Some cycle and path related strongly*-graphs

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Abstract — A graph with \( n \) vertices is said to be strongly*-graph if its vertices can be assigned the values \{1, 2, \ldots, n\} in such a way that when an edge whose end vertices are labeled \( i \) and \( j \), is labeled with the value \( i + j + ij \) such that all edges have distinct labels. Here we derive different strongly*-graphs in context of some graph operations.

Key words : Strongly*-labeling, Strongly*-graph.

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I. Introduction

By a graph \( G \), we mean a simple, finite, undirected graph.

Definition I.1. [1] A graph \( G \) with \( n \) vertices is said to be strongly*-graph if there is a bijection \( f : V(G) \rightarrow \{1, 2, \ldots, n\} \) such that the induced edge function \( f^* : E(G) \rightarrow \mathbb{N} \) defined as \( f^*(e = uv) = f(u) + f(v) + f(u) \cdot f(v) \) is injective. Here \( f \) is called strongly*-labeling of graph \( G \).

Definition I.2. [2] A chord of cycle \( C_n \), \( n \geq 4 \), is an edge joining two non-adjacent vertices of \( C_n \).

Definition I.3. [1] Two chords of cycle \( C_n \), \( n \geq 5 \), are said to be twin chords if they form a triangle with an edge of \( C_n \).

For positive integers \( n \) and \( p \) with \( 3 \leq p \leq (n - 2) \), \( C_{n,p} \) is the graph consisting of a cycle \( C_n \) with twin chords where the chords form the cycles \( C_{p} \), \( C_3 \) and \( C_{n+1-p} \) without chords with the edges of \( C_n \).

Definition I.4. [1] The cycle with triangle is a cycle with three chords which by themselves form a triangle.

For positive integers \( p \), \( q \), \( r \) and \( n \geq 6 \) with \( p + q + r + 3 = n \), \( C_n(p, q, r) \) denotes the cycle with triangle whose edges form the edges of cycles \( C_{p+2} \), \( C_{q+2} \) and \( C_{r+2} \) without chords.

Definition I.5. [1] The crown \( (C_n \bigodot K_1) \) is obtained by joining a pendant vertex to each vertex of cycle \( C_n \) by an edge.

Definition I.6. [1] Tadpole, \( T(l, r) \) is the graph in which path of length \( r \) is attached to any one vertex of cycle \( C_l \) by a bridge. \( T(l, r) \) has \( l + r \) vertices and \( l + r \) edges.

Definition I.7. [1] The middle graph of a graph \( G \) (with two or more vertices), denoted by \( M(G) \), is the graph whose vertex set is \( V(G) \cup E(G) \) in which two vertices are adjacent if and only if either they are adjacent edges of \( G \) or one is a vertex of \( G \) and the other is an edge incident on it.

Definition I.8. [1] The total graph of a graph \( G \) (with two or more vertices), denoted by \( T(G) \), is the graph whose vertex set is \( V(G) \cup E(G) \) and two vertices are adjacent whenever they are adjacent vertices or adjacent edges in \( G \) or one is a vertex of \( G \) and the other is an edge incident on it.
Definition I.9. [1] The split graph of a graph $G$, denoted by $spl(G)$, is the graph obtained by adding a new vertex $v'$ to each vertex $v$ such that $v'$ is adjacent to every vertex that is adjacent to $v$ in $G$.

Definition I.10. [1] The shadow graph $D_2(G)$ of a connected graph $G$ is constructed by taking two copies of $G'$ and $G''$ of graph $G$. Join each vertex $v'$ of $G'$ to the neighbors of the corresponding vertex $v''$ of $G''$.

Adiga and Somashekara [3] proved that all trees, cycles and grids are strongly*-graphs. In the same paper they considered the problem of determining the maximum number of edges in any strongly*-graph of given order and relates it to the corresponding problem for strongly multiplicative graphs.

For all others standard terminology and notations we follow Harary [2].

