ON COMMUTATIVITY OF *-PRIME NEAR-RINGS WITH DERIVATIONS

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Abstract: The primary purpose of this paper is to introduce the notion of *-prime near-rings, which is a special class of distributive near-rings and to investigate their commutativity. Let \( N \) be a left near-ring. \( N \) is called distributive near-ring if \((x + y)z = xz + yz\) for all \( x, y, z \in N \). Further, an additive mapping \( x \mapsto x^* \) on \( N \) is said to be an involution on \( N \) if (i) \((x^*)^* = x\) and (ii) \((xy)^* = y^*x^*\) hold for all \( x, y \in N \). A near-ring equipped with an involution ‘*’ is called a *-near-ring. A *-near-ring \( N \) is called *-prime near-ring if \( xNy = xNy^* = \{0\}\) implies that either \( x = 0 \) or \( y = 0 \). Analogues of some ring theoretic results, regarding commutativity have been obtained in the setting of *-prime near-rings satisfying some properties and identities involving derivations.

1. INTRODUCTION

Throughout the present paper, unless otherwise mentioned, \( N \) will denote a left near-ring. \( N \) is called a prime near-ring if \( xNy = \{0\}\) implies \( x = 0 \) or \( y = 0 \). It is called semiprime if \( xNx = \{0\}\) implies \( x = 0 \). Given an integer \( n > 1 \), near-ring \( N \) is said to be \( n \)-torsion free, if for \( x \in N, nx = 0 \) implies \( x = 0 \). If \( K \) is a nonempty subset of \( N \), then a normal subgroup \((K, +)\) of \((N, +)\) is called a right ideal (resp. a left ideal) of \( N \) if \((x + k)y - xy \in K\) (resp. \(xk \in K\)) holds for all \( x, y \in N \) and for all \( k \in K \). \( K \) is called an ideal of \( N \) if it is both a left ideal as well as a right ideal of \( N \). The symbol \( Z \) will denote the multiplicative center of \( N \), that is, \( Z = \{x \in N \mid xy = yx \text{ for all } y \in N\} \). For any \( x, y \in N \) the symbol \([x, y] = xy - yx\) stands for multiplicative commutator of \( x \) and \( y \), while the symbol \( xoy \) will represent \( xy + yx \). For terminologies concerning near-rings, we refer to G.Pilz [1, 2]. Following [3], an additive mapping \( d : N \rightarrow N \) satisfying \( d(xy) = xd(y) + d(x)y \) for all \( x, y \in N \) is called a derivation on \( N \). A *-near ring \( N \) is called *-prime near-ring if \( xNy = xNy^* = \{0\}\) implies that either \( x = 0 \) or \( y = 0 \). Let \( N \) be a *-near-ring. An ideal \( I \) of \( N \) is called *-ideal if \( I^* = I \). An element \( x \in N \) is called a symmetric element if \( x^* = x \) and an element \( x \in N \) is called a skewsymmetric element if \( x^* = -x \). We denote the collection of all symmetric and skewsymmetric elements

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of $N$ by $Sa_*(N)$ i.e.; $Sa_*(N) = \{x \in N \mid x^* = \pm x\}$. There has been a lot of work on commutativity of $*$-prime rings costrained with derivations (see 4 – 7, where further references can be found). Motivated by these works, we have investigated commutativity of $*$-prime near-rings constrained with derivations.

2. PRELIMINARY RESULTS

We begin with the following lemmas which are essential for developing the proofs of our main results.

**Lemma 2.1.** Let $N$ be a $*$-near-ring. Then

(i) $N$ is a distributive near-ring.

(ii) $xy + zt = zt + xy$ for all $x, y, z, t \in N$.

(iii) $n(xy) = (nx)y = x(ny)$ for all $x, y \in N$ and $n \in Z$, where $Z$ stands for the set of integers.

(iv) $[x, y + z] = [x, y] + [x, z]$ and $[x + y, z] = [x, z] + [y, z]$ for all $x, y, z \in N$.

(v) $[x, y]z = y[x, z] + [x, y]z$ and $[xy, z] = x[y, z] + [x, z]y$ for all $x, y, z \in N$.

(vi) If $I$ is an ideal of $N$ then $NI \subseteq I$ and $IN \subseteq I$.

*Proof.* (i) For all $x, y, z \in N$ we have $\{(y + z)x\}^* = x^*y^* + x^*z^*$, now taking the image of both the sides under $*$ we get $(y + z)x = yx + zx$. This means that $N$ is a distributive near-ring.

(ii) Since $N$ has both distributive properties, expanding $(x + z)(t + y)$ for all $x, y, z, t \in N$, we have $xt + xy + zt + zy = xt + zt + xy + zy$. This implies our required result.

(iii) Since $(N, +)$ is a group and $N$ has both distributive properties, the result is obvious.

(iv) Using both distributive properties of $N$ and (ii), we get the result.

(v) Same trick as used in (iv).

