Common Fixed Point Theorems for Weakly Compatible Maps in Intuitionistic Fuzzy Metric Space

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Abstract — The aim of this paper is to prove a common fixed point theorem in an intuitionistic fuzzy metric space by using the notion of property E.A.

Keywords — Intuitionistic fuzzy metric space, property E.A., fixed point, weakly compatible maps.

I. INTRODUCTION


Definition 2.2.[13] A binary operation : [0,1] × [0,1] → [0,1] is continuous t-conorm if satisfies the following conditions:
(i) ◊ is commutative and associative
(ii) ◊ is continuous
(iii) ◊ a ◊ b = a for all a ∈ [0, 1]
(iv) ◊ a ◊ b ≤ c ◊ d whenever a ≤ c and b ≤ d for all a, b, c, d ∈ [0, 1].

Alaca et al. [2] using the idea of Intuitionistic fuzzy sets, defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norm and continuous t-conorms as a generalization of fuzzy metric space to Kramosil and Michalek [7] as :

Definition 2.3. [2] A 5-tuple (X, M, N, *, ◊) is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, * is a continuous t-norm, ◊ is a continuous t-conorm and M, N are fuzzy sets on X² × [0, ∞) satisfying the following conditions:
(i) M(x, y, t) + N(x, y, t) ≤ 1 for all x, y ∈ X and t > 0
(ii) M(x, y, 0) = 0 for all x, y ∈ X
(iii) M(x, y, t) = 1 for all x, y ∈ X and t > 0
if and only if x = y
(iv) M(x, y, t) = M(y, x, t) for all x, y ∈ X and t > 0
(v) M(x, y, t) * M(y, z, s) ≤ M(x, z, t + s) for all x, y, z ∈ X and s, t > 0
(vi) for all x, y ∈ X, M(x, y, .) : [0, ∞)→ [0, 1] is left continuous
(vii) lim —— M(x, y, t) = 1; for all x, y ∈ X and t > 0
t→∞
(viii) N(x, y, 0) = 1; for all x, y ∈ X
(ix) N(x, y, t) = 0; for all x, y ∈ X and t > 0
if and only if x = y
(x) N(x, y, t) = N(y, x, 0); for all x, y ∈ X and t > 0

II. PRELIMINARIES

The concept of triangular norms (t-norms) and triangular connorms (t-conorm) were originally introduced by Schweizer and Sklar (1960) [13] in the study of statistical metric spaces.

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(iii) a * 1 = a for all a ∈ [0, 1]
(iv) a * b ≤ c * d whenever a ≤ c and b ≤ d for all a, b, c, d ∈ [0, 1].
(xi) $N(x, y, t) \land N(y, z, s) \geq N(x, z, t+s)$; for all $x, y, z \in X$ and $t, s > 0$

(xii) for all $x, y \in X$, $N(x, y, t) : [0, +\infty) \to [0, 1]$ is right continuous

(xiii) $\lim_{n \to +\infty} N(x, y, t) = 0$; for all $x, y \in X$.

Then $(M, N)$ is called an intuitionistic fuzzy metric space on $X$. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between $x$ and $y$ with respect to $t$ respectively.

**Remark 2.1** [2] Every fuzzy metric space $(X, M, *)$ is an intuitionistic fuzzy metric space of the form $(X, M, 1-M, *, \diamond)$ such that $t$-norm * and t-conorm $\diamond$ are associated as $x \circ y = ((1-x)(1-y)); \forall x, y \in X$

**Remark 2.2** [2] In intuitionistic fuzzy metric space $(X, N, *, \diamond)$, $M(x, y, .)$ is non-decreasing and $N(x, y, .)$ is non-increasing for all $x, y \in X$.

Alaca et al. [2] introduced the following notions:

**Definition 2.4** [2] Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. Then

(a) a sequence $\{x_n\}$ in $X$ is said to be Cauchy sequence if, for all $t > 0$ and $p > 0$, $\lim_{n \to +\infty} M(x_{n+p}, x_n, t) = 1$ and

$\lim_{n \to +\infty} N(x_{n+p}, x_n, t) = 0$

(b) a sequence $\{x_n\}$ in $X$ is said to be convergent to a point $x \in X$ if, for all $t > 0$,

$\lim_{n \to +\infty} M(x_n, x, t) = 1$

and

$\lim_{n \to +\infty} N(x_n, x, t) = 0$

**Definition 2.5** [2] An intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is said to be complete if and only if every Cauchy sequence in $X$ is convergent.

