(u_2, v_1) \]
\[ td^+_{G_1 \times G_2}(u_1, v_1) = td^+_{G_1 \times G_2}(u_2, v_1) \]
\[ td^-_{G_1 \times G_2}(u_1, v_2) = td^-_{G_1 \times G_2}(u_2, v_2) \]
\[ td^+_{G_1 \times G_2}(u_1, v_2) = td^+_{G_1 \times G_2}(u_2, v_2) \]
Fix \( u \in V_1 \) and consider \((u, v_1)\) and \((u, v_2)\) in \(V_1 \times V_2\), where \(v_1, v_2 \in V_2\) are arbitrary. Then,
\[ d^-_{G_1}(u) + c_1 dG^2_2(v_1) = d^-_{G_1}(u) + c_1 dG^2_2(v_2) \]
which implies \( d^+_{G_2}(v_1) = d^+_{G_2}(v_2) \). This is true for all \( v_1, v_2 \in V_2 \). Thus \( G_2 \) is regular.

**Theorem 4.4.**

Let \( G_1 = (A_1, B_1) \) and \( G_2 = (A_2, B_2) \) be two IVFGs such that \( A_1 \leq B_2 \) and \( A_1 \) is a constant function. Then \( G_1 \times G_2 \) is totally regular if and only if \( G_1 \) is totally regular and \( G_2 \) is partially regular.

**Proof:**

Let \( G_1 = (A_1, B_1) \) and \( G_2 = (A_2, B_2) \) be two IVFGs such that \( A_1 \leq B_2 \). Then, by theorem 3.4,
\[ td^-_{G_1 \times G_2}(u_1, v_1) = td^-_{G_1}(u_1) + \mu^-_{A_1}(u_1) dG^2_2(v_1) \]
and
\[ td^+_{G_1 \times G_2}(u_1, v_1) = td^+_{G_1}(u_1) + \mu^+_{A_1}(u_1) dG^2_2(v_1) \]
Let \( A_1 = [c_1, c_2] \), a constant. Then the above equations become
\[ td^-_{G_1 \times G_2}(u_1, v_1) = td^-_{G_1}(u_1) + c_1 dG^2_2(v_1) \]
and
\[ td^+_{G_1 \times G_2}(u_1, v_1) = td^+_{G_1}(u_1) + c_1 dG^2_2(v_1) \]
Suppose $G_1$ is totally regular with degree $[k_1,k_2]$ and $G_2$ is partially regular with degree $m$. Then the above equations become

$$td_{G_1}^+(u_1,v_1) = td_{G_1}^+(u_1) + c_1dG_2^+(v_1).$$

Similarly, $td_{G_1}^+(u_2,v_2) = td_{G_1}^+(u_2) + c_1dG_2^+(v_2).$ This is true for all $v_1, v_2 \in V_2$. Thus $G_2$ is regular. Hence $G_2$ is a PRIVFG.

Now fix $v \in V_2$ and consider $(u_1,v)$ and $(u_2,v)$ in $V_1 \times V_2$, where $u_1, u_2 \in V_1$ are arbitrary. Then,

$$td_{G_1}^+(u_1) + c_1dG_2^+(v) = td_{G_1}^+(u_2) + c_1dG_2^+(v),$$

which implies $dG_2^+(v) = dG_2^+(v_2). This is true for all $v_1, v_2 \in V_2$. Thus $G_2$ is regular. Hence $G_2$ is a PRIVFG.

V. CONCLUSION

Cartesian Product of graphs have applications in many branches like coding theory, network designs, chemical graph theory etc. In this paper, we have obtained the total degree of a vertex in the Cartesian Product of two IVFGs in terms of degree and total degree of vertices of component graphs. This will be very helpful in analyzing many properties of Cartesian Product of IVFGs. We have observed that the Cartesian Product of two TRIVFGs need not be a TRIVFG. Also we derived some necessary and sufficient conditions for the Cartesian product of two TRIVFGs to be totally regular under some restrictions.

ACKNOWLEDGEMENT

I express my sincere gratitude to the University Grants Commission for granting me PhD leave for completing my Ph. D work. I am also extremely grateful to the reviewers and the Editor-in-Chief for their valuable comments and suggestions for improving the paper.

REFERENCES