(vi) Under hypothesis it is a trivial fact.
Lemma 2.2. Let N be a *-near-ring.

(i) If N is a prime near-ring then it is a *-prime near-ring.

(ii) If N is *-prime near-ring then it is a semiprime near-ring.

(iii) N is *-prime near-ring if and only if xNy = x*Ny = {0} yields x = 0 or y = 0.

Proof. (i) Suppose that xNy = x*Ny* = {0}. If first case holds then primeness of N insures that either x = 0 or y = 0. On the other hand if second case holds then primeness of N again provides us either x = 0 or y* = 0. Including both the cases we arrive at either x = 0 or y = 0. Hence N is *-prime near-ring.

(ii) Assume that xNx = {0} then xNx: N*x* = {0}. By *-primeness of N we get that either x = 0 or Nx* = {0}. But Nx* = {0} together with xNx = {0} implies that x = 0.

(iii) Let N be a *-prime near-ring. Further suppose that xNy = x*Ny = {0}. This provides us y*Nx* = y*Nx = {0}. Using *-primeness of N we obtain that either y* = 0 or x = 0. This implies that either x = 0 or y = 0. Converse can be proved in a similar way.

Lemma 2.3. Let N be a *-prime near-ring.

(i) If Z ≠ {0} then N is a ring.

(ii) If z ∈ Z\{0} and x is an element of N such that xz, xz* ∈ Z (resp. xz, x*z ∈ Z) then x ∈ Z.

Proof. (i) Since Z ≠ {0}, there exists 0 ≠ z ∈ Z. By Lemma 2.1 we obtain that zx + zy = zy + zx for all x, y ∈ N. Now we infer that z(x + y − x − y) = 0 for all x, y ∈ N. This implies that zN(x + y − x − y) = {0} and zN(x + y − x − y)* = {0}. Now *-primeness of N provides us x + y = y + x for all x, y ∈ N. Hence (N, +) is abelian. Using Lemma 2.1 again we conclude that N is a ring.

(ii) If xz, xz* ∈ Z, we have xzr = rxz and xz*r = rxz* for all r ∈ N. It is obvious that z* ∈ Z. These facts provide us zN[x, r] = {0} and z*N[x, r] = {0} for all r ∈ N. Using Lemma 2.2 we obtain that x ∈ Z. On the other hand if xz, x*z ∈ Z, we have xzr = rxz and x*zr = rxz* for all r ∈ N. It follows that zN[x, r] = {0} and z*N[x*, r] = {0} for all r ∈ N. Replacing r by r* in the relation zN[x*, r] = {0} we obtain that zN[x*, r*] = {0} i.e.; zN[x*, r*] = {0}. Now we arrive at zN[x, r] = {0} and zN[x, r]* = {0} for all r ∈ N.
Finally $*$-primeness of $N$ finishes the proof.

In the year 2006, L. Oukhtite and S. Salhi [4, Lemma 3.1] proved that if $R$ is a $*$-prime ring possessing a nonzero $*$-ideal $I$ and $x,y \in R$ such that $xIy = \{0\} = xIy^*$, then $x = 0$ or $y = 0$. We have obtained its analogue in the setting of $*$-prime near-rings.

**Lemma 2.4.** Let $N$ be a $*$-prime near-ring and $I$ be a nonzero $*$-ideal of $N$. If $x,y \in N$ satisfy $xIy = xIy^* = \{0\}$ (resp. $xIy = x^*Iy = \{0\}$), then $x = 0$ or $y = 0$.

**Proof.** Assume $x \neq 0$, there exists some $z \in I$ such that $xz \neq 0$. For otherwise $xNy = \{0\}$ and $xNy^* = \{0\}$ for all $y \in I$ and thus $*$-primeness of $N$ gives us $x = 0$. Since $xINy = \{0\}$ and $xINy^* = \{0\}$, we then obtain $xzNy = xzNy^* = \{0\}$. Now $*$-primeness of $N$ provides us $y = 0$. Using similar arguments with necessary variations one can easily prove that $xIy = x^*Iy = \{0\}$ implies that $x = 0$ or $y = 0$.

Recently, L. Oukhtite and S. Salhi [6, Lemma 2 — 5] studied derivations in $*$-prime rings and proved the following: Let $R$ be a $*$-prime ring having nonzero $*$-ideal $I$ then (i) If $d$ is a nonzero derivation on $R$ which commutes with $*$ and $[x,R]Id(x) = \{0\}$ for all $x \in I$, then $R$ is commutative. (ii) If $d$ is a nonzero derivation on $R$ which commutes with $*$ and $[d(x),x] = 0$ for all $x \in I$, then $R$ is commutative. (iii) Let $d$ be a derivation of $R$ satisfying $d^* = \pm *d$. If $d^2(I) = \{0\}$, then $d = 0$. (iv) Let $d_1$ and $d_2$ be derivations of $R$ such that $d_1^* = \pm *d_1$ and $d_2^* = \pm *d_2$. If $d_2(I) \subseteq I$ and $d_1d_2(I) = \{0\}$, then $d_1 = 0$ or $d_2 = 0$. We have obtained the analogues of these results in the setting of $*$-prime near-rings as below.