II. MAIN RESULTS

Theorem II.1. Cycle $C_n$ with one chord is strongly*-graph for all $n \in \mathbb{N}$, where chord forms a triangle with two edges of $C_n$.

Proof. Let $G$ be the cycle $C_n$ with one chord. Let $\{v_1, v_2, \ldots, v_n\}$ denote the successive vertices of $C_n$, where $v_1$ is adjacent to $v_n$ and $v_i$ is adjacent to $v_{i+1}$, $1 \leq i \leq n-1$. Let $e = v_1v_3$ be the chord of $C_n$. Note that $d(v_1) = d(v_3) = 3$ and $d(v_i) = 2$, $2 \leq i \leq n$, $i \neq 3$.

We define vertex labeling function $f : V(G) \to \{1, 2, \ldots, n\}$ as follows.

$$f(v_i) = \begin{cases} 2i - 1; 1 \leq i \leq \lceil \frac{n}{2} \rceil, \\ 2(n - i + 1); (\lceil \frac{n}{2} \rceil + 1) \leq i \leq n. \end{cases}$$

Hence the vertex labels on one half of $C_n$ are monotonically increasing (consecutive) odd numbers, whereas vertex labels on the other half of $C_n$ are monotonically increasing (consecutive) even numbers. Hence the produced edge labels must be different. Further the chord has $(1, 5) = 11$ label which is unique with respect to labeling $f$ of the graph. Hence above defined labeling pattern satisfies the conditions of strongly*-labeling. i.e. Cycle $C_n$ with one chord is strongly*-graph. □

Example 1. Strongly*-labeling of cycle $C_8$ with one chord is shown in Figure 1.

![Fig. 1](image-url)

Corollary 1. Cycle $C_n$ with twin chords $C_{n,3}$ is strongly*-graph for all $n \in \mathbb{N}$.

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ denote the successive vertices of the $C_{n,3}$ where $v_1$ is adjacent to $v_n$, and $v_i$ is adjacent to $v_{i+1}$, $1 \leq i \leq n - 1$. Let $e_1 = v_1v_3$ and $e_2 = v_1v_5$ be two chords of $C_{n,3}$. Note that $d(v_1) = 4$, $d(v_3) = d(v_5) = 3$ and $d(v_i) = 2$, $2 \leq i \leq n$, $i \neq 3, i \neq 5$.

We define vertex labeling function $f : V(C_{n,3}) \to \{1, 2, \ldots, n\}$ same as per the labeling defined in Theorem 1.

Here vertex labels on one half of the $C_{n,3}$ are monotonically increasing (consecutive) odd numbers, whereas vertex labels on the other half of the $C_{n,3}$ are monotonically increasing (consecutive) even numbers. Hence the produced edge labels must be different. Further the vertex with label 1 is adjacent to the vertices with label 5, 3 and 8 which are unique with respect to the labeling $f$ of the given graph. Hence above defined labeling pattern satisfies the
conditions of strongly*-labeling, i.e. $C_{n,3}$ with twin chords is strongly*-graph.

**Example 2.** Strongly*-labeling of cycle $C_8$ with twin chords is shown in Figure 2.

![Figure 2](image)

**Corollary 2.** Cycle with triangle $C_n(1,1,n-5)$ is strongly*-graph for all $n \in \mathbb{N}$.

**Proof.** Let $G$ be the cycle with triangle $C_n(1,1,n-5)$. Let $\{v_1, v_2, \ldots, v_n\}$ denote the successive vertices of the $G$ such that $v_i$ is adjacent to $v_{i-1}$ and $v_{i+1}$, $1 \leq i \leq n-1$. Let $e_1 = v_1v_3$, $e_2 = v_1v_5$ and $e_3 = v_3v_5$ be three chords of $C_n$. Note that $d(v_1) = d(v_3) = d(v_5) = 4$ and $d(v_i) = 2, 2 \leq i \leq n, i \neq 3, i \neq 5$.

