Behaviour of Solutions of Linear Systems

Vijayalakshmi Menon R
Asst. Prof., Dept. of Mathematics, Govt. College, Madappally, Vatakara, Calicut, Kerala, S. India

Abstract
This paper deals with the behaviour of solutions of linear systems. The notions of stability, boundedness and asymptotic behaviour of solutions of a general linear system are studied.

AMS SUBJECT CLASSIFICATION CODE : 34D20

Keywords: Stability, perturbation, boundedness, almost-constant, trace, perturbed equation, perturbing matrix.

1. INTRODUCTION
We consider the behaviour of the solutions of the linear differential equation
\[ \frac{d}{dt} z = [A + B(t)]z \quad \text{(1)} \]
where \( A \) is a constant matrix and \( B(t) \) is small as \( t \to \infty \).
Two particularly important cases are those where \( ||B(t)|| \to 0 \) or where \( \int_0^\infty ||B(t)|| dt < \infty \)

The solutions of (1) share many properties with the solutions of
\[ \frac{dy}{dt} = Ay \quad \text{(2)} \]
so far as their behaviours are concerned.

In this paper, section 2 deals with the concept of stability of linear equations. In section 3, the boundedness property of solutions and the sufficient conditions for boundedness of solutions are studied in detail. Section 4 illustrates the asymptotic behaviour of solutions of linear systems.

2. STABILITY OF LINEAR EQUATIONS

2.1 Defn: The solutions of \( \frac{dy}{dt} = A(t)y \to [3] \)
are stable with respect to a property P and perturbations \( B(t) \) of type T if the solutions of
\[ \frac{d}{dt} z = [A(t) + B(t)]z \to (4) \]
also possess property P. If this is not true, the solutions of (3) are said to be unstable with respect to property P under perturbations of type T.
To illustrate the above definition, we consider two simple differential equations:
\[ \frac{du}{dt} = -au \to (5) \]
and
\[ \frac{dv}{dt} = [-a + b(t)]v \to (6) \]

\( a > 0 \) and \( b(t) \to 0 \) as \( t \to \infty \).

Considering the solutions of (5) and (6)
\[ \frac{du}{dt} = -au \Rightarrow \frac{du}{dt} = -ad t \]
\[ \Rightarrow \log u = -at + c \]
\[ \Rightarrow u = ce^{-at} \Rightarrow \lim_{t \to \infty} u = 0 \]
\[ \frac{dv}{dt} = [-a + b(t)]v \]
\[ \Rightarrow \frac{dv}{v} = [-a + b(t)]dt \]
\[ \Rightarrow \log v = -at + \int b(t)dt \]
\[ \Rightarrow v = e^{-at} + \int b(t)dt \]
\[ \Rightarrow \lim_{t \to \infty} v = 0 \]

Thus, both solutions \( u \) and \( v \) tend to zero as \( t \to \infty \).

Also, \( \lim_{t \to \infty} \frac{\log u}{t} = \lim_{t \to \infty} \frac{\log v}{t} = -a \)

So, both solutions \( u \) and \( v \) have the following properties:

1) \( \lim_{t \to \infty} u = \lim_{t \to \infty} v = 0 \)

and

2) \( \lim_{t \to \infty} \frac{\log u}{t} = \lim_{t \to \infty} \frac{\log v}{t} = -a \)

Now suppose \( a=0 \) and \( b(t) = \frac{1}{t} \).

Then \( \frac{du}{dt} = -au \Rightarrow u = ce^{-at} \)
\[ \Rightarrow u = c \text{ when } a = 0 \]
and
\[ \frac{dv}{dt} = [-a + b(t)]v \]
\[ \Rightarrow \frac{dv}{v} = [-a + b(t)]dt \]
\[ \Rightarrow \log v = -at + \log t \]
\[ \Rightarrow v = te^{-at} \]
which is unbounded as \( t \to \infty \).

With respect to property (ii), since \( a=0 \),
\[ \lim_{t \to \infty} \frac{\log u}{t} = 0 \]
Also,
\[ \lim_{t \to \infty} \frac{\log v}{t} = 0 \]

Thus, there is stability with respect to property (ii) but instability with respect to property (i) – the property of boundedness.

2.2 Note:
If we replace \( \frac{1}{t} \) by a function which is integrable over \((t_0, \infty)\), then boundedness will be preserved.

2.3 Note:
The most important property of solutions is that of boundedness. If a solution is bounded, we are interested in knowing whether or not it approaches zero at \( t \to \infty \).