**Lemma 2.5.** Let $N$ be a $*$-prime near-ring admitting a nonzero derivation $d$, which commutes with $*$. If $I$ is a nonzero $*$-ideal of $N$ and $[x,N]Id(x) = \{0\}$ for all $x \in I$, then $N$ is a commutative ring.

**Proof.** Let $x \in I$. Since $y = x - x^* \in I$, then $[y,z]Id(y) = 0$ for all $z \in N$. As $y \in S_{a*}(N)$, then using Lemma 2.1 we arrive at $[y,z]Id(y) = [y,z]^*Id(y) = \{0\}$ for all $z \in N$. By Lemma 2.4 we obtain that $d(y) = 0$ or $[y,z] = 0$ for all $z \in N$. If $d(y) = 0$, then $d(x) = d(x^*) = (d(x))^*$. Therefore $[x,z]Id(x) = [x,z]Id(x)^* = \{0\}$ and by Lemma 2.4 we infer that either $d(x) = 0$ or $x \in Z$. On the other hand if $[y,z] = 0$ for all $z \in N$, then $y \in Z$ and hence $[x-x^*,z] = 0$ for all $z \in N$. By Lemma 2.1 we have that $[x,z] = [x^*,z]$ for all $z \in N$. Therefore using Lemma 2.1 again we obtain that $[x,z]Id(x) = [x,z]^*Id(x) = \{0\}$. Again using Lemma 2.4 we get either $d(x) = 0$ or $x \in Z$. Now we conclude that for each $x \in I$ either $d(x) = 0$ or $x \in Z$.
Let us consider $H = \{x \in I \mid d(x) = 0\}$ and $K = \{x \in I \mid x \in Z\}$. Using Lemma 2.1 it can be easily shown that $H$ and $K$ are additive subgroups of $I$ such that $I = H \cup K$. But a group can not be a union of two of its proper subgroups and hence $I = H$ or $I = K$. If $I = H$, then $d(x) = 0$ for all $x \in I$. For any $t \in N$, replacing $x$ by $xt$ we get $xd(t) = 0$, for all $x \in I$ i.e.; $Id(t) = \{0\}$ for all $t \in N$. In particular $pId(t) = p^*Id(t) = \{0\}$ for all $t \in N$, where $0 \neq p \in N$. Now Lemma 2.4 gives us $d = 0$, a contradiction. Hence $I = K$ so that $I \subseteq Z$. $I \neq \{0\}$ implies that $Z \neq \{0\}$. Hence by Lemma 2.3, $N$ is a ring. Let $z, t \in N$ and $x \in I$. From $ztx = zxt = tzx$ we conclude that $[z, t]I = \{0\}$ and then $[z, t]Ip = [z, t]Ip^* = \{0\}$, where $0 \neq p \in N$. In view of Lemma 2.4, we conclude that $[z, t] = 0$ for all $z, t \in N$. Therefore, $N$ is a commutative ring.

**Lemma 2.6.** Let $N$ be a $*$-prime near-ring admitting a nonzero derivation $d$, which commutes with $*$. If $I$ is a nonzero $*$-ideal of $N$ and $[d(x), x] = 0$ for all $x \in I$, then $N$ is a commutative ring.

**Proof.** Let $x, y \in I$. Linearizing $[d(x), x] = 0$ with the help of Lemma 2.1 and using the hypothesis we get

$$[d(x), y] + [d(y), x] = 0$$

(2.1)

for all $x, y \in I$. Now replacing $y$ by $yx$ and using Lemma 2.1 we obtain that

$$[d(x), y]x + [d(y), x]x + [y, x]d(x) = 0$$

(2.2)

for all $x, y \in I$. Relations 2.1 and 2.2 yield $[x, y]d(x) = 0$, for all $x, y \in I$. Thus, for any $z \in N$, we have $[x, zy]d(x) = [x, z]yd(x) = 0$ by Lemma 2.1 and therefore $[x, N]Id(x) = \{0\}$ for all $x \in I$. Finally by Lemma 2.5 we get the required result.

**Lemma 2.7.** Let $N$ be a 2-torsion free $*$-prime near-ring admitting a derivation $d$ such that $d* = \pm * d$. If $I$ is a nonzero $*$-ideal of $N$ and $d^2(I) = \{0\}$, then $d = 0$.