We define vertex labeling function $f : V(G) \rightarrow \{1, 2, \ldots, n\}$ same as per the labeling defined in Theorem 1. Here vertex labels on one half of the $C_n(1,1,n-5)$ are monotonically increasing (consecutive) odd numbers, whereas vertex labels on the other half of the $C_n(1,1,n-5)$ are monotonically increasing (consecutive) even numbers. Hence the produced edge labels must be different. Further the chords have $(1,5) = 11$, $(1,6) = 13$ and $(5,6) = 41$ labels which are unique with respect to labeling $f$ of the graph $G$. Hence above defined labeling pattern satisfies the conditions of strongly*-labeling, i.e. $C_n$ with triangle is strongly*-graph.

**Example 3.** Strongly*-labeling of cycle $C_7$ with triangle is strongly*-graph shown in Figure 3.

![Figure 3](image)

**Theorem III.2.** The crown $C_n \odot K_1$ is strongly*-graph for all $n \in \mathbb{N}$.

**Proof.** Let $\{v_1, v_2, \ldots, v_n, v'_1, v'_2, \ldots, v'_n\}$ be the vertices of the crown $C_n \odot K_1$, where $\{v_1, v_2, \ldots, v_n\}$ are the vertices corresponding to cycle $C_n$ and $\{v'_1, v'_2, \ldots, v'_n\}$ are the pendant vertices. Here $v'_i$ is adjacent to $v_i$, $i = 1, 2, \ldots, n$.

To define vertex labeling function $f : V(C_n \odot K_1) \rightarrow \{1, 2, 3, \ldots, 2n\}$ we consider the following cases.

**Case 1:** $n$ is odd.

\[
\begin{align*}
f(v_i) &= 2i; 1 \leq i \leq n, \quad f(v'_i) = 2i - 1; 1 \leq i \leq n.
\end{align*}
\]

When $n$ is odd, the labels for the vertices $v_i$ are consecutive odd numbers, whereas the labels for the vertices $v'_i$ are consecutive even numbers. Therefore the labels produced for the edges $v_iv_{i+1}$ are odd and in increasing order. Further the edges $v_iv_{i+1}$, $1 \leq i \leq n$ are labeled with even labels in increasing order. Hence all edges will have different labels.

**Case 2:** $n$ is even.

\[
\begin{align*}
f(v_i) &= 4i - 1; 1 \leq i \leq \frac{n}{2}, \\
&\quad 4(n-i+1); \left(\frac{n}{2}+1\right) \leq i \leq n. \\
f(v'_i) &= 4i - 3; 1 \leq i \leq \frac{n}{2}, \\
&\quad 4(n-i) + 2; \left(\frac{n}{2}+1\right) \leq i \leq n.
\end{align*}
\]

When $n$ is even, for any two edge labels produced in the graph, \{(4j-3) + (4j-1) + (4j-3) \cdot (4j-1) \neq \}
\{(4j-1)+(4j+3)+(4j-1)\cdot(4j+3)\} \neq \\
\{2n+(2n-1)+2n\cdot(2n-1)\} \neq \\
\{4j+(4j-2)+4j\cdot(4j-2)\} \neq \\
\{4j+4(j+1)+4j\cdot(4j+1)\} \neq 19, \\
1 \leq j \leq \frac{n}{2}.

Hence above defined labeling pattern satisfies the conditions of strongly* labeling. i.e. Crown \(C_n \odot K_1\) is strongly* graph for all \(n\).

\(\square\)

**Example 4.** Strongly* labeling of crown \(C_6 \odot K_1\) is shown in Figure 4.

![Fig. 4](image)

**Theorem II.3.** Tadpoles \(T(l, r)\) are strongly* graph for all \(l, r \in \mathbb{N}\).

**Proof.** Let \(\{v_1, v_2, \ldots, v_l\}\) be the vertices of the cycle \(C_l\) and \(\{v_{l+1}, v_{l+2}, \ldots, v_{l+r}\}\) be the vertices of path \(P_r\) attached to vertex \(v_l\) of \(C_l\) in the \(T(l, r)\). Let \(e = v_lv_{l+1}\) be the bridge joining vertex \(v_l\) of \(C_l\) and vertex \(v_{l+1}\) of the \(P_r\). Note that \(|V(T(l, r))| = |E(T(l, r))| = l + r\).

