Uniform Continuity and its Example

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ABSTRACT

The aim of this paper is to study about uniform continuity and its some examples.

1. INTRODUCTION:

The first published definition of uniform continuity was by Heine in 1870, and in 1872 he published a proof that a continuous function on a open interval need not be uniformly continuous. The proofs are almost verbatim given by Dirichlet in his lectures on definite integrals in 1854. The definition of uniform continuity appears earlier in the work of Bolzano where he also proved that continuous functions on an open interval do not need to be uniformly continuous. In addition he also states that a continuous function on a closed interval is uniformly continuous, but he does not give a complete proof.

Uniform continuity is a much stronger condition than continuity and it is used in lots of places. One very fundamental usage of uniform continuity is in the proof that every continuous function of a closed interval is Riemann integrable. Another one of the reasons it is useful is that if we are working with a continuous function on a closed interval, we get uniform continuity for free.

2. UNIFORM CONTINUITY:

Continuity itself is a local property of a function—that is, a function $f$ is continuous, or not, at a particular point, and this can be determined by looking only at the values of the function in an (arbitrarily small) neighbourhood of that point. When we speak of a function being continuous on an interval, we mean only that it is continuous at each point of the interval. In contrast, uniform continuity is a global property of $f$, in the sense that the standard definition refers to pairs of points rather than individual.

Let $A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$ be continuous. Then for each $a \in A$ and for given $\varepsilon > 0$, there exists a $\delta(\varepsilon, a) > 0$ such that $x \in A$ and $|x - a| < \delta$ imply $|f(x) - f(a)| < \varepsilon$. We emphasize that $\delta$ depends, in general, on $\varepsilon$ as well as the point $a$. Intuitively this is clear because the function $f$ may change its values rapidly near certain points and slowly near other points.

Now we will give definition of uniform continuity.

A function $f : A \to \mathbb{R}$, where $A \subset \mathbb{R}$ is said to be uniformly continuous on $A$ if given $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $x, y \in A$ and $|x - y| < \delta$, we have $|f(x) - f(y)| < \varepsilon$.

Next, we will give sequential definition of uniform continuity.

SEQUENTIAL DEFINITION OF UNIFORM CONTINUITY: Let $f : A \subset \mathbb{R} \to \mathbb{R}$ be a real valued function, then the following are equivalent:

1. $f$ is uniformly continuous on $A$

2. If $<x_n>$ and $<y_n>$ be any two sequence in $A$ such that $|x_n - y_n| \to 0$ implies $|f(x_n) - f(y_n)| \to 0$ when $n \to \infty$
Hence, if there exist two sequences \( <x_n> \) and \( <y_n> \) in \( A \) such that \( |x_n-y_n| \rightarrow 0 \) but \( |f(x_n)-f(y_n)| \not\rightarrow 0 \) as \( n \rightarrow \infty \), then \( f \) is not uniform continuous on \( A \).

Next, we will give some examples which are not uniform continuous.

EXAMPLES:

1. \( f(x)=1/x \), where \( x \in (0,1) \)
   Let \( x_n=1/n \) and \( y_n=1/(n+1) \)
   Now, \( |x_n-y_n|=|(1/n)-(1/(n+1))| \rightarrow 0 \) as \( n \rightarrow \infty \), but \( |f(x_n)-f(y_n)|=|1/n-1/(n+1)| \not\rightarrow 0 \) as \( n \rightarrow \infty \). Hence, \( f(x) \) is not uniform continuous.

Now, we will use some results to check the uniform continuity of functions.

Theorem 1: If \( f:A \rightarrow \mathbb{R} \) be a continuous function on a compact set \( A \), then \( f \) is uniformly continuous on \( A \).

Example:
1. \( f(x)=\sin(x) \) on \( [a,b] \)
2. \( f(x)=x^n \) on \( [a,b] \)

Theorem 2: If \( f(a,b) \rightarrow \mathbb{R} \) where \( a,b \in \mathbb{R} \) be a continuous function, then \( f \) is uniformly continuous on \( (a,b) \) iff \( \lim_{x \to a^+} f(x) \) and \( \lim_{x \to b^-} f(x) \) both exist finitely.

Example:
1. By Theorem 2, \( f(x)=1/x \) is not uniformly continuous on \( (0,1) \) as \( \lim_{x \to 0} f(x) \) does not exist.
2. \( f(x)=\sin(1/x) \) on \( (0,1) \) is not uniformly continuous as \( \lim_{x \to 0} f(x) \) doesn't exist.

Theorem 3: If \( f: A \subset \mathbb{R} \rightarrow \mathbb{R} \) be a differentiable function with bounded derivative, then \( f \) is uniformly continuous on \( A \), but not conversely.

Example:
1. \( f(x)=\sqrt{x} \) on \( (0,1) \) is uniformly continuous however \( f'(x)=1/2 \sqrt{x} \) is not bounded on \( (0,1) \).
2. \( f(x)=\sin^2(x) \) on \( \mathbb{R} \) is uniformly continuous as it is differentiable on \( \mathbb{R} \) and its derivative \( f'(x)=\sin(2x) \) is bounded on \( \mathbb{R} \).

Theorem 4: If \( f \) is a real valued continuous function which is periodic, then \( f \) is uniformly continuous.

Example:
1. \( f(x)=\sin(x) \)
2. \( f(x)=\cos(x) \)
3. \( f(x)=\cos(\sin(x)) \)
4. \( f(x)=e^{\sin(\cos(x))} \), where \( x \in \mathbb{R} \).
Reference:-