**Proof.** For any $x \in I$, we have $d^2(x) = 0$. Putting $xy$ for $x$ where $y \in I$, and using Lemma 2.1 we arrive at $d^2(x)y + 2d(x)d(y) + xd^2(y) = 0$ for all $x, y \in I$. 2-torsion freeness of $N$ and $d^2(I) = \{0\}$, provide us $d(x)d(y) = 0$. Replacing $x$ by $xz$ where $z \in I$ in the last relation and using Lemma 2.1, we get $d(x)zd(y) = 0$ for all $x, y, z \in N$ i.e.; $d(x)Id(y) = \{0\}$. Since $d* = \pm * d$, by Lemma 2.4 I infer that $d(x) = 0$ for all $x \in I$. Now replacing $x$ by $xt$ where $t \in N$ we have $xd(t) = 0$ and therefore $INd(t) = \{0\}$ for all $t \in N$. Since $I$ is a nonzero $*$-ideal and $N$ is a $*$-prime near-ring, We get $d(z) = 0$ for all $z \in N$ and consequently $d = 0$. 

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Lemma 2.8. Let \( N \) be a 2-torsion free \(*\)-prime near-ring admitting derivations \( d_1 \) and \( d_2 \) such that \( d_1^* = \pm * d_1 \) and \( d_2^* = \pm * d_2 \). If \( I \) is a nonzero \(*\)-ideal of \( N \) such that \( d^2(I) \subseteq I \) and \( d_1d_2(I) = 0 \), then \( d_1 = 0 \) or \( d_2 = 0 \).

Proof. Let \( x, y \in I \). Then \( d_1d_2(xy) = d_1(x)d_2(y) + d_2(x)d_1(y) = 0 \). Replacing \( x \) by \( d_2(x) \) we get \( d^2_2(x)d_1(y) = 0 \) for all \( x, y \in I \). Now putting \( yz \) where \( z \in I \) for \( y \) we obtain that \( d^2_2(x)yd_1(z) = 0 \) for all \( x, y, z \in I \). It then gives us \( d^2_2(x)Id_1(z) = \{0\} \) for all \( x, z \in I \). The conditions \( I^* = I \) and \( d_1^* = \pm * d_1 \) provide us \( d^2_2(x)Id_1(z) = d^2_2(x)Id_1(z) \) and by Lemma 2.4 it follows that either \( d_1(z) = 0 \) for all \( z \in I \) or \( d^2_2(x) = 0 \) for all \( x \in I \). If \( d_1(z) = 0 \) for all \( z \in I \), then replacing \( z \) by \( zt \) where \( t \in N \), we obtain that \( zd_1(t) = 0 \) i.e.; \( Id_1(t) = \{0\} \). As \( d_1^* = \pm * d_1 \), this implies that \( pId_1(t) = p\{d_1(t)\}^* = \{0\} \) for \( 0 \neq p \in N \). In view of Lemma 2.4 we obtain that \( d_1 = 0 \). On the other hand if \( d^2_2(x) = 0 \) for all \( x \in I \), we obtain by Lemma 2.7 that \( d_2 = 0 \).

3. MAIN RESULTS

In the year 2006, L. Oukhtite and S. Salhi [4, Theorem 3.2] obtained the following: Let \( d \) be a nonzero derivation of a 2-torsion free \(*\)-prime ring \( R \) and \( I \) a nonzero \(*\)-ideal of \( R \). If \( r \in Sa_s(R) \) satisfies \([d(x), r] = 0 \) for all \( x \in I \), then \( r \in Z(R) \). Furthermore, if \( d(I) \subseteq Z(R) \), then \( R \) is commutative. We have obtained its analogue for \(*\)-prime near-rings with derivation.

Theorem 3.1. Let \( N \) be a 2-torsion free \(*\)-prime near-ring admitting a nonzero derivation \( d \) and a nonzero \(*\)-ideal \( I \). If \( t \in Sa_s(N) \) satisfies \([d(x), t] = 0 \) for all \( x \in I \), then \( t \in Z \). Furthermore, if \( d(I) \subseteq Z \), then \( N \) is a commutative ring.

Proof. Since \([d(xy), t] = 0 \) for all \( x, y \in I \), using Lemma 2.1 it provides us \( d(xy)t + xd(y)t - td(x)y - txd(y) = 0 \). Conditions \([d(x), t] = [d(y), t] = 0 \) give us

\[
d(x)[y, t] + [x, t]d(y) = 0
\]

(3.1)

for all \( x, y \in I \). Replacing \( y \) by \( yt \) and using Lemma 2.1, we conclude that \([x, t]Id(t) = \{0\} \). The fact that \( I \) is a \(*\)-ideal together with \( t \in Sa_s(N) \) and Lemma 2.1, provide \([x, t]Id(t) = \{0\} \). Applying Lemma 2.4, either \( d(t) = 0 \) or \([x, t] = 0 \). If \( d(t) \neq 0 \), then \([x, t] = 0 \) for all \( x \in I \). Let \( s \in N \), from \([sx, t] = 0 \) it follows by Lemma 2.1 that \([s, t]x = 0 \). Let \( 0 \neq x_0 \in I \), as \([s, t]Nx_0 = [s, t]N dx_0^* \). Since \( N \) is \(*\)-prime near-ring, which proves that \([s, t] = 0 \) i.e.; \( t \in Z \). On the other hand if \( d(t) = 0 \), then \( d([x, t]) = [d(x), t] = 0 \) and consequently

\[
d([I, t]) = \{0\}.
\]