To define vertex labeling function \(f : V(T(l, r)) \rightarrow \{1, 2, 3, \ldots, l + r\}\), We consider the following cases.

**Case 1:** \(l \neq 2 + \sum_{i=3}^{m} i\), where \(m \in \mathbb{N} \setminus \{1, 2\}\).

\[f(v_i) = i, 1 \leq i \leq l + r.\]

**Case 2:** \(l = 2 + \sum_{i=3}^{m} i\), where \(m \in \mathbb{N} \setminus \{1, 2\}\).

\[f(v_i) = i + 1, f(v_{i+1}) = l.\]

\[f(v_i) = i, 1 \leq i \leq l + r, i \neq l, l + 1.\]

Since the labeling \(f\) defined above is strictly increasing, for any two edges with end vertices labeled by vertex labels \((i, j)\) and \((s, t)\) respectively, \((i \neq j \neq s \neq t)\) we have \((i+j+ij) \neq (s+t+st)\). Hence above defined labeling pattern satisfies the conditions of strongly* labeling. i.e. \(T(l, r)\) is strongly* graph.

**Example 5.** Strongly* labeling of Tadpole \(T(5, 3)\) is shown in Figure 5.

![Fig. 5](image)

**Theorem II.4.** Middle graph of cycle \((C_n)\) is strongly* graph for all \(n \in \mathbb{N}\).

**Proof.** Let \(\{v_1, v_2, \ldots, v_n, v'_1, v'_2, \ldots, v'_n\}\) be the vertices of the graph \(M(C_n)\), where \(\{v_1, v_2, \ldots, v_n\}\) be the vertices corresponding to the cycle \(C_n\) and \(\{v'_1, v'_2, \ldots, v'_n\}\) be the vertices corresponding to the edges of \(C_n\).

We define vertex labeling function \(f : V(M(C_n)) \rightarrow \{1, 2, 3, \ldots, 2n\}\) as follows.

\[f(v_i) = \begin{cases} 
4i - 3; 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil. \\
4(n-i+1); \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n. 
\end{cases}\]

\[f(v'_i) = \begin{cases} 
4i - 1; 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil. \\
4(n-i) + 2; \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n. 
\end{cases}\]

Here vertex labels in one half of the \(M(C_n)\) are consecutive even numbers in increasing order, whereas the vertex labels in other half of the \(M(C_n)\) are consecutive odd numbers in increasing order.
When \( n \) is even, for any even edge label produced in graph, 
\[
\{(4k + (4k + 2) + (4k) \cdot (4k + 2)) \neq (4k + 2) + (4k + 1)) + (4k + 2) \cdot (4k + 1)) \neq
\{(4k + 1) + (4k) \cdot (4k + 1)) \neq (2) + (4) + (2) \cdot (4)\}
\]
and for any odd edge label produced in graph, 
\[
\{(4k - 3) + (4k - 1) + (4k - 3) \cdot (4k - 1)) \neq
\{(4k - 1) + (4k + 1) \cdot (4k + 1)) \neq (2n - 3) + (2n - 1) + (2n - 3) \cdot (2n - 1)) \neq
\{(2n - 1) + (2n) + (2n - 1) \cdot (2n)) \neq
\{(1) + (2) + (1) \cdot (2)) \neq (1) + (4) \cdot (4)\}.
\]