3.1 BOUNDEDNESS OF SOLUTIONS([1],[2])

3.1.1 Definition:
We call the coefficient matrix \( A(t) \) of the differential equation \( \frac{dz}{dt} = A(t)z \) almost constant if
\[
\lim_{t \to \infty} A(t) = A \text{ a constant matrix.}
\]

3.1.2 Lemma (Fundamental lemma):
If \( u, v \geq 0 \), if \( C_1 \) is a positive constant,
and if
\[
u \leq C_1 + \int_0^t uv dt_1 \tag{7}\]
then \( u \leq C_1 \exp(\int_0^t vd_1) \tag{8}\)
Proof: From (7), we have
\[
\frac{uv}{C_1 + \int_0^t uv dt_1} \leq v \to 9
\]
Integrating both sides of (9) between 0 and \( t \), we get
\[
\log[C_1 + \int_0^t uv dt_1] - \log C_1 \leq \int_0^t v dt_1
\]
\[
\text{i.e.,} \log(\frac{C_1 + \int_0^t uv dt_1}{C_1}) \leq \int_0^t v dt_1
\]
\[
C_1 + \int_0^t uv dt_1 \leq C_1 \exp(\int_0^t v dt_1) \to 10
\]
\[
\therefore \quad u \leq C_1 + \int_0^t uv dt_1 \leq C_1 \exp(\int_0^t v dt_1) \to 11
\]
which is the fundamental lemma.

3.1.3 Theorem:
If all solutions of
\[
\frac{dy}{dt} = Ay \to 12
\]
where \( A \) is a constant matrix, are bounded as \( t \to \infty \), then the same is true of the solutions of
\[
\frac{dz}{dt} = [A + B(t)]z \to 13
\]
provided \( \int_0^\infty ||B(t)|| dt < \infty \)
Proof:
Equation (13) can be written as
\[
\frac{dz}{dt} = Az + B(t)z \to 14
\]
Every solution of (14) satisfies a linear integral equation
\[
z = y + \int_0^t Y(t - t_1)B(t_1)z(t_1) dt_1 \to 15
\]
where \( y \) is the solution of (12) for which \( y(0) = z(0) \) and
\( Y \) is the matrix solution of
\[
\frac{dy}{dt} = Ay, \quad y(0) = I \to 16
\]
we have \( y = Yz(0) \)
Let \( C_1 = \max_{t \geq 0} ||Y(t)||, \quad \sup_{t \geq 0} ||Y(t)|| \)
Then, from (15), we get
\[
||z|| \leq ||y|| + \int_0^t ||Y(t - t_1)|| ||B(t_1)|| ||z(t_1)|| dt_1
\]
\[
\leq C_1 + C_1 \int_0^t ||B(t_1)|| ||z(t_1)|| dt_1 \tag{17}
\]
Applying the fundamental lemma in (17), we get
\[
||z|| \leq C_1 \exp(\int_0^t ||B(t_1)|| dt_1) \to 18
\]
Since \( \int_0^\infty ||B(t)|| dt < \infty \) from (18), it follows that \( ||z|| \) is bounded.
\( \therefore \) the solutions of equation (13) are bounded.
Hence, the theorem.

3.1.4 Theorem
If all the solutions of the equation
\[
\frac{dy}{dt} = Ay \to 19
\]
approach zero as \( t \to \infty \), the same holds for the solutions of
\[
\frac{dz}{dt} = [A + B(t)]z \to 20
\]
provided that \( ||B(t)|| \leq C_1 \) for \( t \geq t_0 \), where \( C_1 \) is a constant which depends upon \( A \).

Proof
Every solution of (20) satisfies a linear integral equation
\[
z = y + \int_0^t Y(t - t_1)B(t_1) \to 21
\]
\[
z(t_1) dt_1 \to 22
\]
where \( Y \) is the matrix solution of
\[
\frac{dy}{dt} = Ay, \quad y(0) = I \to 23
\]
\[
\Rightarrow \log Y = At + B \to 24
\]
\[
\Rightarrow Y = e^{At+B} = Ce^{At} \to 25
\]
Since \( ||Y|| \to 0 \) as \( t \to \infty \), \( \exists \) a positive constant \( C \) such that
\( ||Y|| \leq C_2 e^{-Ct} \) for \( t \geq 0 \) \to (22)
By theorem 3.1.3, we have \( y = Yy(0) : ||y|| \leq C_2 e^{-Ct} \) for \( t \geq 0 \) \to (23)
Hence, from (21),
\[
||z|| \leq C_2 e^{-Ct} + C_2 \int_0^t e^{C(t-t_1)} ||B(t_1)|| ||z(t_1)|| dt_1 \to 26
\]
Since \( ||B(t)|| \leq C_1 \) for \( t \geq t_0 \), we get
\[
||z|| e^{Ct} \leq C_2 + C_1 C_2 \int_0^t e^{C(t-t_1)} ||z(t_1)|| dt_1 \to 27
\]
Applying fundamental lemma in equation (25), we get
\[
||z|| e^{Ct} \leq C_2 e^{Ct} \to 28
\]
i.e., \( ||z|| e^{Ct} \leq C_2 e^{Ct} e^{Ct} \to 29 \)
If $C_1C_2 \ll \infty$, then the above equation gives $||z|| \to 0$ as $t \to \infty$. But the constants $C_2$ and $\omega$ depend upon the characteristic roots of $A$. Hence, it follows that $C_1$ depends upon $A$. Hence the theorem.