(3.2)
Replacing $y$ by $yz$ where $z \in I$ in the relation (3.1) and using Lemma 2.1, we see that
\[ d(x)y[z,t] + [x,t]yd(z) = 0. \quad (3.3) \]
Taking $[z,t]$ instead of $z$ in relation (3.3) and applying relation (3.2) and Lemma 2.1 we then arrive at $d(x)y[[z,t],t] = 0$ so that $d(x)I[[z,t],t] = \{0\} = d(x)I[[z,t],t]^*$. Hence $d(I) = \{0\}$ or $[[z,t],t] = 0$ for all $z \in I$, by Lemma 2.4. If $d(I) = \{0\}$, then for any $s \in S$ we get $d(sx) = d(s)x = 0$ for all $x \in I$. Therefore $d(s)NI = \{0\} = d(s)NI^*$ and as $I$ is nonzero, then *-primeness of $N$ provides us $d(t) = 0$ which implies that $d = 0$, a contradiction. Thus we conclude that $[[z,t],t] = 0$. Now putting $x \in Z$ and using Lemma 2.1 we obtain that $0 = [[zx,t],t] = [z,t][x,t] + [z,t][x,t]$. It follows that $[z,t][x,t] = 0$, by 2-torsion freeness of $N$. Replacing $z$ by $z$ where $s \in N$ and using Lemma 2.1 again we obtain that $0 = [sz,t][x,t] = [s,t]z[x,t]$ and consequently $[s,t]I[x,t] = \{0\}$ for all $x \in I$. Therefore by Lemma 2.1, $[s,t]I[x,t] = [s,t]I[x,t]^* = \{0\}$. Once again using Lemma 2.4 we arrive at $[s,t] = 0$ or $[x,t] = 0$. If $[s,t] = 0$, then $t \in Z$. If $[x,t] = 0$ for all $x \in I$, then for any $s \in N$ we have $0 = [sx,t] = s[x,t] + [s,t]x = [s,t]x$ by Lemma 2.1. Hence $\{0\} = [t]I = [s,t]Ip = [s,t]Ip^*$, where $0 \neq p \in N$. Using Lemma 2.4 once again we conclude that $[s,t] = 0$, which proves that $t \in Z$.

Now suppose that $d(I) \subseteq Z$. Hence from the first part of the theorem we conclude that $Sa_*(N) \subseteq Z$. It is obvious that for each $s \in N$, $s - s^* \in Sa_*(N)$. Now for any given $s \in N$, there are two possibilities either $s - s^* \neq 0$, or $s - s^* = 0$. If first case occurs, then $0 \neq s - s^* \in Z$. Therefore in this case $Z \neq \{0\}$ and by Lemma 2.3, $N$ becomes a ring. If first case does not occur, then $s = s^*$ for all $s \in N$. Which implies that $s_1s_2 = (s_1s_2)^* = s_2^*s_1 = s_2s_1$ for all $s_1, s_2 \in N$ and we conclude that in this case $N = Z$ i.e.; $Z \neq \{0\}$. Again Lemma 2.3 shows that $N$ is a ring. Finally we get a fact that if $d(I) \subseteq Z$ holds, then $N$ is a ring. As $N$ is a ring, $s + s^*, s - s^* \in Sa_*(N)$. We then obtain $s + s^*, s - s^* \in Z$ and hence $2s \in Z$. Since $N$ is 2-torsion free, then $s \in Z$ proving the commutativity of $N$. Hence $N$ is a commutative ring.

Recently, L.Oukhtite and S.Salhi [4, Theorem 3.3] proved the following: Let $d$ be a nonzero derivation of a 2-torsion free *-prime ring $R$ and let $t \in Sa_*(R)$. If $d([R,t]) = 0$, then $t \in Z(R)$. In particular, if $d[x,y] = 0$, for all $x, y \in R$, then $R$ is commutative. We have obtained its analogue in the setting of *-prime near-rings with derivation.

**Theorem 3.2.** Let $N$ be a 2-torsion free *-prime near-ring admitting a nonzero derivation $d$ and let $t \in Sa_*(N)$. If $d([N,t]) = 0$, then $t \in Z$. In particular, if $d[x,y] = 0$, for all $x, y \in N$, then $N$ is commutative ring.