When \( n \) is odd, for any even edge label produced in graph, 
\[
\{(4k) + (4k + 2) + (4k) \cdot (4k + 2)) \neq (4k + 2) + (4k + 1)) + (4k + 2) \cdot (4k + 1)) \neq
(4k + 1) + (4k + 1)) \cdot (4k) \neq (2n - 2) + (2n) + (2n - 2) \cdot (2n) \neq
\{(2) + (4) + (2) \cdot (4), \text{ where } 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \}
\]
and for any odd edge label produced in graph, 
\[
\{(4k - 3) + (4k - 1) + (4k - 3) \cdot (4k - 1)) \neq
\{(4k - 1) + (4k + 1) \cdot (4k + 1)) \neq (2n - 2) + (2n - 1) + (2n - 2) \cdot (2n - 1)) \neq
\{(2n - 1) + (2n) + (2n - 1) \cdot (2n)) \neq
\{(1) + (2) + (1) \cdot (2)) \neq (1) + (4) \cdot (4)\},
\]
where \( 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor \).

Hence above defined labeling pattern satisfies the conditions of strongly*-labeling, i.e. \( M(C_n) \) is strongly*-graph.

- **Example 6.** Strongly*-labeling of \( M(C_k) \) is shown in Figure 6.

- **Theorem II.5.** Middle graph of path \( P_n \) is strongly*-graph for all \( n \in \mathbb{N} \).

**Proof.** Let 
\[
\{v_1, v_2, \ldots, v_n, v'_1, v'_2, \ldots, v'_{n-1}\}
\]
be the vertices of the \( M(P_n) \), where \( \{v_1, v_2, \ldots, v_n\} \) be the vertices corresponding to path \( P_n \) and \( \{v'_1, v'_2, \ldots, v'_{n-1}\} \) be the vertices corresponding to the edges \( \{e_1, e_2, \ldots, e_{n-1}\} \) of \( P_n \).

We define vertex labeling \( f : V(M(P_n)) \rightarrow \{1, 2, 3, \ldots, 2n - 1\} \) as follows. 
\[
f(v_i) = 2i - 1, \quad 1 \leq i \leq n.
\]
\[
f(v'_i) = 2i, \quad 1 \leq i \leq n - 1.
\]
The labels for the vertices \( v_i \) are consecutive odd numbers, whereas the labels for the vertices \( v'_i \) are consecutive even numbers. Therefore the labels produced for the edges \( v_iv'_i \) are odd and in increasing order. Further the edges \( v'_iv'_{i+1} \) and \( 1 \leq i \leq n - 2 \) are labeled with even label in increasing order. Hence all edges will have different label. So, labeling pattern defined above satisfies the conditions of strongly*-labeling, i.e. \( M(P_n) \) is strongly*-graph.

- **Example 7.** Strongly*-labeling of \( M(P_7) \) is shown in Figure 7.
Theorem II.6. The total graph of $P_n$ is strongly*-graph for all $n \in \mathbb{N}$.

Proof. Let \( \{v_1, v_2, \ldots, v_n, v_1', v_2', \ldots, v_{n-1}'\} \) be the vertices of $T(P_n)$, where \( \{v_1, v_2, \ldots, v_n\} \) be the vertices of path $P_n$ and \( \{v_1', v_2', \ldots, v_{n-1}'\} \) be the vertices corresponding to the edges \( \{e_1, e_2, \ldots, e_{n-1}\} \) of $P_n$.

We define vertex labeling function $f : V(T(P_n)) \to \{1, 2, 3, \ldots, 2n - 1\}$ as follows.

$$f(v_i) = 2i - 1, \quad 1 \leq i \leq n.$$  

$$f(v_i') = 2i, \quad 1 \leq i \leq n - 1.$$  

The labels for the vertices $v_i$ are consecutive odd numbers, whereas the labels for the vertices $v_i'$ are consecutive even numbers. Therefore the labels produced for the edges $v_iv_i'$ and $v_iv_{i+1}$ are odd and in increasing order. For the label of any of the remarking edges with common end vertices, we have \( \{(2i - 1) + (2i + 1) + (2i - 1) \cdot (2i + 1)\} \neq \{(2i + 2i + 1) + (2i - 1) \cdot (2i + 1)\} \neq \{(2i) + (2i - 1) + (2i) \cdot (2i + 1)\}, \quad 1 \leq i \leq n - 1. \) Further the edges incident with $v_i'$ and $v_{i+1}'$ (\( 1 \leq i \leq n - 2 \)) are labeled with even labels in increasing order. Hence all edges will have different labels. So, labeling pattern defined above satisfies the conditions of strongly*-labeling. i.e. $\text{M}(P_n)$ is strongly*-graph. \( \square \)