**Theorem**

Let $(X, M, N, *, \diamond)$ be a complete intuitionistic fuzzy metric space with continuous $t$-norm and continuous $t$-conorm defined by $a \land a \geq a$ and $(1-a) \circ (1-a) \leq (1-a)$ where $a \in [0, 1]$. Let $g$ and $h$ be two weakly compatible mappings of $X$ satisfying the following conditions:

(a) $g$ and $h$ satisfy the property E.A.
(b) for $x, y \in X$, $t > 0$ there exist $0 < k < 1$ such that

$M(hx, hy, kt) \geq \min[M(hx, gx, t) \land M(gx, hy, t)]$

for $x, y \in X, k > 0$.

(c) $g(X)$ and $h(X)$ are complete subspaces of $X$. Then $g$ and $h$ have a unique common fixed point.

**Proof** – Let $\{x_n\}$ is a sequence in $X$ such that

$\lim_{n \to +\infty} gx_n = \lim_{n \to +\infty} hx_n = u$ for some $u \in X$ by property E.A.

Since $g(X)$ is a complete subspace of $X$, therefore, every convergent sequence of points of $g(X)$ has a limit point in $g(X)$.

Now we prove that $g = h$

Take $x = x_0$ and $y = a$ in (3.1) and (3.2)

$M(hx_0, hx_0, kt) \geq \min[M(hx_0, gx_0, t) \land M(gx_0, hx_0, t)]$

and

$N(hx_0, hx_0, kt) \leq \max[N(hx_0, gx_0, t) \land N(gx_0, hx_0, t)]$

Taking limit $n \to +\infty$ on both the sides, we get

$M(ga, ga, t) \geq \min[M(ga, ga, t) \land M(ga, ga, t)]$ and

$N(ga, ga, t) \leq \max[N(ga, ga, t) \land N(ga, ga, t)]$

$\Rightarrow M(ga, ga, t) \geq \min[1 \land M(ga, ga, t)]$ and

$N(ga, ga, t) \leq \max[0 \land N(ga, ga, t)]$

$\Rightarrow M(ga, ga, t) \geq M(ga, ga, t)$ and

It is easy to see that two compatible maps are weakly compatible.
\[ N(ga, ha, kt) \leq N(ga, ha, t), \text{[by Definition (2.1) and (2.2)]} \]

\[ \Rightarrow ga = ha \quad \text{[by Lemma (2.1)]} \]

Therefore \( u = ga = ha \) \hspace{0.5cm} (3.3)

This shows that a coincident point of \( h \) and \( g \).

Since \( h \) and \( g \) are weakly compatible.

Therefore \( hga = gha = gha \) \hspace{0.5cm} (3.4)

Now we show that \( h \) is the common fixed point of \( h \) and \( g \).

Now we take \( x = a \) and \( y = ha \)

\[ M(ha, hha, kt) = \min[M(ha, ga, t) \ast M(ga, hha, t), M(ga, ha, t) \ast M(ga, hha, t)] \]

and

\[ N(ha, hha, kt) \leq \max[N(ha, ga, t) \odot N(ga, hha, t), N(ha, ha, t) \odot N(ga, hha, t)] \]

\[ \Rightarrow M(ha, hha, kt) \geq \min[M(ha, ga, t) \ast M(ga, hha, t), M(ga, ha, t) \ast M(ga, hha, t)] \]

and

\[ N(ha, hha, kt) \leq \max[N(ha, ga, t) \odot N(ga, hha, t), N(ha, ha, t) \odot N(ga, hha, t)] \]

\[ \Rightarrow M(ha, hha, kt) \geq \min[M(ha, ga, t) \ast M(ga, hha, t), M(ha, hha, t) \ast M(ha, hha, t)] \]

and

\[ N(ha, hha, kt) \leq \max[N(ha, ga, t) \odot N(ha, hha, t), N(ha, ha, t) \odot N(ha, hha, t)] \]

\[ \Rightarrow M(ha, hha, kt) \geq \min[1 \ast M(ha, hha, t), M(ha, hha, t) \ast 1] \] \hspace{0.5cm} (3.5)

and by (3.2)

\[ N(u, v, kt) = N(hu, hv, kt) \leq \max[N(hu, gu, t) \odot N(gu, hv, t), N(gu, hv, t) \odot N(gu, hv, t)] \]

\[ \Rightarrow N(u, v, t) \leq N(u, v, t) \]

Hence \( h \) and \( g \) have a unique common fixed point.

This completes the proof.

**REFERENCES**