**Proof.** If $d(t) = 0$, from our hypothesis and using Lemma 2.1, for any $x \in N$ we obtain,
\[ 0 = d([x,t]) = d(x)t + xd(t) - dt(x) = d(A)d(t) - td(x) = d(x)t - td(x) = [d(x),t]. \] Hence we arrive at
\[ [d(x), t] = 0 \text{ for all } x \in N. \] Applying Theorem 3.2, this gives \( t \in Z \) and in this case proof finishes. Now assume that \( d(t) \neq 0 \). For all \( x \in N \), we have \( 0 = d([tx, t]) = d(t[x, t]) = td[x, t] + d(t)[x, t] \). This implies that

\[ d(t)[x, t] = 0. \tag{3.4} \]

Putting \( xy \) where \( y \in N \) for \( x \), and using Lemma 2.1 we arrive at \( 0 = d(t)[xy, t] = d(t)x[y, t] + d(t)[x, t]y \). Which reduces to \( d(t)x[y, t] = 0 \) i.e.; \( d(t)R[y, t] = \{0\} \) for all \( y \in R \), with the help of relation (3.4). Since \( t \in Sa_*(N) \), then by Lemma 2.1 we have \( d(t)R[y, t] = d(t)R[y, t]^* = \{0\} \). Now \(*\)-primeness of \( N \) insures that \( [y, t] = 0 \) i.e.; \( t \in Z \). Now suppose that \( d(x, y) = 0 \), for all \( x, y \in N \) and using first part of the theorem, we conclude that \( Sa_*(N) \subseteq Z \). Further onward using the same argument as used in the Theorem 3.1, we obtain that \( N \) is a commutative ring.

Recently, L.Oukhtite and S.Salhi [6, Theorem 1] proved the following: Let \( R \) be a 2-torsion free \(*\)-prime ring, admitting a nonzero derivation \( d \), which commutes with \(*\) and \( I \) a nonzero \(*\)-ideal. If \( [d(x), x] \in Z(R) \), for all \( x \in I \), then it is commutative. We have proved its analogue for \(*\)-prime near-rings with derivation.

**Theorem 3.3.** Let \( N \) be a 2-torsion free \(*\)-prime near-ring, admitting a nonzero derivation \( d \), which commutes with \(*\). If \( [d(x), x] \in Z \), for all \( x \in I \), then \( N \) is a commutative ring.

**Proof.** Linearizing \( [d(x), x] \in Z \) with the help of Lemma 2.1, we arrive at

\[ [d(x), y] + [d(y), x] \in Z \text{ for all } x, y \in I. \]

Replacing \( y \) by \( x^2 \) and using Lemma 2.1, we obtain that \( 4x[d(x), x] \in Z \). Now 2-torsion freeness of \( N \) forces \( x[d(x), x] \in Z \) for all \( x \in I \). Thus for any \( t \in N \), we have that \( tx[d(x), x] = x[d(x), x]t = xt[d(x), x] \) and so by Lemma 2.1 we arrive at \( [t, x][d(x), x] = 0 \), for all \( x \in I \) and for all \( t \in N \). Replacing \( t \) by \( d(x) \), we obtain \( [d(x), x]^2 = 0 \) for all \( x \in I \). Since \( [d(x), x] \in Z \), then \( [d(x), x]N[d(x), x][d(x), x]^* = \{0\} \). As \( [d(x), x][d(x), x]^* \in Sa_*(N) \). Now \(*\)-primeness of \( N \) provides us \( [d(x), x] = 0 \) or \( [d(x), x][d(x), x]^* = 0 \). Suppose \( [d(x), x][d(x), x]^* = 0 \) holds then the condition \( [d(x), x] \in Z \) gives us \( [d(x), x]N[d(x), x]^* = \{0\} \). Including the both cases we infer that \( [d(x), x]N[d(x), x]^* = \{0\} = [d(x), x]N[d(x), x] \). \(*\)-primeness of \( N \) yields \( [d(x), x] = 0 \) for all \( x \in I \). Now using Lemma 2.6, we get our required result.

In the year 2007, L.Oukhtite and S.Salhi [7, Theorem 1.2-1.3] obtained the following results: Let \( R \) be a 2-torsion free \(*\)-prime ring admitting a nonzero derivation \( d \), which commutes with \(*\) and \( I \) a nonzero \(*\)-ideal. If \( R \) satisfies any one of the following conditions: (i) \( [d(x), d(y)] = 0 \), for all \( x, y \in I \), (ii) \( d([x, y]) = 0 \) for all \( x, y \in I \), then
it is commutative. we have proved analogues of these results in the setting of *-prime near-rings with derivation. Finally it is also shown that the restriction of 2-torsion freeness of \( R \) used by authors while proving above \((ii)\) is redundant.

**Theorem 3.4.** Let \( N \) be a 2-torsion free *-prime near-ring and \( I \) a nonzero *-ideal of \( N \). If \( N \) admits a nonzero derivation \( d \) such that \([d(x), d(y)] = 0\), for all \( x, y \in I \) and \( d \) commutes with \(*\), then \( N \) is a commutative ring.