3.2 SUFFICIENT CONDITIONS FOR BOUNDEDNESS OF SOLUTIONS (11)

In this section, the sufficient conditions required for the solutions of a linear system to be bounded, are being dealt with.

The boundedness of the solution of

$$\frac{dy}{dt} = A(t)y \to (28)$$

together with the condition $||B(t)|| \to 0$ as $t \to \infty$ is not sufficient to ensure the boundedness of all solutions of

$$\frac{dz}{dt} = [A(t) + B(t)]z \to (29)$$

Even if we amend the condition $||B(t)|| \to 0$ as $t \to \infty$, by the condition $\int_0^\infty ||B(t)|| dt < \infty$, the sufficiency remains unjustified. This fact is illustrated in the following theorem:

3.2.1 Theorem

There is an equation of the type

$$\frac{dy}{dt} = A(t)y$$

with the property that all solutions approach zero as $t \to \infty$, and a matrix $B(t)$ for which $\int_0^\infty ||B(t)|| dt < \infty$, such that all solutions of the equation

$$\frac{dz}{dt} = [A(t) + B(t)]z$$

are not bounded.

Proof: Consider the equations

$$\frac{dy_1}{dt} = -ay_1 \to (30)$$

and

$$\frac{dy_2}{dt} = [\sin(\log t) + \cos(\log t) - 2a]y_2 \to (31)$$

We solve for $y_1$ and $y_2$

Equation (30) $\to \frac{dy_1}{dy_1} = -a dt$

Integrating, $\log y_1 = -at + \log C_1$

i.e., $y_1 = C_1e^{-at}$

Equation (31)

$\to \frac{dy_2}{dt} = [\sin(\log t) + \cos(\log t) - 2a]dt$

Integrating,

$$\log y_2 = [tsin(\log t) - 2at] + \log C_2$$

$\therefore y_2 = C_2e^{tsin(\log t)-2at}$

Thus, the general solutions of (30) and (31) are respectively

$$y_1 = C_1e^{-at} \to (32)$$

and

$$y_2 = C_2e^{tsin(\log t)-2at} \to (33)$$

If ‘a’ is any negative constant, then every solution of (32) and (33) approach zero as $t \to \infty$

Let

$$B(t) = \begin{bmatrix} 0 & 0 \\ -a & 0 \end{bmatrix}$$

be the perturbing matrix.

The perturbed equation has the form

$$\frac{dz}{dt} = -az_1 \to (34)$$

and

$$\int_0^t e^{-t_1 \sin(\log t_1)} dt_1 > \int_0^t e^{-t_1 \sin(\log t_1)} dt_1 \to (35)$$

Equation (34) gives $z_1 = C_1e^{-at}$ and Equation (35) gives $z_2 = e^{-[\sin(\log t) - 2a]}$

$$\begin{cases}
C_2 + C_1 \int_0^t e^{-t_1 \sin(\log t_1)} dt_1 \\
C_2 + C_1 \int_0^t e^{-t_1 \sin(\log t_1)} dt_1
\end{cases}$$

Let $t = e^{[\log t + a]}$

If $1 < 2a < 1 + e^{-\frac{\pi}{2}}$, then

$$\int_0^t e^{-t_1 \sin(\log t_1)} dt_1 > \int_0^t e^{-t_1 \sin(\log t_1)} dt_1 > t(e^{\frac{\pi}{2}} - e^{-a}) \exp \left(-\frac{e^{-a}}{2}\right),$$

which implies the solutions of (34) and (35) will be bounded only if $C_1 = 0$

Now, $C_1 = 0 \Rightarrow z_1(0) = 0$

Thus, if $z_1(0) \neq 0$, the solutions of (35) are not bounded.