Example 8. Strongly*-labeling of $T(P_5)$ is shown in Figure 8 as an illustration for Theorem 6.

\[ \text{Fig. 8} \]

Theorem II.7. The split graph of $P_n$ is strongly*-graph for all $n \in \mathbb{N}$.

Proof. Let \( \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\} \) be the vertices of the spl$(P_n)$, where \( \{v_1, v_2, \ldots, v_n\} \) be the vertices of the path $P_n$ and \( \{u_1, u_2, \ldots, u_n\} \) are newly added vertices corresponding to the vertices of $P_n$ to obtain spl$(P_n)$.

We define vertex labeling function $f : V(\text{spl}(P_n)) \to \{1, 2, 3, \ldots, 2n\}$ as follows.

$$f(v_i) = 2i - 1, \quad 1 \leq i \leq n.$$  

$$f(u_i) = 2i, \quad 1 \leq i \leq n.$$  

Here for the label of any two edges with common vertex, we have \( \{(2i - 1) + (2i + 1) + (2i - 1) \cdot (2i + 1)\} \neq \{(2i - 1) + (2i + 2) + (2i - 1) \cdot (2i + 2)\} \neq \{2i + (2i + 1) + 2i \cdot (2i + 1)\}. \) Therefore above defined labeling pattern satisfies the conditions of strongly*-graph. i.e. spl$(P_n)$ is strongly*-graph.

\[ \square \]

Example 9. Strongly*-labeling of spl$(P_7)$ is shown in Figure 9 as an illustration for Theorem 7.

\[ \text{Fig. 9} \]

Theorem II.8. The shadow graph $D_2(P_n)$ is strongly*-graph for all $n \in \mathbb{N}$.

Proof. Let \( \{v_1, v_2, \ldots, v_n, v_1', v_2', \ldots, v_n'\} \) be the vertices of the $D_2(P_n)$, where \( \{v_1, v_2, \ldots, v_n\} \) are the vertices of path $P_n$ and \( \{v_1', v_2', \ldots, v_n'\} \) are the vertices added corresponding to the vertices \( \{v_1, v_2, \ldots, v_n\} \) in order to obtain $D_2(P_n)$.

We define vertex labeling function $f : V(D_2(P_n)) \to \{1, 2, 3, \ldots, 2n\}$ as follows.

$$f(v_i) = 2i - 1, \quad 1 \leq i \leq n.$$  

$$f(v_i') = 2i, \quad 1 \leq i \leq n.$$  

\[ \square \]
The labels for the vertices \( v_i \) are consecutive odd numbers, whereas the labels for the vertices \( v'_i \) are consecutive even numbers. Therefore the labels produced for the edges \( v'_i v'_{i+1} \) are even labels in increasing order the labels for edges \( v_i v_{i+1}, v'_i v'_{i+1} \) and \( v'_i v_{i+1} \) are odd labels in increasing order. Also for any two edge labels produce in the graph, 
\[
{(2i - 1) + (2i + 1) + (2i - 1) \cdot (2i + 1)} \neq \\
{(2i - 1) + (2i + 2) + (2i - 1) \cdot (2i + 2)} \neq \\
{2i + (2i + 1) + 2i \cdot (2i + 1)}
\]
for any \( i \). So, the labeling pattern defined above satisfies the conditions of strongly\(^*\)-graph. i.e. \( D_2(P_n) \) is strongly\(^*\)-graph.

\[\square\]

**Example 10.** Strongly\(^*\)-labeling of \( D_2(P_6) \) is shown in Figure 10 as an illustration for Theorem 8.

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