**Proof.** By hypothesis we have, \([d(x), d(y)] = 0\), for all \( x, y \in I \). Now replacing \( y \) by \( xy \) and using Lemma 2.1, we obtain that \( d(x)[d(x), y] + [d(x), x]d(y) = 0 \), for all \( x, y \in I \). Putting \( yz \) where \( z \in N \) for \( y \) in the last expression and using Lemma 2.1, we get \( d(x)y[d(x), z] + [d(x), x]yd(z) = 0 \), for all \( x, y, z \in N \). Replacing \( z \) by \( d(t) \) where \( t \in I \) and using hypothesis, we arrive at \( d(x), yd^2(t) = 0 \), for all \( x, y, t \in I \). This implies that \([d(x), x]Id^2(t) = \{0\}\). Since \( d^* = d \) and \( d \) is a *-ideal, we get \([d(x), x]Id^2(t) = [d(x), x]*Id^2(t) = \{0\}\) by Lemma 2.1. Applying Lemma 2.4, either \( d^2(t) = 0 \) for all \( t \in I \) or \([d(x), x] = 0 \) for all \( x \in I \). If \( d^2(t) = 0 \) for all \( t \in I \), then by Lemma 2.7 we conclude that \( d = 0 \), a contradiction. Thus we conclude that \([d(x), x] = 0 \) for all \( x \in I \) and hence Lemma 2.6 finishes the proof.

**Theorem 3.5.** Let \( N \) be a *-prime near-ring and \( I \) a nonzero *-ideal of \( N \). If \( N \) admits a nonzero derivation \( d \) such that \([d(x, y)] = 0\), for all \( x, y \in I \) and \( d \) commutes with \(*\), then \( N \) is a commutative ring.

**Proof.** By hypothesis we have \([d(x, y)] = 0\), for all \( x, y \in I \). Now replacing \( y \) by \( yx \), we obtain that \([d(x, yx)] = 0\). Using hypothesis and Lemma 2.1 we obtain that \([x, y]d(x) = 0\) for all \( x, y \in I \). This implies that, for any \( z \in N \), replacing \( y \) by \( zy \) and using Lemma 2.1 again we arrive at \([x, z]yd(x) = 0\) for all \( x, y \in I \) i.e.: \([x, N]Id(x) = \{0\}\) for all \( x \in I \). Now by Lemma 2.5, the result follows.

**Theorem 3.6.** Let \( N \) be a *-prime near-ring and \( I \) a nonzero *-ideal of \( N \). If \( N \) admits a nonzero derivation \( d \), which commutes with \(*\) and one of the following conditions hold \((i)\) \( d([x, y])] = \pm [x, y]\), \((ii)\) \( d([x, y]) = \pm (xoy)\), \((iii)\) \( d(xoy) = 0\), \((iv)\) \( d(xoy) = \pm (xoy)\) and \((v)\) \( d(xoy) = \pm [x, y]\) for all \( x, y \in I \), then \( N \) is a commutative ring.

**Proof.** It can be proved using the same techniques, as in Theorem 3.5.

The following example justifies the existence of *-primeness in the hypotheses of the Theorems 3.5 and 3.6.
Example 3.1. Let $S$ be a left near-ring. Suppose $N = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid x, y, 0 \in S \right\}$. Define $d, \ast : N \rightarrow N$ such that

$$d \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\left( \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right)^\ast = \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

It is straightforward to check that $N$ is $\ast$-near-ring and $I = \left\{ \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid x, 0 \in S \right\}$ is a $\ast$-ideal of $N$. If we set $p = \begin{pmatrix} 0 & 0 & s \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ with $0 \neq s \in S$, then $pNp = \{0\} = pNp^\ast$ proving that $N$ is not $\ast$-prime near-ring. Furthermore $d$ is a nonzero derivation, which commutes with $\ast$ and satisfies the following conditions:

(i) $d([x, y]) = 0$, (ii) $d([x, y]) = \pm [x, y]$, (iii) $d([x, y]) = \pm (xoy)$, (iv) $d(xoy) = 0$, (v) $d(xoy) = \pm (xoy)$ and (vi) $d(xoy) = \pm [x, y]$ for all $x, y \in I$. However $N$ is not a commutative ring.

Recently, L.Oukhtite and S.Salhi [6, Theorem 2] obtained the following result: Let $R$ be $\ast$-prime ring with characteristic not 2 and $I$ be a nonzero $\ast$-ideal of $R$. Suppose there exist derivations $d_1$ and $d_2$ which commute with $\ast$ such that $d_1(x)x - x d_2(x) \in Z(R)$ for all $x \in I$. If $d_2 \neq 0$, then $R$ is commutative. The main purpose of the following theorem is to prove its analogue for $\ast$-prime near-rings.

Theorem 3.7. Let $N$ be a 2-torsion free $\ast$-prime near-ring and $I$ a nonzero $\ast$-ideal of $N$. Suppose there exist derivations $d_1$ and $d_2$ which commute with $\ast$ such that $d_1(x)x - x d_2(x) \in Z$ for all $x \in I$. If $d_2 \neq 0$, then $N$ is a commutative ring.