3.2.2 Theorem:

If all the solutions of the equation

$$\frac{dy}{dt} = A(t)y$$

are bounded, then all the solutions of the equation

$$\frac{dz}{dt} = [A(t) + B(t)]z$$

are bounded, provided

(a) $\int_0^\infty ||B(t)|| dt < \infty$

&

(b) $\lim_{t \to \infty} \int_0^t tr(A) dt > -\infty$

Proof: Expressing $z$ in terms of $y$, we have

$$z = y + \int_0^t Y(t) Y^{-1}(t) B(t) z(t) dt_1$$

Thus,

$$||z|| \leq ||y|| + \int_0^t ||Y(t)|| ||Y^{-1}(t)|| ||B(t)|| ||z(t)|| dt_1$$

Since $detY = \exp \left[\int_0^t tr(A) dt \right]$ if condition (b) is satisfied, then

$$||Y^{-1}(t)||$$

is bounded as $t \to \infty$

Thus,

$$||z|| \leq C_1 + C_2 \int_0^t ||B(t)|| ||z(t)|| dt_1$$

Applying the fundamental lemma in the above equation, we get

$$||z|| \leq C_1 \exp \left(C_1 \int_0^t ||B(t)|| ||z(t)|| dt_1 \right)$$

Since $\int_0^t ||B(t)|| dt < \infty$, $||z||$ is bounded.

Hence, the theorem.
4. ASYMPTOTIC BEHAVIOUR OF SOLUTIONS

In the previous section, we have dealt with the necessary and sufficient conditions for the boundedness of a solution. If a solution is bounded, we are interested in knowing whether or not it approaches zero as \( t \to +\infty \), which depicts the asymptotic behaviour of the solutions.

Consider the linear system
\[
\frac{dy}{dt} = A(t)y, \quad t \geq 0 \to (36)
\]
where \( A(t) \) is a real-valued continuous \( nxn \) matrix on \( 0 \leq t < \infty \).

We want to find the behaviour of solutions of (36) as \( t \to +\infty \).

If the eigen values of the matrix \( A \) are known, all solutions of (36) are completely determined. Hence, the eigen values determine the behaviour of solutions as \( t \to +\infty \).

4.1 Theorem:

Let \( A(t) \) be a real-valued, continuous, \( nxn \) matrix on \( [0, \infty) \).

Let \( M(t) \) be the largest eigen value of \( (t) + A^T(t) \), where \( A^T(t) \) is the transpose of the matrix \( A(t) \).

If \( \lim_{t \to +\infty} \int_{t_0}^{t} M(s)ds = -\infty \) \( (t_0 > 0 \text{ is fixed}) \to (37) \), then every solution of (36) tends to zero as \( t \to +\infty \).

Proof:

Let \( \phi(t) \) be a solution of (36).

Then \( |\phi(t)|^2 = \phi^T(t)\phi(t) \)

\[
\frac{d}{dt}|\phi(t)|^2 = \phi^T(t)\phi(t) + \phi^T(t)A(t)\phi(t) + A^T(t)\phi^T(t)\phi(t)
\]

\[
= \phi^T(t)[A(t) + A^T(t)]\phi(t)
\]

Since \( M(t) \) is the largest eigen value of the symmetric matrix \( A(t) + A^T(t) \), we get

\[
|\phi^T(t)[A(t) + A^T(t)]\phi(t)| \leq M(t)|\phi(t)|^2
\]

Thus,

\[
0 \leq |\phi(t)|^2 \leq |\phi(t_0)|^2 \left( \exp \left[ \int_{t_0}^{t} M(s)ds \right] \right) \to (38)
\]

By condition (37), the right side of (38) tends to zero.

Hence, \( \lim_{t \to +\infty} \phi(t) = 0 \).

Hence, the theorem.

4.2 Theorem:

Let \( m(t) \) be the smallest eigen value of \( A(t) + A^T(t) \). If

\[
\lim_{t \to +\infty} \int_{t_0}^{t} m(s)ds = +\infty \quad (t_0 > 0 \text{ is fixed}) \to (39)
\]

then every non-zero solution of (36) is unbounded as \( t \to +\infty \).

Proof: Let \( \phi(t) \) be a solution of (36).

As in the previous theorem, we have

\[
\frac{d}{dt}|\phi(t)|^2 = \phi^T(t)[A(t) + A^T(t)]\phi(t) \to (40)
\]

Since \( m(t) \) is the smallest eigen value of \( (t) + A^T(t) \), we get

\[
\frac{d}{dt}|\phi(t)|^2 \geq m(t)|\phi(t)|^2
\]

Thus, \( \frac{d}{dt}\left\{ e^{-\int_{t_0}^{t} m(s)ds} |\phi(t)|^2 \right\} \geq 0 \)

\[
|\phi(t)|^2 \geq |\phi(t_0)|^2 e^{\int_{t_0}^{t} m(s)ds} \to (41)
\]

By condition (39), the right side of (41) tends to \(+\infty \) as \( t \to +\infty \).

\( \lim_{t \to +\infty} |\phi(t)| = +\infty \)

i.e., the solution \( \phi(t) \) is unbounded.

Hence, the theorem.

5. CONCLUSION

This paper is a work on the behaviour of solutions of linear systems, when the time is increased indefinitely; which is a kind of stability property. This provides an insight into the necessary steps to be taken to avoid unwanted phenomena or criteria in a system.

REFERENCES