Proof. If $I \cap Z = \{0\}$; as $d_1(x)x - x d_2(x) \in I \cap Z$, then $d_1(x)x = x d_2(x)$ for all $x \in I$. Linearizing this relation with the help of Lemma 2.1 we get $d_1(x)y + d_1(y)x = x d_2(y) + y d_2(x)$ for all $x, y \in I$. Replacing $y$ by $yx$ in the last relation and using the same again, combined with the fact that $d_1(x)x = x d_2(x)$ and Lemma 2.1, we arrive at $[x, y d_2(x)] = 0$ for all $x, y \in I$. Now putting $ty$, where $t \in N$ and using Lemma 2.1, to get $[x, t y d_2(x) = 0$ i.e.; $[x, N]Id_2(x) = \{0\}$ for all $x \in I$. Since $d_2 \neq 0$, from Lemma 2.5 we conclude that $N$
is a commutative ring. Next, assume that $I \cap Z \neq \{0\}$. This implies that $Z \neq \{0\}$. Therefore by Lemma 2.3, $N$ is a ring. Choose $0 \neq z \in I \cap Z$ in such way $z^* = \pm z$. If $z^* = z$, then nothing to do, otherwise we consider $t = z - z^*$, then $t \in I \cap Z$ and $t^* = -t$. Linearizing $d_1(x)x - xd_2(x) \in Z$, we get

$$d_1(x)y + d_1(y)x - xd_2(y) - yd_2(x) \in Z$$

(3.5) for all $x, y \in I$. Replacing $y$ by $z$ and using $d_2(z) \in Z$ in the relation (3.5), we arrive at

$$z(d_1(x) - d_2(x)) + (d_1(z) - d_2(z))x \in Z$$

(3.6) for all $x \in I$. Putting $y = z^2$ in the relation (3.5) and using the relation (3.6), we conclude that $z(d_1(z) - d_2(z))x \in Z$ for all $x \in I$. This implies that $z(d_1(z) - d_2(z))N[x, t] = \{0\} = z(d_1(z) - d_2(z))N[x, t]^*$ for all $x \in I, t \in N$. Which leads us to $I \subseteq Z$ and by Lemma 2.6, $N$ is a commutative ring or $d_1(z) = d_2(z)$. If $d_1(z) = d_2(z)$, then by relation (3.6) we conclude that $z(d_1(x) - d_2(x)) \in Z$ and so $(d_1(x) - d_2(x)) \in Z$ for all $x \in I$. Hence, $d(I) \subseteq Z$ where $d = d_1 - d_2$. Then it follows by Lemma 2.6 that $N$ is a commutative ring. If $d_1 = d_2$ then $d_1(x)x - xd_1(x) = [d_1(x), x] \in Z$ for all $x \in I$ and by Theorem 3.3, we conclude that $N$ is a commutative ring.

Recently, L.Oukhtite and S.Salhi [5, Theorem 3.3] obtained the following result for prime near-rings: Let $N$ be a prime near-ring, which admits a nonzero derivation $d$. If $d$ acts as a homomorphism on $N$, then $d$ is the identity map. Motivated by this result we investigated its analogue in the setting of $*$-prime near-rings under some constraints.

**Theorem 3.8.** Let $N$ be $*$-prime near-ring, admitting a derivation $d$ and a nonzero $*$-ideal $I$. If $d$ acts as a homomorphism on $I$ and $d* = *d$, then $d = 0$.

**Proof.** Assume that $d$ acts as a homomorphism on $I$. Then one obtains that $d(xy) = d(x)d(y) = d(x)y + xd(y)$ for all $x, y \in I$. Replacing $y$ by $yz$, where $z \in I$, we obtain that $d(xy)d(yz) = d(x)yz + xyd(yz)$. Since $d$ acts as a homomorphism on $I$, we deduce that $d(xy)d(z) = d(x)yz + xyd(z) + xyd(yz).$ Using Lemma 2.1 we arrive at $xd(y)d(z) + d(xy)d(yz) = d(x)yz + xyd(z) + xyd(yz) + xyd(yz)$. This implies that $xyd(z) + xyd(yz) + d(xy)d(yz) = d(xy)d(yz)$ i.e.; $d(xy)(d(z) - z) = 0$ for all $x, y, z \in I$. Since $I$ is a $*$-ideal and $d* = *d$, we conclude that $d(xy)(d(z) - z) = \{0\} = \{d(x)\}^*I(d(z) - z)$ for all $x, z \in I$. By Lemma 2.4 we infer that either $d(z) = z$ or $d(x) = 0$. If first case holds, then replacing $z$ by $zx$ we obtain that $d(zx) = zx$ i.e.; $zd(x) + d(z)x = zx$. This implies that $zd(x) = 0$ i.e.; $Id(x) = \{0\}$. Finally we get $tId(x) = t^*Id(x) = \{0\}$, where $0 \neq t \in N$. 

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By Lemma 2.4 we obtain that $d(x) = 0$. Now combining both the cases we conclude that $d(x) = 0$ for all $x \in I$. Putting $xt$ for $x$ where $t \in N$, we obtain that $xd(t) + d(x)t = 0$ i.e.; $xd(t) = 0$. Finally we get that $sId(t) = \{0\} = s^*Id(t)$, where $0 \neq s \in N$. Again Lemma 2.4 insures that $d = 0$.

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